THE GROWTH OF POLYNOMIALS ORTHOGONAL ON THE UNIT CIRCLE WITH RESPECT TO A WEIGHT w THAT SATISFIES $w, w^{-1} \in L^\infty(\mathbb{T})$

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ABSTRACT. We consider the polynomials $\{\phi_n(z,w)\}$ orthogonal on the circle with respect to a weight w that satisfies $w, w^{-1} \in L^\infty(\mathbb{T})$ and show that $\|\phi_n(e^{i\theta},w)\|_{L^\infty(\mathbb{T})}$ can grow in n at a certain rate.

1. Introduction

Consider a finite positive measure σ defined on the unit circle and assume that it has infinitely many growth points. Let the polynomials $\{\phi_k(z,\sigma)\}$ be orthonormal with respect to measure $d\sigma$, i.e.,

$$\int_{-\pi}^{\pi} \phi_n(e^{i\theta})\, \overline{\phi_m(e^{i\theta})}\, d\sigma = \delta_{n,m}\,, \quad n,m = 0,1,2\ldots, \quad \operatorname{coeff}(\phi_k,k) > 0\,, \tag{1.1}$$

where $\operatorname{coeff}(Q,k)$ denotes the coefficient in front of z^k in the polynomial Q. Besides the orthonormal polynomials, we can define the monic orthogonal ones $\{\Phi_n(z,\sigma)\}$ by requiring

$$\deg \Phi_n = n,\ \operatorname{coeff}(\Phi_n,n) = 1,\ \int_{-\pi}^{\pi} \Phi_n(e^{i\theta},\sigma)\overline{\Phi_m(e^{i\theta},\sigma)}\, d\sigma = 0\,, \quad m < n\,.$$

Later, we will need to use the following notation. Let $\mathbb{P}_n$ denote the space of polynomials of degree at most n, i.e.,

$$Q \in \mathbb{P}_n \ \Leftrightarrow\ Q(z) = q_n z^n + \ldots + q_1 z + q_0,\ q_j \in \mathbb{C}, j = 0,\ldots,n.$$

For each $n \in \mathbb{Z}^+$, we introduce the operation $(*)$ defined on $\mathbb{P}_n$ as follows (see, e.g., [16], formula (1.1.6)):

$$Q(z) \xrightarrow{(*)} Q^*(z) := \bar{q}_0 z^n + \ldots + \bar{q}_n\,.$$

This $(*)$ depends on n. Since $Q^*(z) = z^n\overline{Q(\overline{z}^{-1})}$, we can make a trivial observation that

$$Q^*(z) = z^n\overline{Q(z)},\ |Q(z)| = |Q^*(z)|\,, \tag{1.2}$$

if $z \in \mathbb{T}$. Moreover, $(Q^*)^* = Q$.

One of the central questions in the theory of orthogonal polynomials is to understand the behavior of $\{\phi_n(z,\sigma)\}$ as $n \to \infty$, $z \in \mathbb{T}$ given some regularity assumption on σ. Consider the following class of measure.

Given $\delta \in (0,1)$*, we say that* $\sigma \in S_\delta$ *if* σ *is a probability measure and*

$$\sigma' \geqslant \delta/(2\pi), \quad \text{a.e. } \theta \in [-\pi,\pi]\,. \tag{1.3}$$

We will call S_δ the Steklov class. One version of the celebrated Steklov conjecture [17] is to decide whether the sequence $\{\phi_n(z,\sigma)\}$ is bounded in n for every $z \in \mathbb{T}$ provided that $\sigma \in S_\delta$. This conjecture was solved negatively by Rakhmanov in 1979 [14] (see [6, 7, 8, 9, 10, 18]).

This development lead to more general problem (dubbed "problem by Steklov") which asks to obtain the bounds on $\|\phi_n(z,\sigma)\|_{L^\infty(\mathbb{T})}$ that would be sharp when $n \to \infty$ and $\sigma \in S_\delta$. Some upper estimates are usually not very difficult to come by. For example, one has

Lemma 1.1. *If $\sigma \in S_\delta$, then*

$$\|\phi_n(z,\sigma)\|_{L^\infty(\mathbb{T})} = o(\sqrt{n}), \quad n \to \infty\,. \tag{1.4}$$

This result follows from, e.g., [12], Theorem 4 (see also [13],[20] for the real-line case).

If n is fixed, then the following Lemma is true.

Lemma 1.2. ([2]) *Define $M_{n,\delta}$ by*

$$M_{n,\delta} := \sup_{\sigma \in S_\delta} \|\phi_n(z,\sigma)\|_{L^\infty(\mathbb{T})}\,.$$

Then, we have

$$M_{n,\delta} \leqslant \min\left\{\sqrt{\frac{n+1}{\delta}}, \frac{1}{\sqrt{\delta}}\left(1+\sqrt{\frac{n(1-\delta)}{\delta}}\right)\right\}\,.$$

The question then naturally arises whether these results are optimal when δ is fixed and $n \to \infty$. In his second paper [15], Rakhmanov obtained the estimates that were nearly sharp. However, the full solution to the problem has been given quite recently and by a different method.

Theorem 1.1. ([2]) *If $\delta \in (0,1)$ is fixed, then*

$$M_{n,\delta} > C_{(\delta)}\sqrt{n}\,. \tag{1.5}$$

Theorem 1.2. ([2]) *Let $\delta \in (0,1)$ be fixed. Then, for every positive sequence $\{\beta_n\}$ that satisfies $\lim_{n\to\infty}\beta_n = 0$, there is an absolutely continuous probability measure $\sigma^* : d\sigma^* = \sigma^{*\prime} d\theta$ such that $\sigma^* \in S_\delta$ and*

$$\|\phi_{k_n}(z,\sigma^*)\|_{L^\infty(\mathbb{T})} \geqslant \beta_{k_n}\sqrt{k_n} \tag{1.6}$$

for some sequence $\{k_n\} \subset \mathbb{N}$.

We need to make three remarks here.

Remark 1. The paper [2] proves more than just the inequality (1.5). It shows that for every $p \in [1,\infty)$ and every $L_1 > 1$, the following bound holds

$$\sup_{w \geqslant \delta/(2\pi),\, \|w\|_{L^1(\mathbb{T})}=1,\, \|w\|_{L^p(\mathbb{T})}<L_1} \|\phi_n(z,w)\|_{L^\infty(\mathbb{T})} > C_{(\delta,p,L_1)}\sqrt{n}\,. \tag{1.7}$$

Here $\{\phi_n(z,w)\}$ are orthonormal with respect to the weight w. In general, we call function w a weight if $w \in L^1(-\pi,\pi)$, $w \geqslant 0$, and $w \not\equiv 0$.

Remark 2. We will say that the measure σ belongs to the Szegő class, if

$$\int_{-\pi}^{\pi} \log \sigma' d\theta > -\infty. \tag{1.8}$$

It is known ([16], formulas (1.5.78) and (2.3.1)), that

$$\exp\left(\frac{1}{4\pi}\int_{\mathbb{T}} \log(2\pi\sigma'(\theta))d\theta\right) \leqslant \left|\frac{\Phi_n(z,\sigma)}{\phi_n(z,\sigma)}\right| \leqslant 1, \quad \forall z \in \mathbb{C}\,. \tag{1.9}$$

and the left hand side is positive if and only if (1.8) holds. Thus, for measures in Steklov class, we have a bound

$$\sqrt{\delta} \leqslant \left|\frac{\Phi_n(z,\sigma)}{\phi_n(z,\sigma)}\right| \leqslant 1, \quad \forall z \in \mathbb{C}\,. \tag{1.10}$$

Therefore, if $\{\Phi_n\}$ or $\{\phi_n\}$ grows in n, then the other sequence grows as well.

Remark 3. The requirement that the measure is a probability one is not so restrictive due to the following scaling

$$\phi_n(z,\sigma) = \alpha^{1/2}\phi_n(z,\alpha\sigma), \quad \Phi_n(z,\sigma) = \Phi_n(z,\alpha\sigma), \quad \alpha > 0\,. \tag{1.11}$$

Suppose we are given a function $h(t)$ defined on $[0,2\pi]$ which satisfies

$$h(0) = \lim_{t\to 0} h(t) = 0.$$

We will say that a function w, defined on $\mathbb{R}$ and 2π-periodic there, has h as its modulus of continuity with a constant $L \in [0,\infty)$, if the estimate

$$|w(x_2) - w(x_1)| \leqslant Lh(|x_2 - x_1|) \tag{1.12}$$

holds for all $x_1, x_2 \in \mathbb{R}$ that satisfy $|x_1 - x_2| \leqslant 2\pi$.

If w is defined on $[-\pi,\pi]$ and $w \in C[-\pi,\pi]$, $w(-\pi) = w(\pi)$, we can extend it to all of $\mathbb{R}$ as 2π-periodic function. Then, if (1.12) holds, we will write $w \in C_{h,L}(\mathbb{T})$.

Theorem 1.3. (([19]), *Theorem 12.1.3) Assume that* $d\sigma = w(\theta)d\theta$ *and* $w \in C_{h,L}(\mathbb{T})$, $h(t) = |\log(t/8)|^{-1-\epsilon}$ *with some* $\epsilon > 0$ *and* L*. Then, as long as* $w \geqslant \delta > 0$ *for all* $\theta \in [-\pi,\pi]$*, we have*

$$\phi_n(e^{i\theta}) = e^{in\theta}\overline{\Pi(e^{i\theta})} + \epsilon_n(e^{i\theta}), \quad n \to \infty\,, \tag{1.13}$$

where $\epsilon_n(e^{i\theta}) \to 0$ *uniformly over* $\theta \in [-\pi,\pi]$*. The function* $\Pi(z)$ *is defined as the outer function in* $\mathbb{D}$ *that satisfies* $|\Pi(e^{i\theta})|^{-2} = 2\pi w(\theta)$, $\Pi(0) > 0$.

In [1], given arbitrary $\epsilon > 0$ and $L > 0$, Ambroladze constructed a positive weight $\widehat{w} \in C_{h,L}(\mathbb{T})$, $h(t) = |\log(t/8)|^{-1+\epsilon}$ for which $\{\|\phi_n(z,\widehat{w})\|_{L^\infty(\mathbb{T})}\}$ is unbounded in n. This showed sharpness of regularity assumption in Theorem 1.3.

The bound (1.7) raises a question: what regularity of a weight can improve the $\sqrt{n}$ upper bound? In this paper, we give partial answer to this question. We consider measures given by the weights w that satisfy $w, w^{-1} \in L^\infty(\mathbb{T})$ and obtain upper and lower estimates for the possible growth of the supremum norms. In the second section, we provide an argument due to Fedor Nazarov that gives the upper bounds for $\|\phi_n(e^{i\theta},w)\|_{L^\infty(\mathbb{T})}$. The third section contains the proofs of the lower bounds. The Appendices have some auxiliary statements that are used in the main text.

We will use the following definitions and notation.

- The Schwartz kernel $C(z,\xi)$ is defined as

$$C(z,\xi) := \frac{\xi + z}{\xi - z}, \quad \xi \in \mathbb{T},\, z \in \mathbb{C}\,.$$

- The function analytic in $\mathbb{D} = \{z : |z| < 1\}$ is called Caratheodory function if its real part is nonnegative in $\mathbb{D}$.
- Given a set Ω, χ_Ω denotes the characteristic function of Ω.

