Orthogonal polynomials and a generalized Szegő condition

Polynômes orthogonaux et la condition de Szegő généralisée

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Abstract

Asymptotical properties of orthogonal polynomials from the so-called Szegő class are very well-studied. We obtain asymptotics of orthogonal polynomials from a considerably larger class and we apply this information to the study of their spectral behavior. *To cite this article: S. Denisov, S. Kupin, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Les propriétés asymptotiques des polynômes orthogonaux de la classe de Szegő sont très bien étudiées. Nous obtenons les asymptotiques des polynômes orthogonaux appartenant à une classe considérablement plus large. Ensuite, nous appliquons cette information à l'étude du comportement spectral de ces derniers. *Pour citer cet article : S. Denisov, S. Kupin, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Introduction

In this paper, we prove asymptotics for orthogonal polynomials from the Szegő class with a polynomial weight and we apply the information to the study of their spectral behavior.

Let σ be a non-trivial Borel probability measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. Consider orthonormal polynomials $\{\varphi_n\}$ with respect to the measure,

$$\int_{\mathbb{T}} \varphi_n \overline{\varphi_m}\, d\sigma = \delta_{nm}$$

where δ_{nm} is the Kronecker's symbol. It is very well-known [3,4,6,7] that polynomials $\{\varphi_n\}$ generate a sequence $\{\alpha_k\}, |\alpha_k| < 1$, of the so-called Verblunsky coefficients through special recurrence relations. Conversely, the measure σ (and polynomials $\{\varphi_n\}$) are completely determined by the sequence $\{\alpha_k\}$.

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Hence, it is natural to express properties of the sequence $\{\alpha_k\}$ and polynomials $\{\varphi_n\}$ in terms of σ and vice versa.

We say that σ is a Szegő measure ($\sigma \in$ (S), for the sake of brevity), if $d\sigma = \sigma'_{ac} dm + d\sigma_s$ and the density σ'_{ac} of the absolutely continuous part of σ is such that

$$\int_{\mathbb{T}} \log \sigma'_{ac} \, dm > -\infty$$

Here, the singular part of σ is denoted by σ_s, and m is the probability Lebesgue measure on $\mathbb{T}$, $dm(t) = dt/(2\pi i t) = 1/(2\pi)\, d\theta, t = e^{i\theta} \in \mathbb{T}$.

For instance [3,7], a measure σ belongs to the Szegő class if and only if the corresponding sequence $\{\alpha_k\}$ is in l^2. Moreover, this happens if and only if analytic polynomials are not dense in $L^2(\sigma)$. Asymptotic properties of orthogonal polynomials connected to $\sigma \in$ (S) can be easily described in terms of the function

$$D(z) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} \frac{t+z}{t-z}\, \log \sigma'_{ac}(t)\, dm(t)\right)$$

lying in the Hardy class $H^2(\mathbb{D})$ on the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Namely, we have

$$\lim_{n\to\infty} \int_{\mathbb{T}} |D\varphi_n^* - 1|^2 \, dm = 0$$

and, in particular, $\lim_{n\to\infty} D(z)\varphi_n^*(z) = 1$ for every $z \in \mathbb{D}$. Above, $\varphi_n^*(z) = z^n \overline{\varphi_n(1/\bar{z})}$. A modern presentation and recent advances in this direction can be found in [4,6].

It is extremely interesting and important to obtain similar results for different classes of measures. Consider a trigonometric polynomial $p, p(t) \geq 0, t \in \mathbb{T}$, given by

$$p(t) = \prod_{j=1}^{N} |t - \zeta_j|^{2\kappa_j} \tag{1}$$

Here $\{\zeta_j\}$ are points lying on $\mathbb{T}$ and κ_j are their "multiplicities". We say that σ is in the polynomial Szegő class (i.e., σ is a (pS)-measure or $\sigma \in$ (pS), to be brief), if $d\sigma = \sigma'_{ac} dm + d\sigma_s$, σ_s being the singular part of the measure, and

$$\int_{\mathbb{T}} p(t) \log \sigma'_{ac}(t)\, dm(t) > -\infty \tag{2}$$

The asymptotic behavior of orthogonal polynomials for $\sigma \in$ (pS) is completely described in Theorems 1.2 and 1.3. This information is used to construct wave operators for the so-called CMV-representations in Theorem 1.4. The approximation by analytic polynomials in $L^2(\sigma), \sigma \in$ (pS), is addressed in Theorem 1.5.

1. Results

We fix the polynomial p (1) for the rest of this paper. For the sake of transparency we assume $\kappa_j = 1$; the discussion of the general case follows the same lines. Let $\mathcal{C}$ and $\mathcal{C}_0$ be the CMV-representations connected to σ and m (see [1,6, Ch. 4]), and $\operatorname{rank}(\mathcal{C} - \mathcal{C}_0) < \infty$.

We set $\Phi(\mathcal{C}) = \int_{\mathbb{T}} p(t) \log \sigma'_{ac}(t)\, dm(t)$.

