SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATORS WITH DECAYING POTENTIALS

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Abstract. We review recent advances in the spectral theory of Schrödinger operators with decaying potentials. The area has seen spectacular progress in the past few years, stimulated by several conjectures stated by Barry Simon starting at the 1994 International Congress on Mathematical Physics in Paris. The one-dimensional picture is now fairly complete, and provides many striking spectral examples. The multidimensional picture is still far from clear and may require deep original ideas for further progress. It might hold the keys for better understanding of a wide range of spectral and dynamical phenomena for Schrödinger operators in higher dimensions.

1. Introduction

Schrödinger operators with decaying potentials are used to study the behavior of a charged particle in a local electric field. The operator is defined by

$$H_V = -\Delta + V(x) \tag{1.1}$$

on $L^2(\mathbb{R}^d)$; in one dimension it is common to consider the operator on half-axis with some self-adjoint boundary condition at zero. The spectral and dynamical effects that we are interested in are those depending on the rate of decay of the potential rather than its singularities, so we will often freely assume that V is bounded. If the decay of the potential is sufficiently fast (short range), one expects scattering motion. The corresponding results have been rigorously proved by Weidmann [76] in one dimension (where short-range case means that the potential is in $L^1(\mathbb{R})$) and by Agmon [1] in higher dimensions (where the natural short range class is defined by $|V(x)| \leq C(1+|x|)^{-1-\epsilon}$). These results established pure absolute continuity of the spectrum on the positive semi-axis and asymptotic completeness of the wave operators. There has been a significant amount of work on longer range potentials with additional symbol-like conditions (see e.g. [60] for further references), or oscillating potentials of specific Wigner-von Neumann type structure (see e.g. [3, 24, 78] for further references). However until 1990s there had been very limited progress on understanding slowly decaying potentials with no additional assumptions on behavior of derivatives. The short range or classical WKB methods did not seem to apply in this case, and the possible spectral properties remained a mystery. The celebrated Wigner-von Neumann example [77] provides a potential $V(x)$ behaving like $8\sin(2x)/x + O(x^{-2})$ as $x \to \infty$ and leading to an imbedded eigenvalue $E = 1$, thus showing that surprising things can happen once the potential is not short range. On the other hand,

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the work of Kotani and Ushiroya [42] implied that for potentials decaying at power rate slower than $x^{-\alpha}$, $\alpha \le 1/2$, the spectrum may become purely singular, and thus the scattering picture may be completely destroyed. There was a clear gap in the decay rates where very little information on the possible spectral properties was available. In recent years, there has been a significant progress in the area, largely stimulated by Barry Simon's research and ideas. At the ICMP in Paris in 1994, Simon posed a problem of understanding the spectral properties of Schrödinger operators with potentials satisfying $|V(x)| \le C(1+|x|)^{-\alpha}$, $1 > \alpha > 1/2$. Later, at the 2000 ICMP in London [71], he compiled a list of fifteen problems in Schrödinger operators "for the twenty first century". Two of the problems on the list concern long range potentials.

While there remain many open questions, the recent effort to improve understanding of the long range potentials led to many high quality mathematical works. Fruitful new links between spectral theory of Schrödinger operators and orthogonal polynomials as well as Fourier analysis have been discovered and exploited. Surprising examples of intricate spectral properties have been produced. Advances have been made towards better understanding of effects possible in higher dimensions. In this review, we try to survey recent results in this vital area, as well as underline most active current directions and questions of interest. In the second section we discuss the one-dimensional case, where the picture is much more detailed and complete. The third section is devoted to a number of interesting spectral and dynamical examples, typically one-dimensional, but easily extendable to any dimension by spherically symmetric construction. We briefly mention certain relations to Dirac operators, Jacobi matrices and polynomials orthogonal on the unit circle (OPUC) in the fourth section. In the last section, we consider the higher dimensional case, where the main question - known as Barry Simon's conjecture - is still open and is at the focus of current research.

Most of this work is a compilation and review of known results. There are, however, three nuggets that are new to the best of our knowledge. In Section 3, we provide a new proof of Theorem 3.1, the construction of an example with a dense set of imbedded eigenvalues. In Section 5, Theorem 5.2 and Theorem 6.2 are new.

2. The One-Dimensional Case

The initial progress in understanding slowly decaying potentials started from particular cases, such as random and sparse. Both of these classes have been treated in a single framework by Simon in a joint paper with Last and Kiselev [40]; random decaying potentials in the discrete setting had been pioneered by Simon in a joint work with Delyon and Souillard [14]. Let H_V be a half line Schrödinger operator, and fix some boundary condition at the origin. Let us call $V(x)$ a Pearson potential if $V(x) = \sum_n a_n W(x - x_n)$, where $a_n \to 0$, $x_n/x_{n-1} \to \infty$ as $n \to \infty$, and $W(x) \in C_0^\infty(\mathbb{R})$.

Theorem 2.1. *Let $V(x)$ be a Pearson potential. If $\sum_n a_n^2 < \infty$, the spectrum of H_V on $(0,\infty)$ is purely absolutely continuous. If $\sum_n a_n^2 = \infty$, the spectrum of H_V on $(0,\infty)$ is purely singular continuous.*

This result [40] generalizes the original work of David Pearson [58], who essentially proved Theorem 2.1 under assumption that x_n grow sufficiently fast (with no explicit estimate). See also [61] for related results. In a sense, Pearson's theorem was the first indication of a clear spectral transition at $p = 2$ when potential is viewed in L^p scale. A similar picture is true for the random potentials. Let $V(x) = n^{-\alpha} a_n(\omega) W(x - n)$, where $W \in C_0^\infty(0, 1)$ and $a_n(\omega)$ are random i.i.d variables with mean zero and compactly supported probability density function.

Theorem 2.2. *If $\alpha > 1/2$, then the spectrum of H_V on the positive half axis is purely absolutely continuous with probability one. If $\alpha < 1/2$, the spectrum on $(0, \infty)$ is pure point with probability one. If $\alpha = 1/2$, the spectrum is a mixture of pure point and singular continuous spectrum with probability one.*

Please see [40] for more details in the $\alpha = 1/2$ case, as well as for the proof of a more general theorem. The first result of the type of Theorem 2.2 is due to Delyon, Simon and Souillard [14], who handled the discrete case. In the continuous setting, Kotani and Ushiroya [42] proved a version of Theorem 2.2 for a slightly different model.

Theorems 2.1, 2.2 show a transition which is reminiscent of some classical results on almost everywhere convergence and divergence of Fourier series. Random Fourier series also converge or diverge at almost every point with probability one depending on whether the coefficients are square summable (see, e.g. [32]). A similar result holds for the lacunary Fourier series (see, e.g. [81] for further references). This analogy is not accidental. Indeed, the spectral properties are related to the behavior of solutions of the Schrödinger equation. Although the precise link between solutions and local (in energy) properties of spectral measure is given by the subordinacy condition discovered by Gilbert and Pearson [26], boundedness of the solutions is typically associated with the absolutely continuous spectrum. In particular, it has been shown by Behncke [4], Stolz [74] and by Simon [70] that if all solutions of the equation $-u'' + V(x)u = Eu$ are bounded for each E in a set S of positive Lebesgue measure, then the absolutely continuous part of the spectral measure gives positive weight to S, and the singular part of the spectral measure does not give any weight to S. Establishing the boundedness of the solutions, on the other hand, can be thought of as a nonlinear analog of proving the convergence of Fourier series, at least for the potentials in $L^2(\mathbb{R})$. To clarify this idea, it is convenient to introduce the generalized Prüfer transform, a very useful tool for studying the solutions in one dimension. We will very roughly sketch the idea behind Theorems 2.1, 2.2 following [40].