- If two positive functions f_1 and f_2 are given, we write $f_1 \lesssim f_2$ if there is an absolute constant C such that
$$f_1 < Cf_2$$
for all values of the argument. We define $f_1 \gtrsim f_2$ similarly. Writing $f_1 \sim f_2$ means $f_1 \lesssim f_2 \lesssim f_1$.
- The symbol $P_{[n_1,n_2]}$ denotes the $L^2[-\pi,\pi]$ orthogonal Fourier projection to the Fourier modes $\{n_1,\ldots,n_2\}$:
$$P_{[n_1,n_2]}f := \sum_{j=n_1}^{n_2} (2\pi)^{-1/2} e^{ij\theta} \widehat{f}_j$$
and
$$\widehat{f}_j := (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(\theta) e^{-ij\theta} d\theta$$
stands for the Fourier coefficients of $f \in L^1(-\pi,\pi)$.
- The symbol $\mathcal{F}_n$ will denote the Fejer kernel of degree n given by
$$\mathcal{F}_n := \frac{1}{2\pi n} \left(\frac{\sin(n\theta/2)}{\sin(\theta/2)} \right)^2 .$$
- The Jackson kernel is given by
$$\mathcal{K}_n = c_n \mathcal{F}_n^2 \sim n^{-3} \frac{\sin^4(nx/2)}{\sin^4(x/2)}$$
and c_n is chosen such that
$$\int_{-\pi}^{\pi} \mathcal{K}_n dx = 1 .$$
- Throughout the paper, the symbol C denotes a positive absolute constant. Its actual value might change from formula to formula. If, within a certain proof or statement of the result, we write C_1, C_2, etc., this means that we want to emphasize that these constants might be different. Sometimes we will use c instead of C to emphasize the difference.
- If the proof uses parameters, e.g., $\alpha_1,\ldots,\alpha_n$, we write $C_{(\alpha_1,\ldots,\alpha_n)}$ or $n_{(\alpha_1)}$ to denote a non-negative function defined for all values of these parameters. We use this notation in those cases when the actual dependence on these parameters is not important (or unknown) to us. Similarly to the constants, if we want to distinguish between them to avoid confusion, we write, e.g., $C^{(1)}_{(\alpha_1)}, C^{(2)}_{(\alpha_2)}$, etc.
- If $(\Omega_{1(2)}, \mu_{1(2)})$ are two measure spaces and A is a linear operator, bounded from $L^{p_1}(\Omega_1,\mu_1)$ to $L^{p_2}(\Omega_2,\mu_2)$, then $\|A\|_{p_1,p_2}$ denotes its operator norm.
- The following standard notation will be used several times. If f_1 is non-negative function, then writing $O(f_1)$ denotes a function, say, call it f for a moment, which satisfies
$$|f| \lesssim f_1$$
for the specified range of the arguments. For example, writing
$$O(\epsilon|\theta|), \quad \epsilon \in (0,1), \theta \in [-\pi,\pi)$$

denotes a function, say f, which satisfies

$$|f(\theta,\epsilon)| \lesssim \epsilon|\theta|$$

for $\theta \in [-\pi,\pi), \epsilon \in (0,1)$. In that case, $f_1 = \epsilon|\theta|$.

- If the function f is defined on $\mathbb{T}$, the symbol $\|f\|_p$ means $\|f\|_{L^p(\mathbb{T})}$, $1 \leqslant p \leqslant \infty$.
- Given two functions $f_1, f_2 \in L^1(\mathbb{T})$, we define their convolution by

$$f_1 * f_2 := \int_{\mathbb{T}} f_1(\theta-\phi) f_2(\phi) d\phi \,.$$

- If $z \in \mathbb{C}, z \neq 0$, its argument $\arg z$ is always chosen such that it belongs to $(-\pi,\pi]$.

2. Upper bounds a la Bernstein

In this section, we will apply an idea that was used by S.N. Bernstein when proving an analog of Theorem 1.3 for the case of orthogonality on the segment of real line [3]. In fact, Szegő's proof of Theorem 1.3 is an adaptation of this method.

Lemma 2.1. *The monic polynomial Q_n of degree n is n-th monic orthogonal polynomial with respect to a weight w if and only if*

$$P_{[0,n-1]}(wQ_n) = 0 \,. \tag{2.1}$$

Proof. It is sufficient to notice that (2.1) is equivalent to

$$\int_{-\pi}^{\pi} Q_n e^{-ij\theta} w d\theta = 0, \quad j = 0,\ldots,n-1 \,,$$

which is the orthogonality condition. □

Lemma 2.2. *For every $p \in [2,\infty)$,*

$$\|P_{[0,n]}\|_{p,p} \leqslant 1 + C(p-2) \,. \tag{2.2}$$

Proof. If $\mathcal{P}^+$ is the projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ (Riesz projection), then we can write an identity

$$P_{[0,n]} = \mathcal{P}^+ - z^{n+1}\mathcal{P}^+ z^{-(n+1)} \,.$$

In [11], it was proved that

$$\|\mathcal{P}^+\|_{p,p} = \frac{1}{\sin(\pi/p)}, \quad 1 < p < \infty \,.$$

Thus

$$\|P_{[0,n]}\|_{p,p} \leqslant \frac{2}{\sin(\pi/p)}$$

by triangle inequality. On the other hand, we have $\|P_{[0,n]}\|_{2,2} = 1$ and the Riesz-Thorin Theorem [5] allows to interpolate between $p = 2$ and, e.g., $p = 3$ to get a bound

$$\|P_{[0,n]}\|_{p,p} \leqslant 1 + C(p-2), \quad 2 < p < 3 \,.$$

Since

$$\frac{2}{\sin(\pi/p)} \sim p$$

for $p \geqslant 3$, the proof is finished. □

The proof of the following result is due to Fedor Nazarov (personal communication). We give it here for completeness.

Theorem 2.1. *(F. Nazarov) If* $\epsilon \in (0,1]$*, then*

$$\sup_{1\leqslant w\leqslant 1+\epsilon} \|\Phi_n(e^{i\theta},w)\|_p \lesssim 1, \quad p = C_1\epsilon^{-1}\,. \tag{2.3}$$

If $T>2$*, then*

$$\sup_{1\leqslant w\leqslant T} \|\Phi_n(e^{i\theta},w)\|_p \lesssim 1, \quad p = 2 + C_2T^{-1}\,. \tag{2.4}$$

Proof. Let $\mu = w-1$. Following Bernstein, we use (2.1) to get the formula

$$\Phi_n(z,w) = z^n + (2\pi)^{-1}\sum_{j=0}^{n-1} z^j \int_{\mathbb{T}} \Phi_n(e^{i\theta},w)e^{-ij\theta}d\theta$$

$$= z^n - (2\pi)^{-1}\sum_{j=0}^{n-1} z^j \int_{\mathbb{T}} \Phi_n(e^{i\theta},w)e^{-ij\theta}\mu(\theta)d\theta = z^n - P_{[0,n-1]}(\mu\Phi_n)\,.$$

Thus, Φ_n is a fixed point of the operator defined by

$$f \to z^n - P_{[0,n-1]}(\mu f)$$

on $f \in L^p(\mathbb{T})$. If $1 \leqslant w \leqslant 1+\epsilon$, then $\|P_{[0,n-1]}\mu\|_{p,p} \leqslant Cp\epsilon$ by (2.2). Choosing p such that $Cp\epsilon < 1$ makes $P_{[0,n-1]}\mu$ a contraction in $L^p(\mathbb{T})$. Then, (2.3) follows from the Fixed Point Theorem for contractions.

To prove (2.4), we need to modify this argument a little. From (2.1), we get

$$\Phi_n = z^n + P_{[0,n-1]}(1-\kappa w)\Phi_n\,, \tag{2.5}$$

where κ is an arbitrary real parameter. Taking $\kappa = T^{-1}$, we can write

$$0 \leqslant 1-\kappa w \leqslant 1 - T^{-1}$$

and (2.2) implies

$$\|P_{[0,n-1]}(1-\kappa w)\|_{p,p} \leqslant (1+C(p-2))(1-T^{-1}) = 1-(1-C\lambda)T^{-1} - C\lambda T^{-2}\,,$$

provided that $p = 2+\lambda T^{-1}$. Choosing $\lambda < C^{-1}$ makes the right hand side of (2.5) a contraction in $L^p(\mathbb{T})$. This again proves (2.4) by the Fixed Point Theorem if we let $C_2 := \lambda$. □

By Nikolskii inequality for elements of $\mathbb{P}_n$, we have

Corollary 2.1. *If* $\epsilon \in (0,1]$*, then*

$$\sup_{1\leqslant w\leqslant 1+\epsilon} \|\Phi_n(e^{i\theta},w)\|_\infty \lesssim n^{C_3\epsilon}\,. \tag{2.6}$$

If $T>2$*, then*

$$\sup_{1\leqslant w\leqslant T} \|\Phi_n(e^{i\theta},w)\|_\infty \lesssim n^{\frac{1}{2}-C_4T^{-1}}\,. \tag{2.7}$$

Proof. We have an estimate (see, e.g., [21], p.154, Theorem 6.35)

$$\|\Phi_n\|_\infty \lesssim n^{1/p}\|\Phi\|_p, \quad p \geqslant 2\,.$$

Taking p as in (2.3) and (2.4) finishes the proof. □

Remark. The estimate (2.7) shows that (1.7) is no longer true if w is in Steklov class and belongs to $L^\infty(\mathbb{T})$.

Remark. Due to (1.9) and (1.11), the analogous results for orthonormal polynomials and for probability measures hold as well.

3. LOWER BOUNDS

The following Theorem is the main result of the paper.

Theorem 3.1. *If* $\epsilon \in (0,1]$, *then*

$$\sup_{1\leqslant w\leqslant 1+\epsilon} \|\Phi_n(e^{i\theta},w)\|_\infty > C^{(1)}_{(\epsilon)} n^{C_5\epsilon}\,. \tag{3.1}$$

If $T>2$, *then*

$$\sup_{1\leqslant w\leqslant T} \|\Phi_n(e^{i\theta},w)\|_\infty > C^{(2)}_{(T)} n^{\frac{1}{2}-C_6T^{-1/4}}\,. \tag{3.2}$$

Remark. The constants C in these bounds are different from those in the corollary above. The bound (3.1) implies that for every $\alpha_1 \in (0,C_5)$, there is $n_{(\epsilon,\alpha_1)}\in\mathbb{N}$ such that

$$\sup_{1\leqslant w\leqslant 1+\epsilon} \|\Phi_n(e^{i\theta},w)\|_\infty > n^{\alpha_1\epsilon}$$

for all $n>n_{(\epsilon,\alpha_1)}$. Analogous claim can be made for (3.2).

The main tool for proving this Theorem will be the technique developed in [2] to obtain the sharp estimates in Steklov's problem ([2], [4], see Appendix A), along with the localization principle ([2], Appendix B).

Proof. (of Theorem 3.1). The proof will consist of two parts (estimate (3.1) and estimate (3.2)). In the first one, we will construct a weight having deviation from a constant at most $C\epsilon$ on a small arc around the point $\theta=0$. The corresponding orthogonal polynomial will be large at $z=1$. Then, using the localization principle, which is discussed in Appendix B, we will provide a weight of small deviation over the whole circle and show that the corresponding polynomial still has the required size. In the second part, we will apply a similar strategy to handle the weights with large deviation.

3.1. The case of small deviation, estimate (3.1).

We will follow the strategy used in [2] and explained in Lemma 4.3 in Appendix A. Let us assume that n is even, the case of odd n can be handled similarly. Thus, we will be applying this Lemma taking $2n$ instead of n. Below, we introduce $\widetilde{F}$, ϕ^*_{2n} and check that they satisfy conditions of Lemma 4.3. Then, we will control how the weight σ' in (4.5) deviates from a constant. Notice that, when proving (3.1), it is sufficient to consider $\epsilon\in(0,\epsilon_0)$ where $\epsilon_0\in(0,1)$ is fixed. We can also assume that $n>n_{(\epsilon)}$.

(a) Consider auxiliary function h_n given by

$$h_n := 2(1-e^{i\theta})^\epsilon * \mathcal{F}_n\,.$$

Clearly h_n is a polynomial of degree $n-1$. Since $\operatorname{Re}(1-e^{i\theta})^\epsilon \geqslant 0$ and $\mathcal{F}_n\geqslant 0$, its real part is strictly positive over $\mathbb{T}$ so h_n is zero-free in $\overline{\mathbb{D}}$. In Lemmas 6.1 and 6.2 of Appendix C, more detailed information is obtained, in particular,

$$|\arg h_n| \lesssim \epsilon \tag{3.3}$$

uniformly in $\theta \in [-\pi, \pi)$ and $n > n_{(\epsilon)}$.

Because $\int_{-\pi}^{\pi} \mathcal{F}_n d\theta = 1$, we get

$$\int_{-\pi}^{\pi} h_n d\theta = 2\pi h_n(0) = 4\pi\,. \tag{3.4}$$

Since $\mathcal{F}_n$ is even and $\operatorname{Im}(1 - e^{i\theta})^\epsilon$ is odd on $[-\pi, \pi]$, we know that $\operatorname{Im} h_n$ is odd on $[-\pi, \pi]$ as well.

(b) In Lemma 6.2, choose

$$\widetilde{F} := \frac{2}{h_n}\,. \tag{3.5}$$

From the properties of h_n, we get analyticity of $\widetilde{F}_n$ in $\mathbb{D}$ and infinite smoothness on $\mathbb{T}$. Since

$$\operatorname{Re} \widetilde{F} = 2\frac{\operatorname{Re} h_n}{|h_n|^2}, \tag{3.6}$$

$\widetilde{F}$ is Caratheodory function. Moreover, since $h_n(0) = 2$, we get the normalization $\operatorname{Re} \widetilde{F}(0) = 1$.

From (3.3) and (3.6), we also have

$$1 \leqslant \operatorname{Re} h_n \cdot \operatorname{Re} \widetilde{F} \leqslant 2 \tag{3.7}$$

uniformly in $\theta \in [-\pi, \pi), n > n_{(\epsilon)}$.