Lemma 1.1 *Let* $\operatorname{rank}(\mathcal{C} - \mathcal{C}_0) < \infty$. *Then there is a polynomial P such that*

$$\int_{\mathbb{T}} p \log \sigma'_{ac}\, dm = a_0 t_0 + \operatorname{Re} \operatorname{tr}(P(\mathcal{C}) - P(\mathcal{C}_0)) \tag{3}$$

where $a_0 = 2\int_{\mathbb{T}} p\, dm, t_0 = \sum_k \log \rho_k$, *and* $\rho_k = (1 - |\alpha_k|^2)^{1/2}$.

We denote the right-hand side of equality (3) by $\Psi(\mathcal{C})$ and we rewrite it in a different form. To this end, we consider the shift $S: l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$, given by $Se_k = e_{k+1}$ and, for a bounded operator A on $l^2(\mathbb{Z}_+)$, we look at $\tau(A) = S^*AS$. Consequently, we see that

$$\Psi(\mathcal{C}) = \sum_{k=0}^{2N+1} \{a_0 \log \rho_k + \mathrm{Re}\,((P(\mathcal{C}) - P(\mathcal{C}_0))e_k, e_k)\} + \sum_{k=2N+2}^{\infty} \psi \circ \tau^k(\mathcal{C})$$

where $\psi(\mathcal{C}) = a_0 \log \rho_{2N+2} + \mathrm{Re}\,((P(\mathcal{C}) - P(\mathcal{C}_0))e_{2N+2}, e_{2N+2})$. It turns out that there exist functions η and γ, depending on a finite number of arguments, such that for any $\mathcal{C}$ with $\mathrm{rank}\,(\mathcal{C} - \mathcal{C}_0) < \infty$

$$\Psi(\mathcal{C}) = \tilde{\Psi}(\mathcal{C}) = \sum_{k=0}^{2N+1} \left\{a_0 \log \rho_k + \mathrm{Re}\,((P(\mathcal{C}) - P(\mathcal{C}_0))e_k, e_k)\right\} + \sum_{k=2N+2}^{\infty} \eta \circ \tau^k(\mathcal{C}) + \gamma \circ \tau^{2N+2}(\mathcal{C})$$

and, moreover, η is nonpositive (see [5], Lemma 3.1).

Theorem 1.2 ([5, Theorem 1.4]) *A measure σ is polynomially Szegő (see (2)) if and only if $\tilde{\Psi}(\mathcal{C}) > -\infty$. Moreover, in this case $\Phi(\mathcal{C}) = \tilde{\Psi}(\mathcal{C}) = \Psi(\mathcal{C})$.*

We turn now to the description of asymptotic properties of orthogonal polynomials for (pS)-measures. Consider a modified Schwarz kernel

$$K(t, z) = \frac{t+z}{t-z}\,\frac{q(t)}{q(z)}$$

where $q(t) = C(\prod_j (t - \zeta_j)^2)/t^N$, and the constant $C, |C| = 1$, is chosen in a way that $q(t) \in \mathbb{R}$ for $t \in \mathbb{T}$ (i.e., $C = (\prod_j (-\zeta_j))^{-1}$). Furthermore, define

$$\tilde{D}(z) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} K(t, z) \log \sigma'_{ac}(t)\, dm(t)\right), \quad \tilde{\varphi}_n^*(z) = \exp\left(\int_{\mathbb{T}} K(t, z) \log |\varphi_n^*(t)|\, dm(t)\right)$$

The functions $\{\tilde{\varphi}_n^*\}$ are called (reversed) modified orthogonal polynomials with respect to σ. It can be readily seen that $|\tilde{D}|^2 = \sigma'_{ac}$ and $|\tilde{\varphi}_n^*| = |\varphi_n^*| = |\varphi_n|$ a.e. on $\mathbb{T}$. Furthermore, we see that $\tilde{\varphi}_n^* = \psi_n \varphi_n^*$, where

$$\psi_n(z) = \exp\left(A_{0n} + \sum_{j=1}^{N}\left(A_{jn}\frac{z+\zeta_j}{z-\zeta_j} + B_{jn}\left\{\frac{z+\zeta_j}{z-\zeta_j}\right\}^2\right)\right) \tag{4}$$

and $A_{0n}, B_{jn} \in i\mathbb{R}$, $A_{jn} \in \mathbb{R}$. The coefficients $\{A_{0n}, A_{jn}, B_{jn}\}_{j,n}$ can be expressed in a closed form through Verblunsky coefficients $\{\alpha_k\}$.

Theorem 1.3 *Let $\sigma \in$ (pS). Then*

$$\lim_{n\to\infty} \int_{\mathbb{T}} |\tilde{D}\tilde{\varphi}_n^* - 1|^2\, dm = 0$$

and, in particular, $\lim_{n\to\infty} \tilde{D}(z)\tilde{\varphi}_n^*(z) = \lim_{n\to\infty} \tilde{D}(z)\psi_n(z)\varphi_n^*(z) = 1$ *for any* $z \in \mathbb{D}$.