Let $u(x, k)$ be a solution of the eigenfunction equation

$$-u'' + V(x)u = k^2 u. \tag{2.1}$$

The modified Prüfer variables are introduced by

$$u'(x, k) = kR(x, k)\cos\theta(x, k); \;\; u(x, k) = R(x, k)\sin\theta(x, k). \tag{2.2}$$

The variables $R(x, k)$ and $\theta(x, k)$ satisfy

$$(\log R(x, k)^2)' = \frac{1}{k} V(x) \sin 2\theta(x, k) \tag{2.3}$$

$$\theta(x, k)' = k - \frac{1}{k} V(x)(\sin\theta)^2. \tag{2.4}$$

Fix some point x_0 far enough, and set $\theta(x_0) = \theta_0$, $R(x_0) = R_0$. From (2.4), we see that

$$\theta(x) = \theta_0 + k(x-x_0) - \frac{V(x)}{k}(\sin(k(x-x_0)+\theta_0))^2 + O(V^2) = \theta_0 + k(x-x_0) + \delta\theta + O(V^2). \quad (2.5)$$

Then

$$\sin 2\theta(x,k) = \sin\left(2\theta_0 + 2k(x-x_0)\right) + 2\cos\left(2\theta_0 + 2k(x-x_0)\right)\delta\theta + O(V^2).$$

From (2.5) and the equation (2.3) for the amplitude, we find

$$\frac{d}{dx}\left(\log(R^2(x)\right) = t_1 + t_2 + O(V^3), \quad (2.6)$$

where

$$t_1 = \frac{V(x)}{k}\sin(2(\theta_0 + k(x-x_0))) - \frac{V(x)}{2k^2}\left(\int_{x_0}^{x} V(y)\,dy\right)\cos(2(\theta_0 + k(x-x_0))),$$

and

$$t_2 = \frac{1}{4k^2}\frac{d}{dx}\left[\int_{x_0}^{x} V(y)\cos(2(\theta_0 + k(y-x_0)))\,dy\right]^2.$$

In both the random and sparse cases, we obtain the asymptotic behavior of $R(x)$ by summing up contributions from finite intervals. In the random case, these intervals correspond to the independent random parts of the potential, while in the sparse case $R(x)$ remains unchanged between the neighboring bumps, and we only have to add the contributions of the bumps. In both cases, for different reasons, the contributions of the t_1 terms can be controlled and are finite (with probability one in the random case). For random potentials, one uses the linearity of terms entering t_1 in V and the independence of different contributions; the argument is then similar to the Fourier transform case and gives convergence as far as $V \in L^2$ by Kolmogorov three series theorem. We note that for the second term in t_1 one actually has to use a bit more subtle reasoning also taking into account the oscillations in energy. In the sparse case, one uses the fact that contributions from different steps are oscillating in k with very different frequency due to the large distance between x_n and x_{n+1}. Again, the argument is related to the techniques used to study the lacunary Fourier series. On the other hand, the sum of t_2 terms is finite if $V \in L^2$, leading to the boundedness of the solutions and absolutely continuous spectrum. If V is not L^2, the sum of t_2 terms diverges and can be shown to dominate the other terms due to lack of sign cancelations.

The question whether the absolutely continuous spectrum is preserved for general L^2 potentials remained open longer. The initial progress in this direction focused on proving boundedness of solutions for almost every energy. There are many examples, starting from the celebrated Wigner and von Neumann [77] construction of an imbedded eigenvalue, which show that spectrum does not have to be purely absolutely continuous if $V \notin L^1$, and imbedded singular spectrum may occur. We will discuss some of these examples in Section 3. Thus there can be exceptional energies with decaying and growing solutions. Again, one can think of a parallel with Fourier transform, where the integral $\int_{-N}^{N} e^{ikx} g(x)\,dx$ may diverge for some energies if $g \in L^2$. It was conjectured by Luzin early in the twentieth century that nevertheless the integral converges for a.e. k. The question turned out to be difficult, and required an extremely subtle analysis by Carleson to be solved positively in 1955 in a famous paper [7]. If $g \in L^p$ with $p < 2$, the problem is significantly simpler, and had been solved by Zygmund in 1928 [80] (see also Menshov [51] and Paley [57] for the discrete case). As the equation (2.3) suggests, the problem of boundedness of solutions to Schrödinger equation may be viewed as a question about a.e. convergence of a nonlinear Fourier transform. Research in this direction started from work of Christ, Kiselev, Molchanov and Remling on power decaying potentials [36, 37, 52, 8, 62]. An elegant and simple paper by Deift and Killip [15] used a completely different idea, sum rules, to prove the sharp result, the preservation of the absolutely continuous spectrum for L^2 potentials. The two approaches can be regarded as complementary: the study of solutions gives more precise information about the operator and dynamics, but has so far been unable to handle the borderline case $p = 2$. The sum-rule methods give the sharp result on the nature of the spectrum, but less information about the nature of the eigenfunctions and dynamical properties. We will briefly sketch the most current results in both areas, starting with the solutions approach.

Let H_V be the whole line Schrödinger operator. Recall that the modified wave operators are defined by

$$\Omega_m^{\pm} g = L^2 - \lim_{t \to \mp\infty} e^{iH_V t} e^{-iW(-i\partial_x, t)} g, \tag{2.7}$$

where the operator $e^{-iW(-i\partial_x,t)}$ acts as a multiplier on the Fourier transform of g. Let

$$W(k,t) = k^2 t + \frac{1}{2k} \int_0^{2kt} V(s)\, ds. \tag{2.8}$$

The following theorem was proved in [9].

Theorem 2.3. *Assume that $V \in L^p$, $p < 2$. Then for a.e. k, there exist solutions $u_\pm(x,k)$ of the eigenfunction equation* (2.1) *such that*

$$u_\pm(x,k) = e^{ikx - \frac{i}{2k}\int_0^x V(y)\,dy}(1 + o(1)) \tag{2.9}$$

as $x \to \pm\infty$. Moreover, the modified wave operators (2.7) *exist.*

Assume, in addition, that $V(x)|x|^\gamma \in L^p$ for some $p < 2$ and $\gamma > 0$. Then the Hausdorff dimension of the set of k where (2.9) *fails cannot exceed $1 - \gamma p'$ (where p' is the Hölder conjugate exponent to p).*

The asymptotic behavior in (2.9) as well as the phase in (2.7) coincide with the WKB asymptotic behavior, which has been known for a long time for potentials satisfying additional conditions on the derivatives. The main novelty of (2.7) is that no such condition is imposed. Note that if the integral $\int_0^N V(s)\,ds$ converges, the asymptotic behavior of $u_\pm$ becomes identical to the solutions of the unperturbed equation, and modified wave operators can be replaced by the usual Möller wave operators. The proof of Theorem 2.3 proceeds by deriving an explicit series representation for the solutions $u_\pm$ via an iterative procedure. The terms in the series may diverge for some values of k, but converge almost everywhere. The first term in the series is a generalization of the Fourier transform, $\int_0^N \exp(ikx - \frac{i}{k}\int_0^x V(y)\,dy)V(x)\,dx$.

The main difficulty in the proof comes from proving the estimates for the multilinear higher order terms such that the series can be summed up for a.e. k. The estimate (2.9) implies that all solutions of (2.1) are bounded for a.e. k if $V \in L^p$, $p < 2$, and can be thought of as a nonlinear version of Zygmund's result for the Fourier transform. The question whether (2.9) holds and whether the modified operators exist for $V \in L^2$ is still open, and appears to be very hard, especially the a.e. boundedness of the eigenfunctions. Indeed, proving (2.9) would be the nonlinear analog of the Carleson theorem. Moreover, Muscalu, Tao and Thiele showed [55] that the method of [9] has no chance of succeeding when $p = 2$ (since some terms in the multilinear series expansion may diverge on a set of positive measure). The techniques behind Theorem 2.3 have been used to prove related results on slowly varying potentials (with derivatives in L^p, $p < 2$) and perturbations of Stark operators. See [10, 11] for more details.