(c) Notice that $\left(\operatorname{Re} \widetilde{F}\right) * \mathcal{F}_n$ is a positive trigonometric polynomial of degree at most $n-1$. Let $q_n \in \mathbb{P}_{n-1}$ be zero free in $\mathbb{D}$ and satisfy

$$|q_n|^2 = \left(\operatorname{Re} \widetilde{F}\right) * \mathcal{F}_n, \quad z \in \mathbb{T}, \quad q_n(0) > 0\,. \tag{3.8}$$

The existence and uniqueness of this polynomial q_n is well-known ([19], Theorem 1.2.2). We will need this q_n in the next argument.

In Lemma 6.2, we also need to choose ϕ_{2n}. We will first specify ϕ_{2n}^* and then will take $\phi_{2n} := (\phi_{2n}^*)^*$. In this calculation, the operation $(*)$ acts on $\mathbb{P}_{2n}$.

We let

$$\phi_{2n}^* := \alpha_n(q_n + q_n^* + q_n h_n)\,, \tag{3.9}$$

where α_n is the positive normalization parameter chosen to ensure that

$$\|\phi_{2n}^{-1}\|^2_{L^2(\mathbb{T})} = \|(\phi_{2n}^*)^{-1}\|^2_{L^2(\mathbb{T})} = 2\pi\,. \tag{3.10}$$

as required in (4.3). We will obtain the estimates for α_n below. We emphasize again that q_n^* in (3.9) is understood as $(*)$–operation on $\mathbb{P}_{2n}$. Thus, $\deg \phi_{2n}^* \leqslant 2n$.

To make sure that ϕ_{2n}^* has no zeroes in $\overline{\mathbb{D}}$, we write

$$q_n + q_n^* + q_n h_n = q_n \left(1 + h_n + \frac{q_n^*}{q_n}\right)\,. \tag{3.11}$$

Then, q_n is zero-free in $\overline{\mathbb{D}}$ and $1 + h_n + q_n^*/q_n$ is analytic in $\mathbb{D}$ having the positive real part since

$$\operatorname{Re} h_n > 0, \quad \operatorname{Re}\left(1 + \frac{q_n^*}{q_n}\right) \geqslant 0, \quad z \in \mathbb{T}\,.$$

The last inequality is the consequence of $|q_n^*| = |q_n|, z \in \mathbb{T}$ (see (1.2)).

At the point $z = 0$, we have

$$q_n(0) + q_n^*(0) + q_n(0)h_n(0) = q_n(0)(1 + h_n(0))\,,$$

because $q_n^*(0) = 0$. Then, (3.4) and (3.8) ensure that the right hand side in the last expression is positive. Since the polynomial $q_n + q_n^* + q_nh_n$ is positive at zero, we get $\deg \phi_{2n} = \deg((q_n + q_n^* + q_nh_n)^*) = 2n$ because, again, $(*)$ acts on $\mathbb{P}_{2n}$.

Next, we will show that

$$\alpha_n \sim 1\,, \tag{3.12}$$

if $n > n_{(\epsilon)}$. Indeed, the following estimate holds

$$|q_n + q_n^* + q_nh_n|^2 = \left|q_n\left(1 + h_n + \frac{q_n^*}{q_n}\right)\right|^2 \geqslant |q_n|^2(\operatorname{Re} h_n)^2 = (\operatorname{Re} h_n)^2(\operatorname{Re} \widetilde{F} * \mathcal{F}_n)\,, \tag{3.13}$$

where we have used (3.8) at the last step. Let us rewrite the last expression using (3.6)

$$(\operatorname{Re} h_n)^2(\operatorname{Re} \widetilde{F} * \mathcal{F}_n) = (\operatorname{Re} h_n)^2 \operatorname{Re} \widetilde{F} \cdot \frac{(\operatorname{Re} \widetilde{F} * \mathcal{F}_n)}{\operatorname{Re} \widetilde{F}} = 2\frac{(\operatorname{Re} h_n)^3}{|h_n|^2} \cdot \frac{(\operatorname{Re} \widetilde{F} * \mathcal{F}_n)}{\operatorname{Re} \widetilde{F}}\,.$$

For the second factor, we have

$$\left|\frac{(\operatorname{Re} \widetilde{F} * \mathcal{F}_n)}{\operatorname{Re} \widetilde{F}}\right| \sim 1\,, \tag{3.14}$$

if $\theta \in [-\pi, \pi)$ and $n > n_{(\epsilon)}$. This is due to Lemma 6.4.

From (3.3), we get

$$\frac{(\operatorname{Re} h_n)^2}{|h_n|^2} \sim 1$$

under the same assumptions as in (3.14).

Thus, $|q_n + q_n^* + q_nh_n|^2 \gtrsim \operatorname{Re} h_n$ if $\theta \in [-\pi, \pi), n > n_{(\epsilon)}$. Therefore, Lemmas 6.1, 6.2, 6.4, and (3.7) give

$$|\theta|^\epsilon \lesssim |q_n + q_n^* + q_nh_n|^2 \lesssim |q_n|^2 = \left(\operatorname{Re} \widetilde{F}\right) * \mathcal{F}_n \lesssim |\theta|^{-\epsilon}$$

for $\theta \in [-\pi, \pi)$, uniformly in $n > n(\epsilon)$. Therefore, we have (3.12).

Now that we checked all conditions in Lemma 4.3, we can apply it. Consider the value of ϕ_{2n}^* at $z = 1$. Since $|q_n(e^{i\theta})|$ is even and $q_n(0) > 0$, we have $q_n(1) \in \mathbb{R}$. Thus, $q_n^*(1) = q_n(1)$ and

$$|\phi_{2n}^*(1, \sigma)|^2 = \alpha_n^2 |q_n(1)(2 + h_n(0))|^2 \sim |\operatorname{Re} \widetilde{F} * \mathcal{F}_n|_{z=1} \geqslant$$

$$|\operatorname{Re} \widetilde{F}|_{z=1} - |\operatorname{Re} \widetilde{F}|_{z=1} \left|\frac{\operatorname{Re} \widetilde{F} * \mathcal{F}_n}{\operatorname{Re} \widetilde{F}} - 1\right|_{z=1} \gtrsim |\operatorname{Re} \widetilde{F}|_{z=1}$$

by Lemma 6.4, provided that $n > n_{(\epsilon)}$. Now, (3.6) and Lemma 6.2 yield

$$|\phi_{2n}(1, \sigma)|^2 = |\phi_{2n}^*(1, \sigma)|^2 \gtrsim n^\epsilon\,,$$

which ensures the necessary growth (compare with (3.1)).

Next, we will study the weight σ' given by the formula (4.5). We can write

$$\sigma'^{-1} = v_n \mathcal{A}\mathcal{B}\mathcal{C}\,,$$

where $\upsilon_n = \pi\alpha_n^2/2$ is n–dependent parameter and

$$\mathcal{A} := \frac{|q_n|^2}{\operatorname{Re}\widetilde{F}}, \quad \mathcal{B} := |2 + \overline{h}_n(1 - \widetilde{F})|^2, \quad \mathcal{C} := \left|\xi + \frac{2 + h_n(1 + \widetilde{F})}{2 + \overline{h}_n(1 - \widetilde{F})}\right|^2, \quad \xi = e^{i(2n\theta - 2\Theta)}, \tag{3.15}$$

where $\Theta = \arg q_n$. The estimate (3.12) guarantees that $\upsilon_n \sim 1$ if $n > n_{(\epsilon)}$.

Now, in what follows, we will consider a small ϵ–dependent interval $I_\epsilon \subset [-\pi, \pi)$ centered at $\theta = 0$ and will control how each of the factors $\mathcal{A}$ and $(\mathcal{B}\mathcal{C})$ deviates from constants on I_ϵ. The size of I_ϵ will be specified later, it will not depend on n if $n > n_{(\epsilon)}$.

By Lemma 6.4 and the choice of q_n,

$$|\mathcal{A} - 1| \lesssim \epsilon \tag{3.16}$$

uniformly in $\theta \in [-\pi, \pi)$ and $n > n(\epsilon)$.

For $\mathcal{B}$, we can use (3.5) to write

$$\mathcal{B} = |2 + \overline{h}_n(1 - \widetilde{F})|^2 = \left|2\left(1 - \frac{\overline{h}_n}{h_n}\right) + \overline{h}_n\right|^2 \lesssim \epsilon^2 + |h_n|^2 \sim \epsilon^2 + n^{-\epsilon} + |\theta|^\epsilon$$

due to estimates on h_n obtained in Lemma 6.2), provided that $n > n_{(\epsilon)}$.

Now, we choose I_ϵ such that $\max_{\theta \in I_\epsilon} |\theta|^\epsilon < \epsilon^2$. Then, we have

$$\mathcal{B} \lesssim \epsilon^2$$

for $\theta \in I_\epsilon$, $n > n_{(\epsilon)}$.

Recall the definition of $\mathcal{C}$ and consider

$$\mathcal{J} := \frac{2 + h_n(1 + \widetilde{F})}{2 + \overline{h}_n(1 - \widetilde{F})}\,.$$

Then, we have

$$\mathcal{B}\mathcal{C} = \mathcal{B}(1 + |\mathcal{J}|^2 + 2\operatorname{Re}(\xi\overline{\mathcal{J}})) = \mathcal{B} + \mathcal{B}|\mathcal{J}|^2 + 2\mathcal{B}\operatorname{Re}(\xi\overline{\mathcal{J}})\,. \tag{3.17}$$

Assume that $\theta \in I_\epsilon$ and $n > n_{(\epsilon)}$. Then, the first term is at most $C\epsilon^2$. For the third one, we use the formula for $\mathcal{B}$ and (3.5) to get

$$|2\mathcal{B}\operatorname{Re}(\xi\overline{\mathcal{J}})| \lesssim \sqrt{\mathcal{B}}|2 + h_n(1 + \widetilde{F})| \lesssim \epsilon|2 + h_n(1 + 2h_n^{-1})| \lesssim \epsilon\,,$$

where at the last step we used an estimate for h_n from Lemma 6.2. By (3.5), the second term in (3.17) can be rewritten as

$$|2 + h_n(1 + \widetilde{F})|^2 = |4 + h_n|^2 = 16 + 8\operatorname{Re} h_n + |h_n|^2\,.$$

Notice that, by the choice of I_n,

$$|8\operatorname{Re} h_n + |h_n|^2| \lesssim \epsilon\,,$$

if $\theta \in I_\epsilon$ and $n > n_{(\epsilon)}$. To summarize, we have that $\mathcal{A}$ deviates from 1 by at most $C\epsilon$ and $\mathcal{B}\mathcal{C}$ deviates from 16 by at most $C\epsilon$, provided that $\theta \in I_\epsilon$ and $n > n_{(\epsilon)}$. Thus,

$$\sigma'(\theta) = \omega_n\left(1 + O(\epsilon)\right)\,,$$

if $\theta \in I_\epsilon$ and $n > n_{(\epsilon)}$. The positive parameter ω_n depends on n only and $\omega_n \sim 1$.

To apply Lemma 5.1 later, we have to check that σ' satisfies the "global lower and upper bounds", i.e., we need to verify that

$$\mathcal{A}\mathcal{B}\mathcal{C} \sim 1\,,$$

if $\theta \in \mathbb{T}$ and $n > n_{(\epsilon)}$. Indeed, for $\mathcal{A}$, we use (3.16). Then, to control $\mathcal{B}\mathcal{C}$, we argue differently:

$$\mathcal{B}\mathcal{C} = \left|4 + h_n + \xi\left(\overline{h}_n + 2\left(1 - \frac{\overline{h}_n}{h_n}\right)\right)\right|^2\,. \tag{3.18}$$

It is clear from Lemma 6.2 that $\mathcal{B}\mathcal{C} \lesssim 1$. For the lower bound, we can use an estimate (see again Lemma 6.2)

$$\left|1 - \frac{\overline{h}_n}{h_n}\right| \lesssim \epsilon$$

to get

$$\begin{aligned} 4 + h_n + \xi\left(\overline{h}_n + 2\left(1 - \frac{\overline{h}_n}{h_n}\right)\right) &= 4 + h_n + O(\epsilon) + \xi\left(\frac{\overline{h}_n}{h_n} - 1\right)h_n + \xi h_n \\ &= 4 + (\xi + 1)h_n + O(\epsilon)\,. \end{aligned}$$

In these estimates, $O(\epsilon)$ is a shorthand for a function, whose absolute value is bounded by $C\epsilon$ for all $\theta \in [-\pi, \pi)$ and $n > n_{(\epsilon)}$.

Since $\arg(1+\xi) \in [-\pi/2, \pi/2]$ and $|\arg h_n| < C\epsilon$, we can say that there is $\epsilon_0 > 0$ such that

$$|4 + (\xi + 1)h_n + O(\epsilon)| \gtrsim 1\,,$$

provided that $\epsilon \in (0, \epsilon_0)$ and $\theta \in [-\pi, \pi), n > n_{(\epsilon)}$. Therefore, we have $\sigma' \sim 1$ for $\epsilon \in (0, \epsilon_0), n > n_{(\epsilon)}, \theta \in [-\pi, \pi)$.