Special versions of this result for Jacobi matrices are obtained in [2,5]. The proof of the theorem is partially based on the sum rules proved in Theorem 1.2. The second important observation is that, for an $\varepsilon > 0$ small enough,

$$|\tilde{D}\tilde{\varphi}_n^*(z)| \le \frac{C_\varepsilon}{\sqrt{1-|z|}}$$

where $z \in \mathbb{D}\backslash(\cup_k B_\varepsilon(\zeta_k))$, $B_\varepsilon(\zeta) = \{z : |z - \zeta| < \varepsilon\}$.

We use asymptotics described above, to construct modified wave operators. Let $\mathcal{F}_0 : L^2(m) \to l^2(\mathbb{Z}_+)$, $\mathcal{F} : L^2(\sigma) \to l^2(\mathbb{Z}_+)$ be the Fourier transforms associated to the CMV-representations $\mathcal{C}$ and $\mathcal{C}_0$, see [6, Ch. 4]. Recall that $\mathcal{C} = \mathcal{F}z\mathcal{F}^{-1}$, $\mathcal{C}_0 = \mathcal{F}_0 z \mathcal{F}_0^{-1}$.

Theorem 1.4 *Let $\sigma \in$ (pS). The limits*

$$\tilde{\Omega}_{\pm} = \mathrm{s} - \lim{}_{n\to\pm\infty} e^{W(2n,\mathcal{C})}\mathcal{C}^n\mathcal{C}_0^{-n}$$

exist. Here

$$W(\mathcal{C}, n) = A_{0n} + \sum_{j=1}^{N}\left(A_{jn}\frac{\mathcal{C}+\zeta_j}{\mathcal{C}-\zeta_j} + B_{jn}\left\{\frac{\mathcal{C}+\zeta_j}{\mathcal{C}-\zeta_j}\right\}^2\right)$$

and coefficients $\{A_{0n}, A_{jn}, B_{jn}\}$ are defined in (4)*. We also have*

$$\mathcal{F}^{-1}\tilde{\Omega}_{+}\mathcal{F}_0 = \chi_{E_{ac}}\frac{1}{\tilde{D}}, \quad \mathcal{F}^{-1}\tilde{\Omega}_{-}\mathcal{F}_0 = \chi_{E_{ac}}\frac{1}{\bar{\tilde{D}}}$$

where $E_{ac} = \mathbb{T}\backslash\operatorname{supp}\sigma_s$.

The proof of the above theorem mainly follows [6, Ch. 10].

We now briefly discuss approximation by analytic polynomials in $L^2(\sigma)$ with $\sigma \in$ (pS). We put $\mathcal{P}_0'$ to be the set of analytic on $\mathbb{D}$ polynomials g with the property $g \neq 0$ on $\mathbb{D}$; normalize them by the condition $g(0) > 0$. Furthermore, for a $g \in \mathcal{P}_0'$, we set

$$\lambda(g) = \exp\left(\int_{\mathbb{T}} p \log|g|\, dm\right)$$

and define $\mathcal{P}_1' = \{g : g \in \mathcal{P}_0',\ \lambda(g) = 1\}$.

Theorem 1.5 *Let $d\sigma = w\,dm + d\sigma_s$. Then*

$$\exp\left(\int_{\mathbb{T}} p\log\frac{w}{p}\,dm\right) \leq \inf_{g\in\mathcal{P}_1'}||g||_{\sigma}^2 = \inf_{g\in\mathcal{P}_0',\ ||g||_{\sigma}\leq 1}\frac{1}{|\lambda(g)|^2} \leq \exp\left(\int_{\mathbb{T}} p\log w\,dm\right)$$

Remind that σ is a Szegő measure if and only if the system $\{e^{iks}\}_{k\in\mathbb{Z}}$ is uniformly minimal in $L^2(\sigma)$. Saying that σ is a (pS)-measure translates into the uniform minimality of another system, $\{e^{ik\nu(s)}\}_{k\in\mathbb{Z}}$, in the same space $L^2(\sigma)$. Above,

$$\nu(s) = C_0\int_0^s p(e^{is'})\,ds'$$

where $s, s' \in [0, 2\pi]$ and the constant C_0 comes from the condition $C_0\int_{\mathbb{T}} p\,dm = 1$, see [5], Lemma 2.2.

We conclude the note with a few examples. For instance, classical Pollaczek polynomials [7] belong to the (pS)-class with $p(e^{i\theta}) = \sin^2\theta$. It was proved recently in [6, Ch. 2] that $\sigma \in (\mathrm{p}_0\mathrm{S})$ with $p_0(e^{i\theta}) = 1 - \cos\theta$ if and only if $\{\alpha_k\} \in l^4$ and $\{\alpha_{k+1} - \alpha_k\} \in l^2$. Theorems 1.2–1.5 also apply to this case and yield explicit formulas for $\{\tilde{\varphi}_n^*\}, \{\psi_n\}, \tilde{D}$ and coefficients $\{A_{0n}, A_{jn}, B_{jn}\}$.

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