The sum rules approach to proving absolute continuity of the spectrum was pioneered by Deift and Killip and led to an explosion of activity in the area and many impressive new results. Assume that $V(x) \in C_0^\infty(\mathbb{R})$. Let us consider the solution $f(x,k)$ of (2.1) such that $f(x,k) = \exp(ikx)$ when x is to the right of the support of V. Then, for x to the left of the support of V, we have $f(x,k) = a(k)\exp(ikx) + b(k)\exp(-ikx)$. The solution $f(x,k)$ is called the Jost solution, and $f(0,k)$ the Jost function. The coefficient $t(k) = a^{-1}(k)$ is the transmission coefficient in classical scattering theory. Denote by E_j the eigenvalues of the operator H_V. The following identity is well known (see e.g. [25]):

$$\int\limits_{-\infty}^{\infty} (\log|a(k)|)k^2\,dk + \frac{2\pi}{3}\sum_j |E_j|^{3/2} = \frac{\pi}{8}\int\limits_{-\infty}^{\infty} V^2(x)\,dx. \tag{2.10}$$

The identity can be proved, for example, by a contour integration in the complex upper half plane of an asymptotic expansion in k^{-1} of the integral equation one can write for $f(x,k)$. There is a whole hierarchy of formulas (sum rules) similar to (2.10). This fact is related to the role that the inverse scattering transform for Schrödinger operators plays in understanding the KdV dynamics. The expressions involving $V(x)$ which appear on the right hand side in such sum rules are the KdV invariants. The inequalities of type (2.10) have been applied in the past to derive bounds on the moments of the eigenvalues of H_V (Lieb-Thirring inequalities). Deift and Killip realized that the coefficient $a(k)$ is directly linked to the spectral measure of H_V. Building a sequence of compact support approximations to $V(x)$, and then passing to the limit, one essentially derives a lower bound on the entropy of the absolutely continuous part of the spectral measure:

$$\int\limits_I \log\mu'(\lambda)d\lambda > -\infty,$$

where $d\mu(\lambda)$ is the spectral measure and I is an arbitrary bounded subinterval of $\mathbb{R}^+$. This proves

Theorem 2.4. *For any $V \in L^2$, the essential support of the absolutely continuous spectrum of the operator H_V coincides with the half axis $(0,\infty)$. That is, the absolutely continuous part* *of the spectral measure μ_{ac} gives positive weight to any set $S \subset (0, \infty)$ of the positive Lebesgue measure.*

Killip [33] later proved a strengthened version of Theorem 2.4, also applicable to potentials from L^3 given additional assumptions on the Fourier transform, and to Stark operators. The key advance in [33] is a local in energy version of (2.10), which is more flexible and useful in different situations. The important fact exploited in [33] is that the Jost function is actually the perturbation determinant of the Schrödinger operator. That yields a natural path to obtaining estimates necessary to control the boundary behavior of Jost function. The square of the inverse of the Jost function, on the other hand, is proportional to the density of the spectral measure (see (5.5) for a similar higher dimensional relation). Therefore, the estimates on Jost function have deep spectral consequences.

The results of [15] have been extended to slowly varying potentials with higher order derivative in L^2 by Molchanov, Novitski and Vainberg [53], using the higher order KdV invariants. Some improvements were made in [22] where the asymptotical methods for ODE were used.

In the discrete setting, the application of sum rules led Killip and Simon [34] to a beautiful result giving a complete description of the spectral measures of Jacobi matrices which are Hilbert-Schmidt perturbations of a free Jacobi matrix. Further extensions to slower decaying perturbations of Jacobi matrices and Schrödinger operators have been obtained in different works by Laptev, Naboko, Rybkin and Safronov, [45, 46, 66, 67]. Recently, Killip and Simon [35] proved a continuous version of their Jacobi matrix theorem, giving a precise description of spectral measures that can occur for Schrödinger operators with L^2 potentials. We will further discuss their result in the following section.

Certain extensions of the sum rules method also have been applied to higher dimensional problems, and will be discussed in Section 5.

We complete this section with a somewhat philosophical remark. The technique of the Deift-Killip proof (and its developments) has a certain air of magic about it. After all, it is based on an identity – sum rule (2.10), a rarity in analysis. Recall the classical von Neumann-Kuroda theorem, which says that given an arbitrary self-adjoint operator A, one can find an operator Y with arbitrary small Hilbert-Schmidt norm (or any Schatten class norm weaker than trace class) such that $A + Y$ has pure point spectrum. Theorem 2.4 says that the situation is very different if one restricts perturbations to potentials in the case of $A = H_0$. The result is so clear cut that one has to wonder if there is a general, operator theory type of result which, for a given A with absolutely continuous spectrum, describes classes of perturbations which would be less efficient in diagonalizing it. Such more general understanding could prove useful in other situations, but currently is completely missing.

3. Striking Examples

Apart from the general results described in the previous section, there are fairly explicit descriptions of decaying potentials leading to quite amazing spectral properties. The examples we discuss here deal with imbedded singular spectrum. Although all examples we mention are constructed in one dimension, in most cases it is not difficult to extend them to an arbitrary dimension using spherically symmetric potentials. The grandfather of all such examples is a Wigner-von Neumann example of a potential which has oscillatory asymptotic behavior at infinity, $V(x) = 8\sin 2x/x + O(x^{-2})$, and leads to an imbedded eigenvalue at $E = 1$. The imbedded singular spectrum for decaying potentials is the resonance phenomenon, and requires oscillation in the potential, similarly to the divergence of Fourier series or integrals. It is also inherently unstable – for example, for a.e. boundary condition in the half line case there is no imbedded singular spectrum. The first examples we are going to discuss are due to Naboko [56] and Simon [69], who provided different constructions for potentials leading to a similar phenomenon.

Theorem 3.1. *For any positive monotone increasing function $h(x) \to \infty$, there exist potentials satisfying $|V(x)| \leq \frac{h(x)}{1+|x|}$ such that the half-line operator H_V (with, say, Dirichlet boundary condition) has dense point spectrum in $(0, \infty)$.*

If $|V(x)| \leq \frac{C}{1+|x|}$, the eigenvalues $E_1, \dots, E_n, \dots$ of H_V lying in $(0, \infty)$ must satisfy $\sum_n E_n < \infty$.

The last statement of Theorem 3.1 was proved in [40].

The construction of Naboko used the first order system representation of the Schrödinger equation, and had a restriction that the square roots of eigenvalues in $(0, \infty)$ had to be rationally independent. Simon's construction can be used to obtain any dense countable set of eigenvalues in $(0, \infty)$. The idea of the latter construction is, roughly, given a set of momenta $k_1, \dots, k_n, \dots$, take

$$V(x) = W(x) + \sum_n \chi_{(x_n,\infty)}(x) B_n \frac{\sin(2k_n x + \beta_n)}{x}.$$

Here x_n, B_n and β_n have to be chosen appropriately, and $W(x)$ is a compactly supported potential whose job is to make sure that the L^2 eigenfunctions at E_n satisfy the right boundary condition at zero. Thus basically, the potential is a sum of resonant pieces on all frequencies where the eigenvalues are planned. To explain the argument better, we will outline a third construction of such an example, which in our view is technically the simplest one to implement. We will only sketch the proof; the details are left to the interested reader.

Proof of Theorem 3.1. Without loss of generality, we assume that $h(x)$ does not grow too fast, say $|h(x)| \leq x^{1/4}$. Recall the Prüfer variables $R(x, k)$, $\theta(x, k)$ and equations (2.4), (2.3) they satisfy. For $x \leq x_1$, x_1 to be determined later, let

$$V(x) = -\frac{h(x)}{2(1+|x|)} \sin 2\theta(x, k_1). \tag{3.1}$$

Here k_1^2 is the first eigenvalue from the list we are trying to arrange. Note that the seeming conflict between defining V in terms of θ and θ in (2.4) in terms of V is resolved by plugging (3.1) into (2.4), solving the resulting nonlinear equation for $\theta(x, k_1)$, and defining $V(x)$ as in (3.1). Now if V is defined according to (3.1) on the whole half-axis, one can see using (2.3), (2.4) and integration by parts that

$$\log(R(x,k_1)^2) = -\int_0^x \frac{h(y)}{2k_1(1+|y|)}\,dy + O(1).$$

Because of our assumptions on $h(x)$, $R(x,k_1)$ is going to be square integrable.