To summarize, we showed that there is $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, there is $n_{(\epsilon)} \in \mathbb{N}$ and I_ϵ, centered at the origin, such that for every $n > n_{(\epsilon)}$ there is a weight σ' (density of absolutely continuous measure σ) for which

- $|\phi_{2n}(1, \sigma)| \gtrsim n^{\epsilon/2}$,
- $\sigma'(\theta) \sim 1$ for $\theta \in [-\pi, \pi)$,
- $\sigma'(\theta) = \omega_n(1 + O(\epsilon))$ if $\theta \in I_\epsilon$, and $\omega_n \sim 1$.

Notice carefully that σ is also n-dependent in this construction. The case of polynomials of odd degree can be handled by a minor modification of the argument (by working in $\mathbb{P}_{2n+1}$).

Finally, we are in a position to use localization principle in Appendix B to produce a new weight whose deviation from the constant is smaller than $C\epsilon$ over the whole $\mathbb{T}$ and the value of the corresponding orthogonal polynomial at point $z = 1$ is still large. To that end, consider w_1:

$$w_1 := \begin{cases} 1, & \theta \notin I_\epsilon\,, \\ \dfrac{\sigma'}{\min\limits_{I_\epsilon} \sigma'}, & \theta \in I_\epsilon\,. \end{cases}$$

We have $0 \leqslant w_1 - 1 \lesssim \epsilon$ over $\mathbb{T}$. Lemma 5.1 and (1.11) gives

$$|\phi_{2n}(1, w_1)| > C_{(\epsilon)}|\phi_{2n}(1, \sigma)| > C_{(\epsilon)}n^{\epsilon/2}\,,$$

if $n > n_{(\epsilon)}$. The analogous estimate for the monic polynomials is true due to (1.10). This proves (3.1) for even n, the case of odd n can be handled similarly.

3.2. **The case of large deviation, estimate** (3.2)**.** Let us introduce a parameter $\alpha \in (1/2, 1)$ and define $\tau := 1 - \alpha$. To clarify the meaning of these parameters, we now mention that T in (3.2) will be related to τ through an estimate $T \sim \tau^{-4}$. Notice that (3.2) will follow, if we show that there is $T_0 > 2$ such that for every $T > T_0$ there is $n_{(T)}$ so that

$$\sup_{1 \leqslant w \leqslant T} \|\Phi_n(e^{i\theta}, w)\|_\infty > C^{(2)}_{(T)} n^{\frac{1}{2} - C_6 T^{-1/4}} \tag{3.19}$$

for all $n > n_{(T)}$. In other words, we can assume that both T and n are "large".

Similarly to the previous subsection, we apply Lemma 4.3. In this Lemma, we take $2n$ instead of n and the $(*)$ operation will act on $\mathbb{P}_{2n}$. The case of odd n is similar.

(a). We define the function H_n by

$$H_n = 2(1 - e^{i\theta})^\alpha * \mathcal{K}_{[n/2]} \,, \tag{3.20}$$

where $\mathcal{K}_l$ is Jackson kernel, i.e.,

$$\mathcal{K}_l = c_l \mathcal{F}_l^2 \sim l^{-3} \frac{\sin^4(lx/2)}{\sin^4(x/2)} \,.$$

Recall that $\mathcal{K}_n \geqslant 0$, $\deg \mathcal{K}_n \leqslant 2n - 2$, and $\|\mathcal{K}_n\|_{L^1[-\pi,\pi]} = 1$.

We have $\operatorname{Re} H_n(z) > 0$ for $z \in \mathbb{T}$ so H_n has no zeros in $\overline{\mathbb{D}}$. Moreover, $H_n \in \mathbb{P}_{n-2}$ and

$$H_n(0) = 2 \,. \tag{3.21}$$

The estimates on H_n and its limiting behavior when $n \to \infty$ are discussed in Appendix D.

(b). The function $\widetilde{F}$ is chosen by

$$\widetilde{F} := 2H_n^{-1} \,. \tag{3.22}$$

We notice that, $\operatorname{Re} \widetilde{F} > 0$ over $\overline{\mathbb{D}}$ and $\operatorname{Re} \widetilde{F}$ is smooth on $\mathbb{T}$. Due to (3.21), we also have $\widetilde{F}(0) = 1$. Thus the normalization condition in Lemma 4.3

$$\int_{-\pi}^{\pi} \operatorname{Re} \widetilde{F} d\theta = 2\pi$$

is satisfied.

(c). As in the case of "small deviations" we will define the polynomial ϕ_{2n} by specifying ϕ_{2n}^* first. Then, we will let $\phi_{2n} = (\phi_{2n}^*)^*$. Take ϕ_{2n}^* as

$$\phi_{2n}^*(z) := \beta_n (Q_n + Q_n^* + Q_n H_n) \,,$$

where Q_n is defined as

$$Q_n := (1 - z)^{-\alpha/2} * \mathcal{F}_n \,. \tag{3.23}$$

The positive parameter β_n is chosen to make sure that we have

$$\|(\phi_n^*)^{-1}\|^2_{L^2(\mathbb{T})} = \|\phi_n^{-1}\|^2_{L^2(\mathbb{T})} = 2\pi \,, \tag{3.24}$$

as required in (4.3). Notice that $Q_n \in \mathbb{P}_{n-1}$ and $\operatorname{Re} Q_n > 0$ in $\overline{\mathbb{D}}$.

Since $(*)$ acts on $\mathbb{P}_{2n}$, $\deg(Q_n + Q_n^* + Q_n H_n) \leqslant 2n$. Moreover,

$$Q_n(0) + Q_n^*(0) + Q_n(0) H_n(0) = 3Q_n(0) > 0 \,.$$

Therefore, $\deg \phi_{2n} = 2n$ as required in Lemma 4.3.

The absence of zeros of ϕ_{2n}^* in the unit disk can be proved as in (3.11).

Next, we proceed with estimating β_n. For the lower bound, we can follow (3.13) to write

$$|Q_n + Q_n^* + Q_nH_n|^2 \geqslant |Q_n|^2(\operatorname{Re} H_n)^2 \gtrsim \begin{cases} \tau^2 n^{-\alpha}, & |\theta| < n^{-1}\,, \\ \tau^2|\theta|^\alpha, & |\theta| > n^{-1}\,, \end{cases}$$

and the last inequality is due to (7.7), (7.8), and Lemma 7.2. This is true, provided that $\tau \in (0, \tau_0)$ and $n > n_{(\tau)}$. For the upper bound,

$$|Q_n + Q_n^* + Q_nH_n|^2 \lesssim |Q_n|^2 \lesssim \begin{cases} n^\alpha, & |\theta| < n^{-1}\,, \\ |\theta|^{-\alpha}, & |\theta| > n^{-1}\,, \end{cases}$$

again by Lemma 7.1 and Lemma 7.2. From (3.24),

$$\beta_n^2 \sim \int_{-\pi}^{\pi} |Q_n + Q_n^* + Q_nH_n|^{-2} d\theta\,,$$

so

$$\beta_n^2 \lesssim \tau^{-2}n^{-\tau} + \int_{1/n}^{1} \frac{1}{\tau^2\theta^\alpha} d\theta \lesssim \tau^{-3}$$

for $\tau \in (0, \tau_0)$ and $n > n_{(\tau)}$. For the lower bound, we can write

$$\beta_n^2 \gtrsim n^{-\tau} + \int_{1/n}^{1} |\theta|^\alpha d\theta \gtrsim 1\,.$$

To summarize, we get an estimate

$$1 \lesssim \beta_n^2 \lesssim \tau^{-3}\,, \tag{3.25}$$

which holds true if $\tau \in (0, \tau_0)$ and $n > n_{(\tau)}$.

Therefore, for the value at $z = 1$, we get

$$|\phi_{2n}(1)| = |\phi_{2n}^*(1)| \gtrsim C_{(\tau)} n^{\alpha/2}$$

by Lemma 7.2, provided that $n > n_{(\tau)}$. This gives the growth required in (3.2) in terms of α.

We apply Lemma 4.3. The identity (4.5) provides the formula for σ', a weight of orthogonality for our $\phi_{2n}(z)$.

We can rewrite (4.5) as follows:

$$\sigma'^{-1} = \upsilon_n \mathcal{E}\mathcal{D}\,, \tag{3.26}$$

where $\upsilon_n = \pi\beta_n^2/2$, and (due to (3.22)),

$$\mathcal{E} := \frac{|Q_n|^2}{\operatorname{Re}\widetilde{F}},\ \mathcal{D} := \left|4 + H_n + \xi\left(\overline{H}_n + 2\left(1 - \frac{\overline{H}_n}{H_n}\right)\right)\right|^2,\ \xi = Q_n^*/Q_n = e^{i(2n\theta - 2\Theta)},\ \Theta = \arg Q_n\,.$$

We will first obtain the lower and upper bounds for $\mathcal{E}$ and $\mathcal{D}$ over some interval I_τ, centered at $\theta = 0$. Then, we will handle all $\mathbb{T}$.

Notice that

$$\operatorname{Re}\widetilde{F} = 2\frac{\operatorname{Re} H_n}{|H_n|^2}\,.$$

Consider $\mathcal{E}$. From Lemma 7.1 and Lemma 7.2, we get

$$\mathcal{E} \sim \tau^{-1} \tag{3.27}$$

if $\tau \in (0,\tau_0)$ and $|\theta| \in (n^{-1}, \tau^2), n > n_{(\tau)}$.

Now, consider $\mathcal{D}$. For the upper bound, we obtain

$$\mathcal{D} \lesssim 1\,, \tag{3.28}$$

if $\tau \in (0,\tau_0)$, and $\theta \in [-\pi,\pi), n > n_{(\tau)}$, as follows from (7.9). For the lower estimate, we notice that $|\xi| = 1$ and write

$$\left|4 + H_n + \xi\left(\overline{H}_n + 2\left(1 - \frac{\overline{H}_n}{H_n}\right)\right)\right| \geqslant 4 - 2\left|1 - \frac{\overline{H}_n}{H_n}\right| - 2|H_n|$$

by triangle inequality.

If we define $u := \arg H_n$, then

$$4 - 2\left|1 - \frac{\overline{H}_n}{H_n}\right| = 2\sqrt{2}(\sqrt{2} - (1-\cos(2u))^{1/2}) \gtrsim \tau^2\,, \tag{3.29}$$

as follows from (7.10). From (7.9), we then get

$$\mathcal{D} \geqslant (C_1\tau^2 - C_2(|\theta|^\alpha + n^{-\alpha}))^2\,,$$

if $\tau \in (0,\tau_0), \theta \in (-\tau^2,\tau^2)$, and $n > n_{(\tau)}$. Now, we choose I_τ such that

- $I_\tau \subset [-\tau^2, \tau^2]$,
- $C_2 \max_{\theta\in I_\tau} |\theta|^\alpha < C_1\tau^2/2$.

That leads to an estimate

$$\mathcal{D} \gtrsim \tau^4\,, \tag{3.30}$$

which holds if $\tau \in (0,\tau_0), \theta \in I_\tau, n > n_{(\tau)}$.

We collect the estimates for $\mathcal{E}$ and $\mathcal{D}$ to get

$$\tau^3 \lesssim \mathcal{E}\mathcal{D} \lesssim \tau^{-1} \tag{3.31}$$

if $1/n < |\theta|, \theta \in I_\tau$.

Now, consider $\theta \in [-1/n, 1/n]$ and write

$$4 - 2\left|1 - \frac{\overline{H}_n}{H_n}\right| - 2|H_n| \leqslant \left|4 + H_n + \xi\left(\overline{H}_n + 2\left(1 - \frac{\overline{H}_n}{H_n}\right)\right)\right| \leqslant 4 + 2\left|1 - \frac{\overline{H}_n}{H_n}\right| + 2|H_n|\,.$$

Since $|H_n| \lesssim n^{-\alpha}$, the estimate (3.29) gives

$$4 - 2\left|1 - \frac{\overline{H}_n}{H_n}\right| \lesssim \left|4 + H_n + \xi\left(\overline{H}_n + 2\left(1 - \frac{\overline{H}_n}{H_n}\right)\right)\right| \lesssim 1$$

if $n > n_{(\tau)}$. Taking the square of both sides and multiplying by $\mathcal{E}$ gives

$$\frac{|Q_n|^2}{\operatorname{Re} H_n}||H_n| - |\operatorname{Im} H_n||^2 \lesssim \mathcal{E}\mathcal{D} \lesssim \frac{|Q_n|^2|H_n|^2}{\operatorname{Re} H_n}$$

and

$$\frac{n^{-2\alpha}\tau^3}{|H_n|^2} \sim n^{2\alpha}\tau^{-1}\frac{(\operatorname{Re} H_n)^4}{|H_n|^2} \lesssim \mathcal{E}\mathcal{D} \lesssim |H_n|^2 n^{2\alpha}\tau^{-1}$$

after Lemma 7.1 and Lemma 7.2 are applied. We apply Lemma 7.1 to bound $|H_n|$ on the interval $[-1/n, 1/n]$. Substituting $|H_n| \sim n^\tau|\theta| + \tau n^{-\alpha}$ and taking the minimum of the left hand side and maximum of the right hand side gives

$$\tau^3 \lesssim \mathcal{E}\mathcal{D} \lesssim \tau^{-1}$$

which matches (3.31). Thus, to summarize, we get

$$\tau^3 \lesssim \mathcal{E}\mathcal{D} \lesssim \tau^{-1} \tag{3.32}$$

if $\theta \in I_\tau$ and $n > n_{(\tau)}$.

Next, we will prove the lower and upper estimates for $\mathcal{E}\mathcal{D}$ which will be true for all of $\mathbb{T}$. We will need it to apply the localization principle. Notice that

$$H_n \to 2(1-e^{i\theta})^\alpha,\ Q_n \to (1-e^{i\theta})^{-\alpha/2},\ \widetilde{F} \to (1-e^{i\theta})^{-\alpha}$$

if $n \to \infty$ and convergence is uniform in $\theta \notin I_\tau$. Therefore,

$$\mathcal{E} \to \frac{|1-e^{i\theta}|^{-\alpha}}{\operatorname{Re}((1-e^{i\theta})^{-\alpha})}, \quad n \to \infty$$

and convergence is uniform over $\theta \notin I_\tau$. So, away from I_τ, the upper and lower bounds for the limiting function imply that $1 \lesssim \mathcal{E} \lesssim \tau^{-1}$. The upper estimate for $\mathcal{D}$ is easy: $|\mathcal{D}| \lesssim 1$ for all $\theta \in \mathbb{T}$. To obtain the lower bound, we recall again that $H_n \to 2(1-e^{i\theta})^\alpha$. In the formula for $\mathcal{D}$, we replace H_n by its limiting value and consider the following function

$$d_\delta := \inf_{|\xi|=1, |z|=1, |1-z|>\delta} \left| 2 + (1-z)^\alpha + \xi \left(\overline{(1-z)^\alpha} + \left(1 - \frac{\overline{(1-z)^\alpha}}{(1-z)^\alpha} \right) \right) \right| .$$

Let us show that $d_\delta > 0$ for positive δ. For shorthand, introduce $t := (1-z)^\alpha$ and write

$$\left| 2 + t + \xi \frac{|t|^2 + 2i \operatorname{Im} t}{t} \right| = \left| \frac{2t + t^2 + \xi(|t|^2 + 2i \operatorname{Im} t)}{t} \right| \geqslant \frac{|2t+t^2| - \left||t|^2 + 2i \operatorname{Im} t\right|}{|t|} =$$

$$|t+2| - \left| |t| + 2i \frac{\operatorname{Im} t}{|t|} \right| \geqslant \frac{|t|^2 + 4 \operatorname{Re} t + 4 - |t|^2 - 4\sin^2(\arg t)}{|t+2| + \left| |t| + 2i \frac{\operatorname{Im} t}{|t|} \right|} \geqslant \frac{4 \operatorname{Re} t}{|t+2| + \left| |t| + 2i \frac{\operatorname{Im} t}{|t|} \right|} .$$

Since $|z| = 1$ and $|1-z| \geqslant \delta$, we have $\operatorname{Re} t \gtrsim \delta^\alpha \tau^2$.

This implies

$$1 \gtrsim \mathcal{D} > C_{(\tau)} ,$$

if $\theta \notin I_\tau$ and $n > n_{(\tau)}$. Combining the bounds, we get

$$c^{(1)}_{(\tau)} < \mathcal{E}\mathcal{D} < c^{(2)}_{(\tau)}$$

for $\theta \in [-\pi, \pi)$ and $n > n_{(\tau)}$. Considering the estimates on β_n and formula (3.26), we obtain

$$C^{(1)}_{(\tau)} < \sigma'(\theta) < C^{(2)}_{(\tau)} \tag{3.33}$$

for $\theta \in [-\pi, \pi)$.

To summarize, we proved that there is $\tau_0 > 0$ such that for all $\tau \in (0, \tau_0)$, there is $n_{(\tau)} \in \mathbb{N}$ and interval I_τ centered at $\theta = 0$ so that for every $n > n_{(\tau)}$ there a weight σ' (defining the absolutely continuous measure σ) such that

- $|\phi_{2n}(1, \sigma)| \gtrsim C_{(\tau)} n^{\alpha/2}$.
- The bound holds
  $$C^{(1)}_{(\tau)} < \sigma'(\theta) < C^{(2)}_{(\tau)} \tag{3.34}$$
  for $\theta \in [-\pi, \pi)$.
- $\upsilon_n^{-1} \tau \lesssim \sigma'(\theta) \lesssim \upsilon_n^{-1} \tau^{-3}$ for $\theta \in I_\tau$.

The last property follows from representation $\sigma' = v_n^{-1}(\mathcal{E}\mathcal{D})^{-1}$ and (3.32).

It is only left to use the localization principle. Since we have (3.34), Lemma 5.1 is applicable. Consider w_1:

$$w_1 = \begin{cases} 1, & \theta \notin I_\tau, \\ \dfrac{\sigma'}{\min\limits_{\theta\in I_\tau} \sigma'}, & \theta \in I_\tau \,. \end{cases}$$

We have $1 \leqslant w_1 \leqslant T$ uniformly over $\mathbb{T}$ and $T := \max_{I_\tau} \sigma' / \min_{I_\tau} \sigma' \lesssim \tau^{-4}$. Lemma 5.1 and (1.11) give

$$|\phi_{2n}(1, w_1)| > C_{(\tau)}^{(3)} |\phi_{2n}(1, \sigma)| > C_{(\tau)}^{(4)} n^{\alpha/2}, \quad n > n_{(\tau)} \,.$$

The estimate (3.2) now follows from (1.9) and (1.11). The case of odd n can be handled similarly.

□

Remark. We can reformulate the Theorem 3.1 in a different way. Let us introduce two functions defined for $t \geqslant 0$:

$$\overline{\nu}(t) := \limsup_{n\to\infty} \frac{\log \sup_{1\leqslant w\leqslant 1+t} \|\Phi_n(z,w)\|_\infty}{\log n}$$

and

$$\underline{\nu}(t) := \liminf_{n\to\infty} \frac{\log \sup_{1\leqslant w\leqslant 1+t} \|\Phi_n(z,w)\|_\infty}{\log n} \,.$$

Then the following two estimates are immediate from combining Theorem 2.1 and Theorem 3.1:

$$\frac{1}{2} - \frac{c_1}{t^{1/4}} \leqslant \underline{\nu}(t) \leqslant \overline{\nu}(t) \leqslant \frac{1}{2} - \frac{c_2}{t}, \quad t > 2$$

and

$$c_3(t-1) \leqslant \underline{\nu}(t) \leqslant \overline{\nu}(t) \leqslant c_4(t-1), \quad 1 < t < 2 \,.$$

where c_1, c_2, c_3, c_4 are some positive absolute constants.

The Theorem 3.1 can now be used to show that for some w that satisfies $w, w^{-1} \in L^\infty(\mathbb{T})$ the sequence $\|\phi_n(e^{i\theta}, w)\|_{L^\infty(\mathbb{T})}$ can grow in n at a certain rate.

Theorem 3.2. *Given $\epsilon \in (0,1]$, there is a weight w and a subsequence $\{k_n\} \subset \mathbb{N}$ that satisfy $1 \leqslant w \leqslant 1+\epsilon$ and*

$$\|\phi_{k_n}(e^{i\theta}, w)\|_{L^\infty(\mathbb{T})} \gtrsim k_n^{c_5\epsilon} \,.$$

If $T \geqslant 2$, then there is a weight w and a subsequence $\{k_n\} \subset \mathbb{N}$ that satisfy $1 \leqslant w \leqslant T$ and

$$\|\phi_{k_n}(e^{i\theta}, w)\|_{L^\infty(\mathbb{T})} \gtrsim k_n^{1/2 - c_6 T^{-1/6}} \,.$$

Here c_5 and c_6 denote positive absolute constants.

Proof. We give the proof only for the first case when the deviation of the weight from the constant is small. The other case of large deviation can be handled similarly.

We consider the sequence of disjoint arcs $\{I_n\}$ in $\mathbb{T}$ that satisfies the following condition

$$\sum_{n=1}^\infty \delta_n < 2\pi, \quad |I_n| = \delta_n \,.$$

Denote the centers of these arcs by $\{e^{i\theta_n}\}$. Take a monotonically increasing sequence $\{k_n\}\subset\mathbb{N}$ so that

$$C^{(1)}_{(\epsilon)}\delta_n^{-1}k_n^{C_5\epsilon}>k_n^{C_5\epsilon/2},\quad n=1,2,\ldots \tag{3.35}$$

and $C^{(1)}_{(\epsilon)}, C_5$ are taken from (3.1). Then, by Theorem 3.1, we have a sequence of weights $\{w_n\}$ so that

$$1\leqslant w_n\leqslant 1+\epsilon$$

and

$$|\phi_{k_n}(1,w_n)|>C^{(1)}_{(\epsilon)}k_n^{C_5\epsilon}\,. \tag{3.36}$$

Now, we define weight w which is equal to 1 away from $\bigcup_{n=1}^\infty I_n$ and

$$w(\theta)=w_n(\theta-\theta_n)$$

on each I_n (we recall that $e^{i\theta_n}$ is the center of I_n). We clearly have $1\leqslant w\leqslant 1+\epsilon$. The localization principle (5.5) now implies that

$$|\phi_{k_n}(e^{i\theta_n},w)|\gtrsim\delta_n^{-1}|\phi_{k_n}(1,w_n)|\,.$$

The application of (3.35) and (3.36) now finishes the proof.

□

4. Appendix A: Method used to prove Theorem 1.1

In this section we explain an idea used in the proof of Theorem 1.1. This material is taken from [4]. We start with recalling some basic facts about the polynomial orthogonal on the unit circle. With any probability measure μ, which is defined on the unit circle and have infinitely many growth points, one can associate the orthonormal polynomials of the first and second kind, $\{\phi_n\}$ and $\{\psi_n\}$, respectively. $\{\phi_n\}$ satisfy the following recursions ([16], p. 57) with Schur parameters $\{\gamma_n\}$:

$$\left\{\begin{array}{ll}\phi_{n+1}=\rho_n^{-1}(z\phi_n-\overline{\gamma}_n\phi_n^*), & \phi_0=1\,,\\ \phi_{n+1}^*=\rho_n^{-1}(\phi_n^*-\gamma_n z\phi_n), & \phi_0^*=1\,,\end{array}\right. \tag{4.1}$$

and $\{\psi_n\}$ satisfy the same recursion but with Schur parameters $\{-\gamma_n\}$, i.e.,

$$\left\{\begin{array}{ll}\psi_{n+1}=\rho_n^{-1}(z\psi_n+\overline{\gamma}_n\psi_n^*), & \psi_0=1\,,\\ \psi_{n+1}^*=\rho_n^{-1}(\psi_n^*+\gamma_n z\psi_n), & \psi_0^*=1\,.\end{array}\right. \tag{4.2}$$

The coefficient ρ_n is defined as

$$\rho_n:=\sqrt{1-|\gamma_n|^2}\,.$$

The following Bernstein-Szegő approximation is valid:

Lemma 4.1. ([6],[16]) *Suppose $d\mu$ is a probability measure and $\{\phi_j\}$ and $\{\psi_j\}$ are the corresponding orthonormal polynomials of the first/second kind, respectively. Then, for any N, the Caratheodory function*

$$F_N(z)=\frac{\psi_N^*(z)}{\phi_N^*(z)}=\int_{\mathbb{T}}C(z,e^{i\theta})d\mu_N(\theta),\ \text{where}\quad d\mu_N(\theta)=\frac{d\theta}{2\pi|\phi_N(e^{i\theta})|^2}=\frac{d\theta}{2\pi|\phi_N^*(e^{i\theta})|^2}$$

has the first N Taylor coefficients identical to the Taylor coefficients of the function

$$F(z)=\int_{\mathbb{T}}C(z,e^{i\theta})d\mu(\theta)\ .$$

In particular, the polynomials $\{\phi_j\}$ *and* $\{\psi_j\}$, $j \leqslant N$ *are the orthonormal polynomials of the first/second kind for the measure* $d\mu_N$.

We also need the following Lemma which can be verified directly:

Lemma 4.2. *The polynomial* $P_n(z)$ *of degree* n *is the orthonormal polynomial for a probability measure with infinitely many growth points if and only if*

1. $P_n(z)$ *has all* n *zeroes inside* $\mathbb{D}$ *(counting the multiplicities).*
2. *The normalization conditions*

$$\int_{\mathbb{T}} \frac{d\theta}{2\pi |P_n(e^{i\theta})|^2} = 1 \ , \quad \operatorname{coeff}(P_n, n) > 0$$

are satisfied.