Now we define our potential $V(x)$ by

$$V(x) = -\sum_{j=1}^{\infty} \frac{h(x)}{2^j(1+|x|)} \chi_{(x_j,\infty)}(x) \sin 2\theta(x,k_j). \tag{3.2}$$

Each $x_n > x_{n-1}$ is chosen inductively, so that the following condition is satisfied: for any $j < n$,

$$\sup_{x > x_n} \left| \int_{x_n}^x \frac{h(y)}{(1+|y|)} \sin 2\theta(y,k_n) \sin 2\theta(y,k_j)\,dy \right| \le 1. \tag{3.3}$$

Using (2.4) and integration by parts, it is easy to see that on each step, the condition (3.3) will be satisfied for all sufficiently large x_n. A calculation using (2.4), (2.3), and integration by parts then shows that $R(x,k_n)$ is square integrable for each n. From (3.2) it also follows that $V(x) \le h(x)/(1+|x|)$. □

Examples with imbedded singular continuous spectrum are significantly harder to construct. The main difficulty is that while to establish point spectrum one just needs make sure that L^2 norm of the solution is finite, it is not quite clear what does one need to control to prove the existence of the singular continuous component of the spectral measure. At the ICMP Congress in London [71], Simon posed the problem of finding a decaying potential leading to imbedded singular continuous spectrum. The first progress in this direction was due to Remling and Kriecherbauer [63, 44]. In particular, they constructed fairly explicit examples of potentials satisfying $|V(x)| \le C(1+|x|)^{-\alpha}$, $\alpha > 1/2$, such that the Hausdroff dimension of the set of singular energies where the WKB asymptotic behavior (2.9) fails is equal to $2(1-\alpha)$. This is sharp according to Theorem 2.3 (and earlier work of Remling [64] on power decaying potentials). The set of singular energies is the natural candidate to support the singular continuous part of the measure, but the actual presence of the singular continuous part of the measure remained open.

The first breakthrough came in a work of Denisov [17] where the following was proved

Theorem 3.2. *There exist potentials $V \in L^2$ such that the operator H_V has imbedded singular continuous spectrum in $(0,\infty)$.*

The method was inspired by some ideas in approximation theory (see the next section) and by inverse spectral theory. The classical inverse spectral theory results (see e.g. [49, 50]) imply that one can find potentials corresponding to spectral measures with imbedded singular continuous component. The standard procedure, however, does not guarantee a decaying potential. In the meantime, one can develop an inverse spectral theory type of construction where one also controls the L^2 norm of the potentials corresponding to certain approximations of the desired spectral measure, in the limit obtaining the L^2 potential. The key control of the L^2 norm appears essentially from the sum rule used by Deift and Killip. The construction in [17] employed some estimates for the Krein systems, a more general system of first order differential equations. The amazing aspect of the construction is the great flexibility on how the singular part of the spectral measure may look. Later, in the paper [23], it was proved that the imbedded singular continuous spectrum can occur for faster decaying potentials, namely if

$$\int_x^\infty q^2(t)\,dt \le C(1+x)^{-1+D+\epsilon},$$

then the spectral measure can have a singular continuous component of exact dimension D.

Killip and Simon [34] realized that the idea of [17] is not tied to the Krein systems. They proved a comprehensive theorem, providing a complete characterization of the spectral measures of Jacobi matrices which are Hilbert-Schmidt perturbations of the free matrix. This theorem should be regarded as an analog of the celebrated Szegő theorem for polynomials orthogonal on the unit circle [75, 72]. Recently, they also extended their result to the continuous case, where it reads as follows [35]. Denote $d\rho(E)$ the spectral measure of H_V, set $d\rho_0(E) = \pi^{-1}\chi_{[0,\infty)}(E)\sqrt{E}dE$, and define a signed measure $\nu(k)$ on $(1,\infty)$ by

$$\frac{2}{\pi}\int f(k^2)kd\nu(k) = \int f(E)[d\rho(E) - d\rho_0(E)].$$

Given a (signed) Borel measure ν, define

$$M_s\nu(k) = \sup_{0<L\le 1}\frac{1}{2L}|\nu|([k-L,k+L]).$$

Denote by $d\mu/d\sigma$ the Radon-Nikodym derivative of μ with respect to σ.

Theorem 3.3. *A positive measure ρ on $\mathbb{R}$ is the spectral measure associated to a $V \in L^2(\mathbb{R}^+)$ if and only if*
(i) $\mathrm{supp}(d\rho) = [0,\infty) \cup \{E_j\}_{j=1}^N$ *with* $E_1 < E_2 < \cdots < 0$ *and* $E_j \to 0$ *if* $N = \infty$.
(ii)

$$\int \log\left[1 + \left(\frac{M_s\nu(k)}{k}\right)^2\right] k^2\,dk < \infty \tag{3.4}$$

(iii)

$$\sum_j |E_j|^{3/2} < \infty$$

(iv)

$$\int_0^\infty \log\left[\frac{1}{4}\frac{d\rho}{d\rho_0} + \frac{1}{2} + \frac{1}{4}\frac{d\rho_0}{d\rho}\right]\sqrt{E}\,dE < \infty.$$

The theorem shows explicitly that the singular part of the spectral measure corresponding to an L^2 potential can be pretty much anything on the positive half-axis, as far as a certain normalization condition (3.4) is satisfied.

The last example that we would like to mention provides the sharp rate of decay for which the imbedded singular continuous spectrum may appear [38].

Theorem 3.4. *For any positive monotone increasing function $h(x) \to \infty$, there exist potentials satisfying $|V(x)| \le \frac{h(x)}{1+|x|}$ such that the half-line operator H_V (with, say, Dirichlet boundary condition) has imbedded singular continuous spectrum. The potential $V(x)$ can be chosen so that the Möller wave operators exist, but are not asymptotically complete due to the presence of the singular continuous spectrum.*

On the other hand, if $|V(x)| \le \frac{C}{1+|x|}$, the singular continuous spectrum of H_V is empty.

The proof is based on building a sequence of approximating potentials V_n which have, respectively, 2^n imbedded eigenvalues E_j^n, approaching a Cantor set. The key is to obtain uniform control of the norms of the corresponding eigenfunctions, $\|u(x, E_j^n)\|_2^2 \le C2^{-n}$. Such estimate allows to control the weights the spectral measure assigns to each eigenvalue, and to pass to the limit obtaining a non-trivial singular continuous component. The estimate of the norms of the eigenfunctions is difficult and is proved using the Prüfer transform, and a Splitting Lemma allowing to obtain two imbedded eigenvalues from one. This lemma is based on a model nonlinear dynamical system providing an elementary block of construction.

4. Dirac operators, Krein systems, Jacobi matrices, and OPUC

It was understood a long time ago that the spectral theory of one-dimensional differential operators (Schrödinger, Dirac, canonical systems) has a lot in common with the classical theory of polynomials orthogonal on the real line. These polynomials are eigenfunctions of the Jacobi matrix, also quite classical object in analysis. So, naturally, to understand the problems for differential operators one might first study analogous problems for the discrete version. Unfortunately, Jacobi matrices are not so easy to study either. That difficulty was encountered by many famous analysts (such as Szegő) and the answer was found in the theory of polynomials orthogonal on the unit circle. It turns out that for many questions (especially in scattering theory) that is a more natural and basic object to study. Then many results and ideas can be implemented for Jacobi matrices. For differential operators, the situation is similar. In many cases, instead of Schrödinger operator it makes sense to consider Dirac operator and for a good reason. Already in 1955 M. Krein [43] gave an outline of the construction that led to the theory of continuous analogs of polynomials orthogonal on the circle. Instead of complex polynomials, one has the functions of exponential type that satisfy the corresponding system of differential equations (the Krein system)

$$\begin{cases} P'(r,\lambda) = i\lambda P(r,\lambda) - \bar{A}(r)P_*(r,\lambda), & P(0,\lambda)=1, \\ P_*'(r,\lambda) = -A(r)P(r,\lambda), & P_*(0,\lambda)=1 \end{cases} \tag{4.1}$$

Although more complicated than the OPUC case, the corresponding theory can be developed. It turns out that the Krein systems happen to be in one-to-one correspondence with the canonical Dirac operators:

$$D\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -b & d/dr - a \\ -d/dr - a & b \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, f_2(0)=0 \tag{4.2}$$

In fact, $a(r) = 2\Re A(2r), b(r) = 2\Im A(2r)$.