Proof. Take $2\pi |P_n(e^{i\theta})|^{-2} d\theta$ itself as a probability measure. The orthogonality is then immediate. □

We continue with a Lemma which paves the way for constructing the measure giving, in particular, the optimal bound (1.5). It is a special case of a solution to the truncated moment problem. In this Lemma, ϕ_n^* is defined as an application of $(*)$ operation to ϕ_n, considered as an element of $\mathbb{P}_n$.

Lemma 4.3. *Suppose we are given a polynomial* ϕ_n *and Caratheodory function* $\widetilde{F}$ *which satisfy the following properties*

1. $\deg \phi_n = n$ *and* ϕ_n^* *has no roots in* $\overline{\mathbb{D}}$.
2. *Normalization on the size and "rotation"*

$$\int_{\mathbb{T}} |\phi_n^*(e^{i\theta})|^{-2} d\theta = 2\pi \ , \quad \phi_n^*(0) > 0 \ . \tag{4.3}$$

3. $\widetilde{F} \in C^\infty(\mathbb{T})$, $\operatorname{Re} \widetilde{F} > 0$ *on* $\mathbb{T}$, *and*

$$\frac{1}{2\pi} \int_{\mathbb{T}} \operatorname{Re} \widetilde{F}(e^{i\theta}) d\theta = 1 \ . \tag{4.4}$$

Consider two probability measures μ_n *and* $\widetilde{\sigma}$ *given by*

$$d\mu_n := \frac{d\theta}{2\pi |\phi_n^*(e^{i\theta})|^2}, \quad d\widetilde{\sigma} = \widetilde{\sigma}' d\theta := \frac{\operatorname{Re} \widetilde{F}(e^{i\theta})}{2\pi} d\theta$$

and denote their Schur (recursion) coefficients by $\{\gamma_j\}$ *and* $\{\widetilde{\gamma}_j\}$, *respectively. Then, the probability measure* σ, *corresponding to Schur (recursion) coefficients*

$$\gamma_0, \ldots, \gamma_{n-1}, \widetilde{\gamma}_0, \widetilde{\gamma}_1, \ldots$$

is purely absolutely continuous with the density (weight) and

$$\sigma' = \frac{4\widetilde{\sigma}'}{|\phi_n + \phi_n^* + \widetilde{F}(\phi_n^* - \phi_n)|^2} = \frac{2 \operatorname{Re} \widetilde{F}}{\pi |\phi_n + \phi_n^* + \widetilde{F}(\phi_n^* - \phi_n)|^2} \ . \tag{4.5}$$

The polynomial ϕ_n *is the orthonormal polynomial for* σ.

The proof of this Lemma is contained in [2]. We, however, prefer to give its sketch here.

Proof. First, notice that $\{\widetilde{\gamma}_j\} \in \ell^1$ by Baxter's Theorem (see, e.g., [16], Vol.1, Chapter 5). Therefore, σ is purely absolutely continuous by the same Baxter's criterion. Define the orthonormal polynomials of the first/second kind corresponding to measure $\widetilde{\sigma}$ by $\{\widetilde{\phi}_j\}, \{\widetilde{\psi}_j\}$. Similarly, let $\{\phi_j\}, \{\psi_j\}$ be orthonormal polynomials for σ. Since, by construction, μ_n and σ have identical first n Schur parameters, ϕ_n is n-th orthonormal polynomial for σ.

Let us compute the polynomials ϕ_j and ψ_j, orthonormal with respect to σ, for the indexes $j > n$. By (4.2), the recursion can be rewritten in the following matrix form

$$\begin{pmatrix} \phi_{n+m} & \psi_{n+m} \\ \phi^*_{n+m} & -\psi^*_{n+m} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_m & \mathcal{B}_m \\ \mathcal{C}_m & \mathcal{D}_m \end{pmatrix} \begin{pmatrix} \phi_n & \psi_n \\ \phi^*_n & -\psi^*_n \end{pmatrix} \tag{4.6}$$

where $\mathcal{A}_m, \mathcal{B}_m, \mathcal{C}_m, \mathcal{D}_m$ satisfy

$$\begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathcal{C}_0 & \mathcal{D}_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{A}_m & \mathcal{B}_m \\ \mathcal{C}_m & \mathcal{D}_m \end{pmatrix} = \frac{1}{\widetilde{\rho}_0 \cdot \ldots \cdot \widetilde{\rho}_{m-1}} \begin{pmatrix} z & -\overline{\widetilde{\gamma}}_{m-1} \\ -z\widetilde{\gamma}_{m-1} & 1 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} z & -\overline{\widetilde{\gamma}}_0 \\ -z\widetilde{\gamma}_0 & 1 \end{pmatrix}$$

and thus depend only on $\widetilde{\gamma}_0, \ldots, \widetilde{\gamma}_{m-1}$. Moreover, we have

$$\begin{pmatrix} \widetilde{\phi}_m & \widetilde{\psi}_m \\ \widetilde{\phi}^*_m & -\widetilde{\psi}^*_m \end{pmatrix} = \begin{pmatrix} \mathcal{A}_m & \mathcal{B}_m \\ \mathcal{C}_m & \mathcal{D}_m \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

Thus, $\mathcal{A}_m = (\widetilde{\phi}_m + \widetilde{\psi}_m)/2$, $\mathcal{B}_m = (\widetilde{\phi}_m - \widetilde{\psi}_m)/2$, $\mathcal{C}_m = (\widetilde{\phi}^*_m - \widetilde{\psi}^*_m)/2$, $\mathcal{D}_m = (\widetilde{\phi}^*_m + \widetilde{\psi}^*_m)/2$ and their substitution into (4.6) yields

$$2\phi^*_{n+m} = \phi_n(\widetilde{\phi}^*_m - \widetilde{\psi}^*_m) + \phi^*_n(\widetilde{\phi}^*_m + \widetilde{\psi}^*_m) = \widetilde{\phi}^*_m \left(\phi_n + \phi^*_n + \widetilde{F}_m(\phi^*_n - \phi_n)\right) , \tag{4.7}$$

where

$$\widetilde{F}_m(z) = \frac{\widetilde{\psi}^*_m(z)}{\widetilde{\phi}^*_m(z)} .$$

Since $\{\widetilde{\gamma}_n\} \in \ell^1$ and $\{\gamma_n\} \in \ell^1$, we have ([16], p. 225)

$$\widetilde{F}_m \to \widetilde{F} \text{ as } m \to \infty \text{ and } \phi^*_j \to \Pi,\ \widetilde{\phi}^*_j \to \widetilde{\Pi} \text{ as } j \to \infty .$$

uniformly on $\overline{\mathbb{D}}$. The functions Π and $\widetilde{\Pi}$ are the Szegő functions of σ and $\widetilde{\sigma}$, respectively, i.e., they are the outer functions in $\mathbb{D}$ that satisfy

$$|\Pi|^{-2} = 2\pi\sigma', \quad |\widetilde{\Pi}|^{-2} = 2\pi\widetilde{\sigma}' \tag{4.8}$$

on $\mathbb{T}$. In (4.7), send $m \to \infty$ to get

$$2\Pi = \widetilde{\Pi}\left(\phi_n + \phi^*_n + \widetilde{F}(\phi^*_n - \phi_n)\right) \tag{4.9}$$

and we have (4.5) after taking the square of absolute values and using (4.8). □

5. Appendix B: the Localization Principle

Given a weight w on $[-\pi,\pi]$, we define

$$\lambda(w) := \exp\left(\frac{1}{4\pi}\int_{\mathbb{T}} \log(2\pi w(\theta))d\theta\right), \quad \Lambda(w) := \sqrt{\|w\|_{L^1(\mathbb{T})}}\,.$$

The following Theorem was proved in [2] (Theorem 5.1).

Theorem 5.1. ([2]) *Let w_1 and w_2 be two weights on $[-\pi,\pi]$ so that*

$$w_1(\theta)=w_2(\theta), \quad \theta\in[-\epsilon,\epsilon]\,. \tag{5.1}$$

Then

$$\left|\frac{\phi_n(1,w_1)}{\phi_n(1,w_2)}\right| \leqslant \frac{\Lambda(w_2)}{\lambda(w_1)} + \frac{4\Lambda(w_1)}{\epsilon\lambda(w_1)}\left(\int_{|\theta|>\epsilon} |\phi_n(e^{i\theta},w_1)\phi_n(e^{i\theta},w_2)|(w_1+w_2)d\theta\right) \tag{5.2}$$

for all n.

We get a simple corollary.

Lemma 5.1. *Assume that*

$$0<m_1\leqslant w_1(\theta)\leqslant m_2,\ 0<m_1\leqslant w_2(\theta)\leqslant m_2 \text{ for } \theta\in[-\pi,\pi) \tag{5.3}$$

and

$$w_1(\theta)=w_2(\theta) \text{ for } \theta\in[-\epsilon,\epsilon]\,. \tag{5.4}$$

Then,

$$\frac{\epsilon m_1}{m_2} \lesssim \left|\frac{\phi_n(1,w_1)}{\phi_n(1,w_2)}\right| \lesssim \frac{m_2}{\epsilon m_1}\,. \tag{5.5}$$

Proof. The lower and upper bounds for the weights imply that

$$\sqrt{m_1}\lesssim\Lambda\lesssim\sqrt{m_2}, \quad \sqrt{m_1}\lesssim\lambda\lesssim\sqrt{m_2}$$

for both weights.

Notice that, e.g.,

$$\int_{-\pi}^{\pi} |\phi_n(e^{i\theta},w_1)\phi_n(e^{i\theta},w_2)|w_1 d\theta \leqslant \left(\int_{-\pi}^{\pi} |\phi_n(e^{i\theta},w_2)|^2 w_1 d\theta\right)^{1/2}$$

by Cauchy-Schwarz and definition of $\phi_n(e^{i\theta},w_1)$. Since

$$1=\int_{-\pi}^{\pi}|\phi_n(e^{i\theta},w_2)|^2 w_2 d\theta \geqslant m_1\int_{-\pi}^{\pi}|\phi_n(e^{i\theta},w_2)|^2 d\theta$$

we get

$$\int_{-\pi}^{\pi} |\phi_n(e^{i\theta},w_1)\phi_n(e^{i\theta},w_2)|w_1 d\theta \leqslant \left(\frac{m_2}{m_1}\right)^{1/2}\,.$$

To prove the lower bound in (5.5), it is sufficient to change the roles of w_1 and w_2. □

6. APPENDIX C: THE PROPERTIES OF AUXILIARY POLYNOMIALS, I

In this Appendix, we study the properties of the polynomials introduced in subsection 3.1. Recall that

$$h_n = 2(1-e^{i\theta})^\epsilon * \mathcal{F}_n, \quad \widetilde{F} = 2h_n^{-1}\,. \tag{6.1}$$

We suppress the dependence of $\widetilde{F}$ on n to make notation consistent with Lemma 4.3.

The standard properties of convolution (Theorem 8.14, [5]) yield the following Lemma.

Lemma 6.1. *For fixed $\delta \in (0,\pi)$ and $\epsilon \in (0,1)$, we have*

$$\lim_{n\to\infty} h_n(\theta) = 2(1-e^{i\theta})^\epsilon, \quad \lim_{n\to\infty} \widetilde{F}(e^{i\theta}) = (1-e^{i\theta})^{-\epsilon}$$

uniformly over $\{\theta : |\theta| \in [\delta,\pi]\}$.

Remark. Notice that $|\arg(1-e^{i\theta})^\epsilon| \leqslant \epsilon\pi/2$ and

$$|\operatorname{Im}(1-e^{i\theta})^\epsilon| \lesssim \epsilon \operatorname{Re}(1-e^{i\theta})^\epsilon\,. \tag{6.2}$$

From that point until the end of this Appendix, we assume that $\epsilon \in (0,\epsilon_0)$ where $\epsilon_0 \in (0,1)$ is a fixed constant to be chosen below. Let us start with the following simple observations about $(1-e^{i\theta})^\epsilon$, the function we are approximating. Taylor expansion for $e^{i\theta}$ around the origin is $e^{i\theta} = 1 + i\theta - \theta^2/2 + O(|\theta|^3), |\theta| < \pi$. Substituting this into $(1-e^{i\theta})^\epsilon$ gives

$$(1-e^{i\theta})^\epsilon = (-i\theta)^\epsilon(1 + i\theta/2 + O(\theta^2))^\epsilon\,.$$

Using Taylor expansion of the logarithm around point 1, we can find an absolute constant $\delta_0 > 0$ such that for every $\theta \in (-\delta_0,\delta_0)$, we get

$$(-i\theta)^\epsilon(1+i\theta/2+O(\theta^2))^\epsilon = |\theta|^\epsilon e^{-i\epsilon\cdot \operatorname{sgn}(\theta)\cdot\pi/2} e^{\epsilon\log(1+i\theta/2+O(\theta^2))} =$$

$$|\theta|^\epsilon \exp(-i\epsilon\cdot\operatorname{sgn}(\theta)\cdot\pi/2 + i\epsilon\theta/2)(1+O(\epsilon\theta^2))\,.$$

For $\theta : |\theta| > \delta_0$, we can write

$$(1-e^{i\theta})^\epsilon = e^{\epsilon\log(1-e^{i\theta})} = 1 + O(\epsilon)\,.$$

These two expansions imply, in particular, that for all $\theta \in [-\pi,\pi)$, we can write the formula for the real part in the following compact form

$$\operatorname{Re}(1-e^{i\theta})^\epsilon = |\theta|^\epsilon\cos(\epsilon\pi/2)(1+O(\epsilon|\theta|))\,. \tag{6.3}$$

The bound (6.2) then yields

$$|\operatorname{Im}(1-e^{i\theta})^\epsilon| \lesssim \epsilon|\theta|^\epsilon\,. \tag{6.4}$$

We will need the following representation for the Fejer kernel

$$\mathcal{F}_n := \frac{1}{2\pi n}\left(\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right)^2 = \frac{2}{\pi n}\left(\frac{\sin^2(n\theta/2)}{\theta^2} + O(1)\right) \tag{6.5}$$

for $|\theta| < \pi$ and $n \in \mathbb{N}$. This expansion follows immediately from the Taylor expansion of $\sin(\theta/2)$ around the origin. We recall that

$$\int_{-\pi}^{\pi} \mathcal{F}_n(\theta)d\theta = 1\,. \tag{6.6}$$

In the proof of the next Lemma, the remainder terms in the formulas (6.3) and (6.5), e.g., $\epsilon O(|\theta|^{1+\epsilon})$ and $n^{-1}O(1)$ contribute the smaller order terms which can be neglected. This will be explained in detail.

Lemma 6.2. *For every $\epsilon \in (0, \epsilon_0)$, there is $n_{(\epsilon)} \in \mathbb{N}$ such that*

$$|\arg h_n(\theta)| \lesssim \epsilon\,, \tag{6.7}$$

$$\operatorname{Re} h_n(\theta) \sim n^{-\epsilon} + |\theta|^\epsilon\,, \tag{6.8}$$

$$|h_n(\theta)| \sim n^{-\epsilon} + |\theta|^\epsilon \tag{6.9}$$

for $\theta \in [-\pi, \pi]$ and all $n > n_{(\epsilon)}$.

Proof. We start by noticing that, since the Fejer kernel and $\operatorname{Re}(1-e^{i\theta})^\epsilon$ are nonnegative,

$$|\operatorname{Im} h_n| \lesssim \epsilon \operatorname{Re} h_n \tag{6.10}$$

as follows from (6.2). Thus, we have (6.7) since $|\tan\arg h_n| \lesssim \epsilon$.

To estimate $\operatorname{Re} h_n$, we can substitute (6.3) and (6.5) into (6.1). For $\theta : |\theta| < \pi$, we have

$$\operatorname{Re} h_n(\theta) = \frac{2}{\pi}\cos(\epsilon\pi/2) h_n^{(1)}(\theta)\,, \tag{6.11}$$

where

$$h_n^{(1)}(\theta) := n^{-1}\int_{-\pi}^{\pi} |x|^\epsilon \frac{\sin^2(n(x-\theta)/2)}{(x-\theta)^2}dx + \epsilon_{(n,\theta)}\,. \tag{6.12}$$

Let us focus on the main term now. We change variables $\widehat{x} := nx, \widehat{\theta} := n\theta$ to write

$$n^{-1}\int_{-\pi}^{\pi} |x|^\epsilon \frac{\sin^2(n(x-\theta)/2)}{(x-\theta)^2}dx = n^{-\epsilon} M_n(\widehat{\theta}), \quad , \quad M_n(\widehat{\theta}) := \int_{-\pi n}^{\pi n} |t|^\epsilon g(\widehat{\theta} - t)dt \tag{6.13}$$

with

$$g(t) := \frac{\sin^2(t/2)}{t^2}\,.$$

The estimation of the last integral shows that

$$M_n(\widehat{\theta}) \sim 1 + |\widehat{\theta}|^\epsilon$$

for $\widehat{\theta} \in [-\pi n, \pi n]$. Indeed, given any $a > 1$, we can write

$$\int_{-a}^{a} |t|^\epsilon g(\xi - t)dt = I_1 + I_2, \tag{6.14}$$

$$I_1 := \int_{t\in[-a,a], |\xi-t|<|\xi|/2} |t|^\epsilon g(\xi-t)dt,\ I_2 := \int_{t\in[-a,a], |\xi-t|>|\xi|/2} |t|^\epsilon g(\xi-t)dt\,.$$

Both I_1 and I_2 are nonnegative and, considering $\xi : |\xi| \in [1, a]$, we have

$$I_1 \sim |\xi|^\epsilon,\ I_2 \lesssim \int_{|t-\xi|>|\xi|/2} \frac{|t-\xi|^\epsilon}{|t-\xi|^2}dt \lesssim 1\,.$$

To get the last estimate, we used trivial observation that $|t| \lesssim |t-\xi|$ provided that $|t-\xi| > |\xi|/2$. For $\xi : |\xi| < 1$, we get

$$I_1 + I_2 \sim 1\,.$$

We are left with controlling $\epsilon_{(n,\theta)}$. We can write

$$|\epsilon_{(n,\theta)}| \lesssim n^{-1} + \epsilon n^{-1} \int_{-\pi}^{\pi} \frac{\sin^2(n(x-\theta)/2)}{(x-\theta)^2} |x|^{1+\epsilon} dx \lesssim n^{-1} + \epsilon n^{-1} \int_{-\pi}^{\pi} \frac{\sin^2(n(x-\theta)/2)}{\sin^2((x-\theta)/2)} |x|^{1+\epsilon} dx\,. \tag{6.15}$$

Now we use (6.6) to obtain

$$\epsilon_{(n,\theta)} \lesssim n^{-1} + \epsilon|\theta|^{1+\epsilon} + \epsilon n^{-1} \int_{-\pi}^{\pi} \frac{\sin^2(n(x-\theta)/2)}{\sin^2((x-\theta)/2)} \left||\theta|^{1+\epsilon} - |x|^{1+\epsilon}\right| dx \lesssim n^{-1} + \epsilon|\theta|^{1+\epsilon} + \epsilon n^{-1}\log n\,.$$

To estimate the last integral, we used the bound $||x|^{1+\epsilon} - |\xi|^{1+\epsilon}| \lesssim |x-\xi|$ which is true for $|x|, |\xi| < \pi$. Notice that

$$|\epsilon_{(n,\theta)}/(n^{-\epsilon}M_n)| \lesssim \epsilon$$

if $\theta \in [-\pi,\pi], n > n_{(\epsilon)}$. Thus, we showed that

$$\operatorname{Re} h_n = \frac{2}{\pi}\cos(\epsilon\pi/2) \cdot n^{-\epsilon} M_n(n\theta) \cdot (1 + O(\epsilon))\,, \tag{6.16}$$

if $\theta \in [-\pi,\pi)$ and $n > n_{(\epsilon)}$. This implies (6.8). Application of the estimate (6.10) gives (6.9). □

Remark. The proof given above provides the following representation

$$h_n(\theta) = \frac{2}{\pi}\cos(\epsilon\pi/2) \cdot n^{-\epsilon} M_n(n\theta) \cdot (1 + O(\epsilon)) \tag{6.17}$$

for $\theta \in [-\pi,\pi]$ and $n > n_{(\epsilon)}$.

The derivative of function $M_n(t)$, introduced in (6.13), is controlled in the following Lemma.

Lemma 6.3. *We have*

$$|M_n'(t)| \lesssim n^{\epsilon-2} + \epsilon(1+|t|)^{\epsilon-1} \tag{6.18}$$

for $|t| < \pi n/2$.

Proof. Differentiating the formula for M_n, we get

$$|M_n'(t)| = \left| \int_{-\pi n}^{\pi n} |s|^{\epsilon} g'(s-t) ds \right|$$

$$\lesssim n^{\epsilon} \left(\frac{1}{(\pi n - t)^2 + 1} + \frac{1}{(\pi n + t)^2 + 1} \right) + \epsilon \left| \int_{-\pi n}^{\pi n} |s|^{\epsilon-1} \operatorname{sgn}(s) \cdot g(s-t) ds \right|\,.$$

Consider the last integral. We can rewrite it as

$$\int_{-\pi n}^{\pi n} |s|^{\epsilon-1}\operatorname{sgn}(s) \cdot g(s-t) ds = \int_0^1 s^{\epsilon-1}\Big(g(s-t) - g(-s-t)\Big) ds + \int_{1<|s|<\pi n} |s|^{\epsilon-1}\operatorname{sgn}(s)\cdot g(s-t) ds\,.$$

Since $|g'(t)| \lesssim (1+t^2)^{-1}$, the Mean Value Theorem applied to the integrand in the first integral gives

$$\left| \int_0^1 s^{\epsilon-1}\Big(g(s-t) - g(-s-t)\Big) ds \right| \lesssim (1+t^2)^{-1} \int_0^1 s^{\epsilon-1} s ds \lesssim (1+t^2)^{-1}\,.$$

For the second integral, we have

$$\left|\int_{1<|s|<\pi n}|s|^{\epsilon-1}\cdot \operatorname{sgn}(s)\cdot g(s-t)ds\right|\lesssim \int_{\mathbb{R}}(1+|s|)^{\epsilon-1}\frac{1}{(t-s)^2+1}ds\lesssim (1+|t|)^{\epsilon-1}\,.$$

At the last step, we argued similarly to (6.14).

Thus, we have

$$|M_n'(t)|\lesssim n^{\epsilon}\left(\frac{1}{(\pi n-t)^2+1}+\frac{1}{(\pi n+t)^2+1}\right)+\epsilon(1+|t|)^{\epsilon-1}$$

and this finishes the proof. □

Lemma 6.4. *For every $\epsilon\in(0,\epsilon_0)$, the bound*

$$\left|\frac{\operatorname{Re}\widetilde{F}*\mathcal{F}_n}{\operatorname{Re}\widetilde{F}}-1\right|\lesssim\epsilon$$

holds if $\theta\in[-\pi,\pi]$ and $n>n_{(\epsilon)}$.

Proof. Notice first that $\operatorname{Re}\widetilde{F}\to\operatorname{Re}(1-e^{i\theta})^{-\epsilon}$ and $\operatorname{Re}\widetilde{F}*\mathcal{F}_n\to\operatorname{Re}(1-e^{i\theta})^{-\epsilon}$ uniformly over $\theta:\pi/2\leqslant|\theta|\leqslant\pi$. Thus, we only need to consider $\theta:|\theta|\leqslant\pi/2$.

Using $|(\operatorname{Re}\widetilde{F})*\mathcal{F}_n-\operatorname{Re}\widetilde{F}|=|\operatorname{Re}(\widetilde{F}*\mathcal{F}_n-\widetilde{F})|\leqslant|\widetilde{F}*\mathcal{F}_n-\widetilde{F}|$, we can rewrite

$$\left|\frac{\operatorname{Re}\widetilde{F}*\mathcal{F}_n}{\operatorname{Re}\widetilde{F}}-1\right|=\left|\frac{\operatorname{Re}\widetilde{F}*\mathcal{F}_n-\operatorname{Re}\widetilde{F}}{\operatorname{Re}\widetilde{F}}\right|\lesssim\frac{|h_n(\theta)|^2}{\operatorname{Re}h_n(\theta)}\cdot\left|\int_{-\pi}^{\pi}\frac{\sin^2(nx/2)}{n\sin^2(x/2)}\Big(h_n^{-1}(\theta-x)-h_n^{-1}(\theta)\Big)dx\right|\,.$$

In the previous Lemma, we showed that $|h_n|\sim\operatorname{Re}h_n$, thus

$$\left|\frac{\operatorname{Re}\widetilde{F}*\mathcal{F}_n}{\operatorname{Re}\widetilde{F}}-1\right|\lesssim\left|\int_{-\pi}^{\pi}\frac{\sin^2(nx/2)}{n\sin^2(x/2)}\frac{h_n(\theta-x)-h_n(\theta)}{h_n(\theta-x)}dx\right|\,.$$

Now, we can substitute (6.5) and (6.17) into this formula.

$$\left|\frac{\operatorname{Re}\widetilde{F}*\mathcal{F}_n}{\operatorname{Re}\widetilde{F}}-1\right|\lesssim\int_{-\pi}^{\pi}\frac{\sin^2(nx/2)}{nx^2}\left|\frac{M_n(n(\theta-x))-M_n(n\theta)}{M_n(n(\theta-x))}\right|dx$$

$$+\epsilon\int_{-\pi}^{\pi}\frac{\sin^2(nx/2)}{nx^2}\frac{|M_n(n(\theta-x))|+|M_n(n\theta)|}{M_n(n(\theta-x))}dx+n^{-1}\int_{-\pi}^{\pi}\left|\frac{h_n(\theta-x)-h_n(\theta)}{h_n(\theta-x)}\right|dx\,.$$

We apply Lemma 6.2 to bound the last term by

$$Cn^{-1}\int_{-\pi}^{\pi}\frac{1}{|\theta-x|^{\epsilon}}dx\lesssim n^{-1}\,,$$

which is smaller than ϵ if $n>n_{(\epsilon)}$.