In his pioneering paper [43], Krein states the following

Theorem 4.1. *If $a(r), b(r) \in L^2(\mathbb{R}^+)$ then $\sigma_{ac}(D) = \mathbb{R}$. Also,*

$$\int_{-\infty}^{\infty} \frac{\ln \sigma'(\lambda)}{1+\lambda^2} > -\infty$$

where σ is the spectral measure for Dirac operator.

This result is actually the corollary from the analogous statement for Krein systems. Just as OPUC are related to Jacobi matrices, Krein systems and Dirac operators generate Schrödinger operators. Assume that $b(r) = 0$ and $a(r)$ is absolutely continuous. Taking the square of the operator D we obtain the 2×2 diagonal matrix with Schrödinger operators on the diagonal. They have potentials $q_{1(2)} = a^2 \pm a'$ and certain boundary conditions. That clearly can be used to reduce the study of Schrödinger operators to the problems in the Krein systems theory. Indeed, assume that the potential q in Schrödinger operator decays at infinity, say, $q \in L^p(\mathbb{R}^+), p < \infty$. Add the large constant γ to the operator and consider one of the corresponding Riccati equations, say:

$$q + \gamma = a^2 + a'$$

We try to find solution in the form $a = \sqrt{\gamma} + \mu$. Then,

$$\mu' + 2\sqrt{\gamma}\mu = q - \mu^2$$

and a decaying solution μ can be found by the contraction principle for the corresponding integral equation as long as γ is large enough. It will also be in $L^p(\mathbb{R}^+)$. Since adding the constant γ does not change the spectral types, the study of Schrödinger operator with decaying potential is essentially equivalent to the study of Dirac operator (and so the Krein system) with $b = 0, a = \sqrt{\gamma} + \mu$ where γ is a constant and μ decays at infinity in essentially the same way as the Schrödinger potential does [21, 22].

We already mentioned the problem of proving the existence of wave operators for L^2 potentials in the second section. In [16], it was proved that for the Dirac operator, wave operators do exist if $a, b \in L^2$. The proof bypasses the question about a.e. in energy behavior of the eigenfunctions, and employs instead integral estimates. The key difference between Dirac and Schrödinger cases is different free evolution. One manifestation of this difference is the fact that no WKB correction is needed in the definition of wave operators; usual Möller wave operators exist for L^2 perturbations of the Dirac operator. Nevertheless, the result may indicate that the L^2 wave operator question for Schrödinger operators is easier to resolve than the question of the asymptotic behavior (2.9).

The study of Dirac operators is often more streamlined than Schrödinger in both one-dimensional and multidimensional cases (see the next section), but it already poses significant technical difficulties whose resolution is far from obvious and has proved to be very fruitful for the subject in general.

5. The Multidimensional Case

As opposed to the one-dimensional theory, the spectral properties of Schrödinger operators with slowly decaying potentials in higher dimensions are much less understood. Early efforts focused on the short range case, $|V(x)| \leq C(1+|x|)^{-1-\epsilon}$, culminating in the proof by Agmon [1] of the existence and asymptotic completeness of wave operators in this case. In the long range case, Hörmander [29] considered a class of symbol like potentials, proving existence and completeness of wave operators. For a review of these results and other early literature, see [60, 79]. For potentials with less regular derivatives, the conjecture by Barry Simon [71] states that the absolutely continuous spectrum of the operator H_V should fill all positive half axis if

$$\int_{\mathbb{R}^d} V^2(x)(1+|x|)^{-d+1}\,dx < \infty. \tag{5.1}$$

To avoid problems with the definition of the corresponding self-adjoint operator (that might appear for dimension high enough because of the local singularity of potential), we also assume that V belongs to, say, Kato class: $V \in K_d(\mathbb{R}^d)$ [13].

Recall that there exist potentials $W(r)$ in one dimension which satisfy $|W(r)| \leq Cr^{-1/2}$ and lead to purely singular spectrum [42, 40]. By taking a spherically symmetric potential $V(x) = W(|x|)$, one can obtain multidimensional examples showing that (5.1) is sharp in many natural scales of spaces. Notice also that the potential satisfying (5.1) does not have to decay at infinity pointwise in all directions: it can even grow along some of them. Nevertheless, it does decay in the average and that makes the conjecture plausible.

Motivated by Simon's conjecture, much of the recent research focused on long range potentials with either no additional conditions on the derivatives, or with weaker conditions than in the classical Hörmander work. The solutions method so far had little success in higher dimensions. There are results linking the behavior of solutions and spectrum which work in higher dimensions, such as for a example the following theorem proved in [39]. In higher dimensional problems, there is no canonical spectral measure, and the spectral multiplicity can be infinite. Given any function $\phi \in L^2(\mathbb{R}^d)$, denote μ^ϕ the spectral measure of H_V corresponding to ϕ, that is, a unique finite Borel measure such that $\langle f(H_V)\phi, \phi\rangle = \int f(E)d\mu^\phi(E)$ for all continuous f with compact support.

Theorem 5.1. *Assume that the potential V is bounded from below. Suppose that there exists a solution $u(x,E)$ of the generalized eigenfunction equation $(H_V - E)u(x,E) = 0$ such that*

$$\liminf_{R\to\infty} R^{-1} \int_{|x|\leq R} |u(x,E)|^2\,dx < \infty. \tag{5.2}$$

Fix some $\phi(x) \in C_0^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \phi(x)u(x,E)\,dx \neq 0.$$

Then we have

$$\limsup_{\delta\to 0} \frac{\mu^\phi(E-\delta, E+\delta)}{2\delta} > 0. \tag{5.3}$$

Notice that if (5.3) holds on some set S of positive Lebesgue measure, this implies that the usual Lebesgue derivative of μ^ϕ is positive a.e. on S, and so presence of the absolutely continuous spectrum. The condition (5.2) corresponds to the power decay $|x|^{(1-d)/2}$, just as spherical wave solutions decay for the free Laplacian. One may ask how precise this condition is, perhaps existence of just bounded solutions on a set S of positive Lebesgue measure is sufficient for the presence of the absolutely continuous spectrum? It turns out that, in general, the condition (5.2) cannot be relaxed.

Theorem 5.2. *There exists a potential V such that for any $\sigma > 0$ there exists an energy interval I_σ with the following properties.*

- *For a.e. $E \in I_\sigma$, there exists a solution $u(x, E)$ of the generalized eigenfunction equation satisfying $|u(x, E)| \le C(E)(1 + |x|)^{\sigma+(1-d)/2}$.*
- *The spectrum on I_σ is purely singular.*

One way to prove Theorem 5.2 is to use one-dimensional random decaying potentials with $|x|^{-1/2}$ rate of decay. The results of [42] or [40] show that the spectrum is singular almost surely, and the eigenfunctions decay at a power rate. Taking spherically symmetric potentials of this type in higher dimensions, it is not difficult to see that one gets examples proving Theorem 5.2.

The link between the behavior of solutions and spectral measures has been made even more general, sharp and abstract in [12]. However, the difficulty is that obtaining enough information about solutions in problems of interest is hard: there seems to be no good PDE analog for the ODE perturbation techniques which can be used to understand the solutions in one dimension. On the other hand, the sum rules in higher dimensions typically involve spaces of potentials which are far from the conjectured class (5.1). Some important progress, however, has been made using the one-dimensional ideas [18, 19, 20, 47, 48, 59, 68]. After reviewing these results, we will discuss random decaying potentials in higher dimensions [5, 6, 65] as well as quickly mention some interesting recent progress on a new class of short range potentials [31] and imbedded eigenvalues [30].

The OPUC, the Krein systems, and the Dirac operators with matrix-valued and even operator-valued coefficients were studied relatively well. This matrix-valued case can give some clues to the understanding of partial differential equations. Indeed, writing up, say, Schrödinger operator in the spherical coordinates one obtains the one-dimensional Schrödinger operator with an operator-valued potential. The difficulty is that this potential $\tilde V$ is not bounded since it involves Laplace-Beltrami operator on the unit sphere. Also $\tilde V(r_1)$ and $\tilde V(r_2)$ do not commute for different values of r.

Consider the three dimensional Dirac operator with the following type of interactions

$$D = -i\alpha \cdot \nabla + V(x)\beta. \tag{5.4}$$

Here

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Matrices σ_j are called the Pauli matrices and

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Then the multidimensional analog of Theorem 4.1 says [18] that if $V(x) \in L^\infty$ and the estimate (5.1) holds then $\sigma_{ac}(D) = \mathbb{R}$. Thus the multidimensional result for Dirac operator is quite satisfactory, and the Dirac analog of Simon's conjecture (5.1) holds. We will sketch the ideas behind this result later. For Schrödinger operator, we first state the following interesting result by Safronov [68]. Let us denote by B_r the ball of radius r centered at the origin.

Theorem 5.3. *Let $d \geq 3$ and suppose $V \in L^\infty(\mathbb{R}^d)$ is such that $V(x) \to 0$ as $x \to \infty$. Assume also that $V(x) \in L^{d+1}(\mathbb{R})$ and for some positive $\delta > 0$, the Fourier transform of V satisfies $\hat{V}(\xi) \in L^2(B_\delta)$. Then, $\sigma_{ac}(H) = \mathbb{R}^+$.*

Note that the L^{d+1} condition corresponds to the $|x|^{-\frac{d}{d+1}}$ power decay. There are several methods to obtain this kind of result. The first one, developed by Laptev, Naboko and Safronov [45], is based on writing the operator in the spherical coordinates. Then one takes the Feshbach projection corresponding to the first harmonic and studies the corresponding one-dimensional Schrödinger operator with non-local operator-valued potential. Instead of trace equality one can get an inequality only which still is enough to conclude the presence of a.c. spectrum. On the other hand, instead of dealing with Feshbach projections, one can carefully study the matrix-valued Dirac or Schrödinger operator and obtain the estimates on the entropy of the spectral measure independent of the size of the matrix [18]. Then, an analogous estimate can be obtained for the corresponding PDE.

Another approach allows to work directly with PDE [19]. It consists of the following observation. Consider, for example, three-dimensional Schrödinger operator with compactly supported potential. Taking $\phi \in L^\infty$ with compact support we introduce $u(x,k) = (H - k^2)^{-1}\phi$ with $k \in \mathbb{C}^+, \Re k > 0$. Then, clearly, $u(x,k)$ has the following asymptotic behavior at infinity:

$$u(x,k) = \frac{\exp(ikr)}{r}(A_\phi(k,\theta) + \bar{o}(1)), r = |x|, \theta = x/r$$

The amplitude $A_\phi(k,\theta)$ can be regarded as an analytic operator on $L^2(\Sigma)$, where Σ is the unit sphere. For the potential V with compact support, it is continuous up to each boundary point $k > 0$ and the following factorization identity holds [79]

$$\mu'_\phi(E) = k\pi^{-1}\|A_\phi(k,\theta)\|^2_{L^2(\Sigma)}, \ E = k^2 > 0. \tag{5.5}$$

Loosely speaking, the density of the spectral measure for any vector ϕ can be factorized on the positive interval via the boundary value of some analytic operator-valued function defined in an adjacent domain in $\mathbb{C}^+$. Therefore, one can try to consider the general potential V, establish existence of $A_\phi(k,\theta)$ for all $k \in \mathbb{C}^+, \Re k > 0$ with some (probably rather crude) bounds on the boundary behavior near the real line. Then the analyticity will be enough to conclude the necessary estimate on the entropy of the spectral measure, similarly to the one-dimensional considerations. Here is the general result.

Theorem 5.4. *Consider a potential $V \in L^\infty(\mathbb{R}^d)$. Let $V_n(x) = V(x)\chi_{|x|<n}$ be its truncation and $A_{\phi,n}(k,\theta)$, the corresponding amplitude. Consider an interval $0 < a < k < b$. Assume that for $k_0 = (a+b)/2 + i\sigma, \sigma > 0$, we have an estimate $\|A_{\phi,n}(k_0,\theta)\|_{L^2(\Sigma)} > \delta > 0$ uniformly in n and $\|A_{\phi,n}(\tau + i\varepsilon,\theta)\|_{L^2(\Sigma)} < C\exp(\varepsilon^{-\gamma}), \gamma > 0$ uniformly in $n, \tau \in [a,b], 0 < \varepsilon < 2\sigma$. Then the spectral measure of the function ϕ has a.c. component whose support contains an interval $[a^2, b^2]$.*

The situation is reminiscent of one in the Nevanlinna theory in the classical analysis when the analyticity and rough bounds close to the boundary are enough to say a lot about the function. Thus the whole difficulty here is to obtain the necessary bounds for the particular PDE. That turns out to be a tricky task but doable in some cases. For example, the following theorem was proved in [19].

Theorem 5.5. *Let $Q(x)$ be a $C^1(\mathbb{R}^3)$ vector-field in $\mathbb{R}^3$ and*

$$|Q(x)| < \frac{C}{1+|x|^{0.5+\varepsilon}}, \quad |div\ Q(x)| < \frac{C}{1+|x|^{0.5+\varepsilon}}, \quad |V_1(x)| < \frac{C}{1+|x|^{1+\varepsilon}}, \varepsilon > 0,$$

Then, $H = -\Delta + div\ Q + V_1$ has an a.c. spectrum that fills $\mathbb{R}^+$.

This theorem is a multidimensional analog of the following result in dimension one [21]: if $V = a' + V_1$ where $V_1 \in L^1(\mathbb{R}^+)$, $a \in W_2^1(\mathbb{R}^+)$, then the a.c. spectrum of one-dimensional Schrödinger with potential V covers the positive half-line. The proof involves Theorem 5.4, and is based on the uniform estimates for the Green's function on complex energies. The PDE approach used to prove Theorem 5.5 succeeds because it allows not to deal with negative eigenvalues and the corresponding Lieb-Thirring inequalities often arising in the sum rule approach (see also [33]).

Although rather elaborate, the conditions on the potential from the Theorem 5.5 are not very difficult to check. Essentially, it means that in addition to decay, one has to have certain oscillation of potential. This condition is related, although not identical, to the condition on the Fourier transform in Theorem 5.3. For example, an application to random potentials is possible. Consider the following model. Take a smooth function $f(x)$ with the support inside the unit ball. Consider

$$V_0(x) = \sum_{j\in\mathbb{Z}^+} a_j f(x - x_j)$$

where the points x_j are scattered in $\mathbb{R}^3$ such that $|x_k - x_l| > 2, k \neq l$, and $a_j \to 0$ in a way that $|V_0(x)| < C/(1+|x|^{0.5+\varepsilon})$. Let us now "randomize" V_0 as follows

$$V(x) = \sum_{j\in\mathbb{Z}^+} a_j \xi_j f(x - x_j) \tag{5.6}$$

where ξ_j are real-valued, bounded, independent random variables with even distribution.

Theorem 5.6. *For V given by (5.6), we have $\sigma_{ac}(-\Delta + V) = \mathbb{R}^+$ almost surely.*

It turns out that for dimension high enough these slowly decaying random potentials fall into the class considered in Theorem 5.5. We note that a similar model has been considered by Bourgain [5, 6] in the discrete setting, and will be discussed below. Theorem 5.6 and Bourgain's results suggest very strongly that Simon's conjecture (5.1) is true at least in a certain "almost sure" sense (however, the assumption that the random variables are mean zero is crucial for the proofs).

The method used in [19] was also applied by Perelman in the following situation [59]:

Theorem 5.7. *For $d = 3$, $\sigma_{ac}(-\Delta + V) = \mathbb{R}^+$ as long as*

$$|V(x)| + |x||\nabla' V(x)| < C/(1 + |x|^{0.5+\varepsilon}), \varepsilon > 0$$

where ∇' means the angular component of the gradient.

Here the oscillation is arbitrary in the radial variable and slow in the angular variable. In this case, the Green function has a WKB-type correction which can be explicitly computed (and has essentially one-dimensional, integration along a ray, form). The notion of the amplitude $A_\phi(k, \theta)$ can be modified accordingly and the needed estimates on the boundary behavior can be obtained.

We now return to random decaying potentials and discuss recent developments in more detail. It is natural to tackle slowly decaying random potentials if one tries to approach one of the most important open problems in mathematical quantum mechanics: the existence of extended states in the Anderson model in higher dimensions. Important progress in understanding random slowly decaying potentials is due to Bourgain [5, 6]. Consider random lattice Schrödinger operator on $\mathbb{Z}^2$: $H_\omega = \Delta + V_\omega$, where Δ is the usual discrete Laplacian and V_ω is a random potential

$$V_\omega = \omega_n v_n$$

with $|v_n| < C|n|^{-\rho}$, $\rho > 1/2$. The random variables ω_n are Bernoulli or normalized Gaussian (and, in particular, are mean zero). Then, [5]

Theorem 5.8. *Fix $\tau > 0$ and denote $I = \{E| \, \tau < |E| < 4 - \tau\}$. Assume that $\rho > 1/2$, and $\sup_n |v_n||n^{-\rho}| < \kappa$. Then for $\kappa \le \kappa(\rho, \tau)$ and for ω outside a set of small measure (which tends to zero as $\kappa \to 0$) we have*
1. H_ω has purely absolutely continuous spectrum on I
2. Denoting $E_0(I)$ the spectral projections for Δ the wave operators $W_\pm(H, \Delta)E_0(I)$ exist and are complete.

Using the fact that the absolutely continuous spectrum and the existence of the wave operators are stable under finitely supported perturbations, one readily obtains absolute continuity and existence of wave operators almost surely for potentials satisfying $|v_n| \le C|n|^{-\rho}$. The method can also be extended effortlessly to dimensions $d > 2$.

Bourgain's approach is based on a careful analysis of Born approximation series for the resolvent. In summation, each of the terms $[R_0(z)V]^s R_0(z)$ is considered. Then a dyadic decomposition of V is introduced: $V = \sum_j V\chi_{2^j<|x|<2^{j+1}}$. In the end, the analysis is reduced to getting multilinear bounds for the resulting terms. An interesting and novel in this context ingredient of the proof is the smart use of a certain entropy bound (the so-called "dual to Sudakov" inequality). Later [6] this approach was further developed to deal with different situations, such as L^p and slower power decaying potentials. The main result of [6] for the slower power decay is the almost sure existence of a bounded, not tending to zero solution at a single energy. This, however, is not yet sufficient for any spectral conclusions. The problem of handling the random decay with the coefficient ρ even a little less than $1/2$ remains an interesting open question. So far, all attempts to deal with this case were not successful.

In another paper on random decaying potentials [65], Rodnianski and Schlag showed existence of modified wave operators for the model with the slow random decay and additional assumptions ensuring slow variation of the derivatives. The standard techniques of scattering theory, but also with averaging over the randomness, are employed. This allows them to prove scattering with weaker assumptions than in the standard Hörmander's case.

Another case for which scattering can be established is the Schrödinger operator on the strip [20]. One can show the presence of a.c. part of the spectrum using the following general result. Assume that we are given two operators H_1 and H_2 that both act in the same Hilbert space $\mathcal{H}$. Take H in the following form:

$$H = \begin{bmatrix} H_1 & V \\ V^* & H_2 \end{bmatrix} \tag{5.7}$$

Theorem 5.9. *Let H_1, H_2 be two bounded self-adjoint operators in the Hilbert space $\mathcal{H}$. Assume that $\sigma_{ess}(H_2) \subseteq [b, +\infty]$ and $[a,b] \subseteq \sigma_{ac}(H_1), (a < b)$. Then, for any Hilbert-Schmidt V, ($V \in \mathcal{J}^2$), we have that $[a,b] \subseteq \sigma_{ac}(H)$, with H given by (5.7).*

This theorem can be effectively applied to study Schrödinger on the strip. Indeed, let

$$L = -\Delta + Q(x,y),$$

considered on the strip $\Pi = \{x > 0, 0 < y < \pi\}$, and impose Dirichlet conditions on the boundary of Π. Consider the matrix representation of L. For $f(x,y) \in L^2(\Pi)$,

$$f(x,y) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \sin(ny) f_n(x), f_n(x) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x,y)\sin(ny)dy$$

and L can be written as follows

$$L = \begin{bmatrix} -\dfrac{d^2}{dx^2} + Q_{11}(x) + 1 & Q_{12}(x) & \dots \\ Q_{21}(x) & -\dfrac{d^2}{dx^2} + Q_{22}(x) + 4 & \dots \\ \dots & \dots & \dots \end{bmatrix} \tag{5.8}$$

$$Q_{lj}(x) = \frac{2}{\pi} \int_0^{\pi} Q(x,y)\sin(ly)\sin(jy)dy$$

Assume $\sup_{0\le y\le\pi} |Q(x,y)| \in L^2(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$. Since $Q_{11}(x)\in L^2(\mathbb{R}^+)$, we can use [15, 54] and Theorem 5.9 to show the presence of a.c. spectrum.

We now turn to the results on the (generalized) short range potentials and imbedded eigenvalues in $\mathbb{R}^d$. Recently, new short-range type results have been obtained for the multidimensional Schrödinger operator with potential from L^p and more general classes [27, 31]. The main goal was to establish limiting absorption estimates for the resolvent acting in certain Banach spaces, which are more detailed and precise than Agmon's classical results. The standard techniques developed in the works of Agmon can be improved if one uses the Stein-Tomas restriction theorem. Here is one of the results in that direction [31]:

Theorem 5.10. *Assume that V is such that*

$$M_q(V)(x)\in L^{(d+1)/2}(\mathbb{R}^d)$$

where

$$M_q f=\left[\int\limits_{|y|<1/2} |f(x+y)|^q dy\right]^{1/q}$$

and $q=d/2$ if $d\ge 3$, $q>1$ if $d=2$. Then, the following is true for the operator $H=-\Delta+V$:

- *The set of nonzero eigenvalues is discrete with the only possible accumulation point at zero. Each nonzero eigenvalue has finite multiplicity.*
- $\sigma_{s.c.}(H)=\emptyset$, *and* $\sigma_{a.c.}(H)=\mathbb{R}^+$
- *The wave operators $\Omega^\pm(H,H_0)$ exist and are complete.*

The actual result is a bit stronger. The authors of [31] present the whole class of "admissible" perturbations for which their method works, including some first order differential operators.

Another direction in which there has been significant recent progress concerns imbedded eigenvalues. The following result, in particular, has been proved by Ionescu and Jerison [30].

Theorem 5.11. *Let $V(x)\in L^{d/2}(\mathbb{R}^d)$ and for $d=2$ we also assume $V(x)\in L^r_{loc}(\mathbb{R}^2), r>1$. Then, $H=-\Delta+V$ does not have positive eigenvalues.*

The paper [30] actually contains a more general result, which allows for slower decay of the potential if its singularities are weaker. The method relies on the Carleman inequality of special type.

Surprisingly, [30] also provides an example of potential V satisfying

$$|V(x)|<C(|x_1|+x_2^2+\ldots+x_d^2)^{-1} \tag{5.9}$$

for which a positive eigenvalue appears (multidimensional analog of Wigner- von Neumann potential). From the point of view of physical intuition, the existence of an imbedded eigenvalue for such potential may seem strange. Indeed, one would expect that tunneling in the directions $x_2,\ldots,x_d$ of fast decay should make the bound state impossible. Yet, Wigner-von Neumann type oscillation and slow Coulomb decay in just one direction turn out to be sufficient. The corresponding eigenfunction decays rather slowly but enough to be from L^2. We note that the potential satisfying (5.9) just misses $L^{(d+1)/2}(\mathbb{R}^d)$. Quite recently, Koch and Tataru [41] improved Theorem 5.11 and showed the absence of embedded eigenvalues for the optimal $L^{(d+1)/2}(\mathbb{R}^d)$ case. They also considered various long-range potentials and more general elliptic operators.

6. The Bethe Lattice

A great example of how the one-dimensional technique works for multidimensional problem is the case of scattering on the Bethe lattice [20]. The step-by-step sum rules used by Simon to study Jacobi matrices [73] can be adjusted to that case. Let us consider this model. Take the Caley tree (Bethe lattice) $\mathbb{B}$ and the discrete Laplacian on it

$$(H_0 u)_n = \sum_{|i-n|=1} u_i$$

Assume, for simplicity, that the degree at each point (the number of neighbors) is equal to 3. It is well known that $\sigma(H_0) = [-2\sqrt{2}, 2\sqrt{2}]$ and the spectrum is purely absolutely continuous. Let $H = H_0 + V$, where V is a potential. Consider any vertex O. It is connected to its neighbors by three edges. Delete one edge together with the corresponding part of the tree stemming from it. What is left will be called $\mathbb{B}_O$. The degree of O within $\mathbb{B}_O$ is equal to 2. The solution to Simon's conjecture in this case is given by the following theorem. We denote by the symbol $c_0(\mathbb{B})$ the functional space of sequences decaying at infinity on $\mathbb{B}$.

Theorem 6.1. *If $V \in \ell^\infty(\mathbb{B}) \cap c_0(\mathbb{B}_O)$ and*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{x\in\mathbb{B}_O, |x-O|=n} V^2(x) < +\infty \tag{6.1}$$

then

$$[-2\sqrt{2}, 2\sqrt{2}] = \sigma_{ac}(H_{|\mathbb{B}_O}) \subseteq \sigma_{ac}(H)$$

The idea of the proof of Theorem 6.1 is based on [73]. This result is sharp in the following sense. Take $V(x) : |V(x)| < C|x-O|^{-\gamma}$. Then for any $\gamma > 0.5$ the condition of the theorem is satisfied (just like in $\mathbb{R}^d$). In the meantime, one can find the spherically symmetric V with slower decay $0 < \gamma < 0.5$, such that there will be no absolutely continuous spectrum at all.

An important role in the proof is played by the well-known recursive relation for $\langle (H_{|\mathbb{B}_O} - z)^{-1}\delta_O, \delta_O\rangle$, where δ_O is the discrete delta-function at the point O. In particular, one can derive the following important and physically meaningful identity

$$\frac{1}{\pi}\int_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{8-\lambda^2}\,\ln[\mu_O'(\lambda)]d\lambda \geq \frac{1}{\pi}\int_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{8-\lambda^2}\,\ln\Big(\frac{\mu_{O_1}'(\lambda)+\mu_{O_2}'(\lambda)}{2}\Big)d\lambda - V^2(O) \tag{6.2}$$

where $\mu'_{O_{1(2)}}$ correspond to the densities of the spectral measures at points $O_{1(2)}$ on the trees obtained from $\mathbb{B}_O$ by throwing away the point O along with the corresponding two edges.

Using inequality $(x+y)/2 \geq \sqrt{xy}$ and iterating (6.2), one proves Theorem 6.1. Formula (6.2) says, in particular, that no matter what happens along one branch of the tree, the scattering is possible through the other branch. It is also clear that the presence of some "bad" points in the tree (say, points where we have no control over potential) should not destroy the scattering as long as these points are rather "sparse". What is an accurate measurement of this sparseness? We suggest the following improvement of Theorem 6.1. Consider the tree $\mathbb{B}_O$ with potential V having finite support, i.e. $V(x) = 0$ for $|x - O| > R$. Consider all paths running from the origin O to infinity without self-intersections. Using diadic decomposition of the real numbers on the interval $[0,1]$, we can assign to each path the real number in the natural way. That is one way of coding the points at infinity. In principle, this map F is not bijection, e.g. sequences $(1,0,0,\ldots)$ and $(0,1,1,\ldots)$ represent the same real number 0.5 but different paths. Fortunately, these numbers have Lebesgue measure zero and will be of no importance for us. Let us define the following functions

$$\phi(t) = \sum_{n=1}^{\infty} V^2(x_n)$$

where x_n are all vertices of the path representing the point $t \in [0,1]$. Since the support of V is within the ball of radius R, function $\phi(t)$ is constant on diadic intervals $[j2^{-R}, (j+1)2^{-R}), j = 0,1,\ldots,2^R - 1$. Notice that F not being bijection cause no trouble in defining $\phi(t)$.

Define the probability measure with density $w(\lambda) = (4\pi)^{-1}(8-\lambda^2)^{1/2}$ on $[-2\sqrt{2}, 2\sqrt{2}]$, and $\rho_O = \sigma'_O(\lambda)[w(\lambda)]^{-1}$, a relative density of the spectral measure at the point O. Define

$$s_O = \int_{-2\sqrt{2}}^{2\sqrt{2}} \ln \rho_O(\lambda) w(\lambda) d\lambda$$

Consider the probability space obtained by assigning to each path the same weight (i.e. as we go from O to infinity, we toss the coin at any vertex and move to one of the neighbors further from O depending on the result). Our goal is to prove the following

Theorem 6.2. *For any V bounded on $\mathbb{B}_O$, the following inequality is true*

$$\exp s_O \geq \mathbb{E}\left\{\exp\left[-\frac{1}{4}\sum_{n=1}^{\infty} V^2(x_n)\right]\right\} = \int_0^1 \exp\left(-\frac{1}{4}\phi(t)\right) dt \qquad (6.3)$$

where the expectation is taken with respect to all paths $\{x_n\}$ going from O to infinity without self-intersections. In particular, if the r.h.s. of (6.3) is positive, then $[-2\sqrt{2}, 2\sqrt{2}] \subseteq \sigma_{ac}(H)$.

Proof. Assume that V has finite support. The estimate (6.2) can be rewritten as

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} \ln \rho_O(\lambda) w(\lambda) d\lambda \geq \int_{-2\sqrt{2}}^{2\sqrt{2}} \ln\left(\frac{\rho_{O_1}(\lambda) + \rho_{O_2}(\lambda)}{2}\right) w(\lambda) d\lambda - V^2(O)/4 \qquad (6.4)$$

Now, we will use the Young's inequality

$$\frac{x^p}{p} + \frac{y^q}{q} \ge xy;\ x, y \ge 0, 1 \le p \le \infty, p^{-1} + q^{-1} = 1$$

in (6.4) to obtain

$$s_O \ge p^{-1} s_{O_1} + q^{-1} s_{O_2} + p^{-1} \ln p + q^{-1} \ln q - \ln 2 - V^2(O)/4 \tag{6.5}$$

Considering s_{O_1} and s_{O_2} to be fixed parameters and maximizing the r.h.s. over $p \in [1, \infty]$, we get the following inequality

$$s_O \ge \ln \frac{\exp s_{O_1} + \exp s_{O_2}}{2} - V^2(O)/4 \tag{6.6}$$

with optimal $p^* = 1 + \exp(s_{O_2} - s_{O_1})$. Iterate (6.6) till we leave the support of V. Thus, we get (6.3). Consider now the general case of bounded V. Define the truncation of V to the ball of radius n: $V_n(x) = V(x)\chi_{\{|x-O|\le n\}}$. For the corresponding $s_O^{(n)}$, we use (6.3), take $n \to \infty$ and apply to the l.h.s. a standard by now argument on the semicontinuity of the entropy ([34], p. 293). Notice that the functions $\phi^{(n)}(t)$ are nonnegative and increasing in n for each t (this is why $\phi(t)$ is always well defined). Therefore, we get (6.3) from the theorem on monotone convergence. $\square$

Using the Jensen's inequality, we obtain

Corollary 6.3. *Assume that A is any subset of $[0, 1]$ of positive Lebesgue measure and $\phi(t) \in L^1(A)$, then*

$$s_O \ge -\frac{1}{4|A|} \int_A \phi(t)dt + \ln |A|$$

In particular, $[-2\sqrt{2}, 2\sqrt{2}] \subseteq \sigma_{ac}(H)$.

It is interesting that the set A does not have to have some special topological structure, say, to be an interval like in the standard scattering theory [2].

There remain many interesting and important open problems regarding the multidimensional slowly decaying potentials. Simon's conjecture (5.1) remains open, and new ideas are clearly needed to make progress. Improving understanding of random slowly decaying potentials is another quite challenging direction. Other difficult and intriguing open questions involve multidimensional sparse potentials, appearance of imbedded singular continuous spectrum, and decaying potentials with additional structural assumptions. This vital area is bound to challenge and inspire mathematicians for years to come.

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