Denote $\widehat{x}:=nx,\widehat{\theta}:=n\theta$. We need to bound

$$\int_{-\pi n}^{\pi n}\frac{\sin^2(\widehat{x}/2)}{\widehat{x}^2}\left|\frac{M_n(\widehat{\theta}-\widehat{x})-M_n(\widehat{\theta})}{M_n(\widehat{\theta}-\widehat{x})}\right|d\widehat{x}+\epsilon\int_{-\pi n}^{\pi n}\frac{\sin^2(\widehat{x}/2)}{\widehat{x}^2}\frac{|M_n(\widehat{\theta}-\widehat{x})|+|M_n(\widehat{\theta})|}{|M_n(\widehat{\theta}-\widehat{x})|}d\widehat{x}\,.$$

We use the estimate for $M_n(x)\sim 1+|x|^{\epsilon}$ to estimate the second term by

$$\epsilon\int_{-\pi n}^{\pi n}\frac{\sin^2(\widehat{x}/2)}{\widehat{x}^2}\cdot\frac{1+|\widehat{\theta}-\widehat{x}|^{\epsilon}+|\widehat{\theta}|^{\epsilon}}{1+|\widehat{\theta}-\widehat{x}|^{\epsilon}}d\widehat{x}\lesssim\epsilon\,,\tag{6.19}$$

because $|\widehat{\theta}| \leqslant |\widehat{\theta}-\widehat{x}| + |\widehat{x}|$ and so $|\widehat{\theta}|^{\epsilon} \leqslant |\widehat{x}|^{\epsilon} + |\widehat{x}-\widehat{\theta}|^{\epsilon}$.

Consider the first term now. It is equal to

$$\int_{|\widehat{x}|<\log n} \frac{\sin^2(\widehat{x}/2)}{\widehat{x}^2}\left|\frac{M_n(\widehat{\theta}-\widehat{x})-M_n(\widehat{\theta})}{M_n(\widehat{\theta}-\widehat{x})}\right| d\widehat{x} + \int_{\pi n>|\widehat{x}|>\log n} \frac{\sin^2(\widehat{x}/2)}{\widehat{x}^2}\left|\frac{M_n(\widehat{\theta}-\widehat{x})-M_n(\widehat{\theta})}{M_n(\widehat{\theta}-\widehat{x})}\right| d\widehat{x}\,. \tag{6.20}$$

If we argue as we did in (6.19), the last integral is bounded by

$$C\int_{|\widehat{x}|>\log n} \frac{1}{|\widehat{x}|^{2-\epsilon}} d\widehat{x} = O(\log^{-1+\epsilon} n) < \epsilon$$

for n sufficiently large.

For the first integral in (6.20), we recall that $\widehat{\theta} : |\widehat{\theta}| < \pi n/2$ and use an estimate (6.18) for the derivative of M_n. Keeping in mind $|\widehat{x}| < \log n$, we obtain

$$|M_n(\widehat{\theta}-\widehat{x})-M_n(\widehat{\theta})| = \left|\int_{\widehat{\theta}}^{\widehat{\theta}-\widehat{x}} M_n'(\xi)d\xi\right| \lesssim n^{\epsilon-2}|\widehat{x}| + \left|\int_{\widehat{\theta}}^{\widehat{\theta}-\widehat{x}} \epsilon(1+|\xi|)^{\epsilon-1}d\xi\right| \lesssim n^{\epsilon-2}\log n + \epsilon(1+|\widehat{x}|^{1/2})$$

by Cauchy-Schwarz, provided that $\epsilon_0 < \frac{4}{10}$. Thus, taking into account the lower bound $M_n \gtrsim 1$, we have an estimate

$$\int_{|\widehat{x}|<\log n} \frac{\sin^2(\widehat{x}/2)}{\widehat{x}^2}\left|\frac{M_n(\widehat{\theta}-\widehat{x})-M_n(\widehat{\theta})}{M_n(\widehat{\theta}-\widehat{x})}\right| d\widehat{x} \lesssim n^{\epsilon-2}\log n + \epsilon\int_{|\widehat{x}|<\log n} \frac{\sin^2(\widehat{x}/2)}{\widehat{x}^2}(1+|\widehat{x}|^{1/2})d\widehat{x} \lesssim \epsilon\,,$$

provided that $\epsilon_0 \in (0, \frac{4}{10})$ and $n > n_{(\epsilon)}$.

□

Finally, we mention that the choice of ϵ_0 is made to have $1 + O(\epsilon) \in (\frac{1}{2}, 2)$ in, e.g., (6.16). We also needed $\epsilon_0 < \frac{4}{10}$.

7. APPENDIX D: THE PROPERTIES OF AUXILIARY POLYNOMIALS, II

In this Appendix, we will study the polynomials introduced in subsection 3.2. Recall that

$$H_n = 2(1-e^{i\theta})^{\alpha} * \mathcal{K}_{[n/2]}\,, \tag{7.1}$$

where $\mathcal{K}_l$ is Jackson kernel and $\alpha \in (\frac{1}{2}, 1)$. Parameter τ was chosen as $\tau = 1-\alpha$.

We again start with estimates for $(1-e^{i\theta})^{\alpha}$. We have

$$\operatorname{Re}(1-e^{i\theta})^{\alpha} \gtrsim \tau|(1-e^{i\theta})|^{\alpha}, \quad \operatorname{Re}(1-e^{i\theta})^{\alpha} \gtrsim \tau|\operatorname{Im}(1-e^{i\theta})^{\alpha}|, \quad |(1-e^{i\theta})^{\alpha}| \sim |\theta|^{\alpha}\,, \tag{7.2}$$

if $\theta \in [-\pi, \pi]$ and $\alpha \in (\frac{1}{2}, 1)$. The function $\operatorname{Im}(1-e^{i\theta})^{\alpha}$ is odd in θ and the function $\operatorname{Re}(1-e^{i\theta})^{\alpha}$ is even. We will also need asymptotical expansion around the origin. We write

$$(1-e^{i\theta})^{\alpha} = (2\sin(\theta/2))^{\alpha}(\sin(\theta/2) - i\cos(\theta/2))^{\alpha} = (2\sin(\theta/2))^{\alpha}e^{-i\alpha\nu}\,,$$

where $\nu := \arctan\cot(\theta/2)$. Therefore, we get

$$\operatorname{Re}(1-e^{i\theta})^{\alpha} = |2\sin(\theta/2)|^{\alpha}\cos(\alpha\arctan\cot(\theta/2)), \tag{7.3}$$

$$\operatorname{Im}(1-e^{i\theta})^{\alpha} = -|2\sin(\theta/2)|^{\alpha}\sin(\alpha\arctan\cot(\theta/2)), \tag{7.4}$$

When $\theta \to 0$,

$$\cot(\theta/2) = 2\theta^{-1}(1+O(\theta^2))\,.$$

When $t \to \infty$,

$$\arctan t = \operatorname{sgn}(t) \cdot \Big(\pi/2 + O(|t|^{-1})\Big)\,.$$

Therefore, expanding $\sin(\theta/2)$ at the origin, we have

$$\operatorname{Re}(1-e^{i\theta})^\alpha = C|\theta|^\alpha(\cos(\alpha\pi/2)+O(\theta)),\ \operatorname{Im}(1-e^{i\theta})^\alpha = -C\operatorname{sgn}(\theta)\cdot|\theta|^\alpha(\sin(\alpha\pi/2)+O(\theta))\,. \tag{7.5}$$

The standard property of the convolution yields

$$H_n(\theta) \to 2(1-e^{i\theta})^\alpha, \quad n\to\infty \tag{7.6}$$

uniformly over $\theta \in [-\pi,\pi]$. Therefore, for every $\delta > 0$, there is $n_{(\delta)} \in \mathbb{N}$, such that

$$1 \gtrsim \operatorname{Re} H_n \gtrsim \tau|\theta|^\alpha, \quad |H_n| \sim |\theta|^\alpha \tag{7.7}$$

if $n > n_{(\delta)}$ and $|\theta| \in [\delta,\pi]$. This follows from (7.2) and uniform convergence (7.6).

Lemma 7.1. *The function* $\operatorname{Im} H_n(\theta)$ *is odd in* θ. *There exists* $\tau_0 > 0$ *such that for every* $\tau \in (0,\tau_0)$ *there is* $n_{(\tau)} \in \mathbb{N}$ *so that* H_n *satisfies the following properties for all* $n > n_{(\tau)}$.

- *For the real part,*

$$\operatorname{Re} H_n \sim \tau(n^{-\alpha}+|\theta|^\alpha), \quad \text{if } |\theta| < \tau^2\,. \tag{7.8}$$

- *For the absolute value,*

$$|H_n| \sim \begin{cases} n^\tau|\theta| + \tau n^{-\alpha}, & |\theta| < 1/n, \\ |\theta|^\alpha, & |\theta| > 1/n\,. \end{cases} \tag{7.9}$$

- *For the argument of* H_n,

$$-\pi/2 + C\tau < \arg H_n < \pi/2 - C\tau, \text{ if } \theta \in [-\pi,\pi)\,. \tag{7.10}$$

Proof. The fact that $\operatorname{Im} H_n$ is odd is immediate because $\operatorname{Im}(1-e^{i\theta})^\alpha$ is odd.

To see (7.10), we notice that the Jackson kernel and $\operatorname{Re}(1-e^{i\theta})^\alpha$ are both nonnegative, so the second estimate in (7.2) yields

$$|\operatorname{Im} H_n| \lesssim \tau^{-1}\operatorname{Re} H_n\,.$$

We then have

$$|\tan\arg H_n| \lesssim \tau^{-1}\,,$$

which gives (7.10).

Fix $\delta > 0$ and consider $|\theta| < \delta$. Denote $l := [n/2]$ and notice that $l \sim n$. The estimates on the Jackson kernel imply

$$\Big((1-z)^\alpha * \mathcal{K}_l\Big)(\theta) = \int_{|x|<\delta} (1-e^{i(\theta-x)})^\alpha \mathcal{K}_l(e^{ix})dx + O(\delta^{-3}n^{-3})\,.$$

If $|x| < \delta$ and $|\theta| < \delta$, then (check (7.3),(7.4),(7.5)) we have

$$\cos\left(\alpha\arctan\cot((\theta-x)/2)\right) = \cos(\alpha\pi/2) + O(\delta),$$

$$\sin\left(\alpha\arctan\cot((\theta-x)/2)\right) = \operatorname{sgn}(\theta-x)\cdot\sin(\alpha\pi/2) + O(\delta)\,.$$

Therefore,

$$\operatorname{Re}\Big((1-z)^\alpha * \mathcal{K}_l\Big)(\theta) = (\cos(\alpha\pi/2)+O(\delta))\int_{|x|<\delta} |2\sin((\theta-x)/2)|^\alpha \cdot \mathcal{K}_l(e^{ix})dx + O(\delta^{-3}n^{-3})$$

and

$$\operatorname{Im}\Big((1-z)^\alpha * \mathcal{K}_l\Big)(\theta) = -\sin(\alpha\pi/2)\int_{|x|<\delta} |2\sin((\theta-x)/2)|^\alpha \cdot \operatorname{sgn}(\theta-x)\cdot \mathcal{K}_l(e^{ix})dx +$$

$$O(\delta)\int_{|x|<\delta}|2\sin((\theta-x)/2)|^\alpha \cdot \mathcal{K}_l(e^{ix})dx + O(\delta^{-3}n^{-3})\,. \tag{7.11}$$

Take $\delta = \tau^2$. Then, we choose τ_0 such that

$$\cos(\alpha\pi/2) + O(\delta) \sim \tau\,. \tag{7.12}$$

for all $\tau \in (0,\tau_0)$. Then,

$$\int_{|x|<\tau^2}|2\sin((\theta-x)/2)|^\alpha \cdot \mathcal{K}_l(e^{ix})dx \sim \int_{|x|<\tau^2}|\theta-x|^\alpha\cdot\frac{\sin^4(lx/4)}{l^3x^4}dx\,.$$

As in the previous Appendix, we introduce $\widehat{x} := lx, \widehat{\theta} := l\theta$. The last integral becomes

$$l^{-\alpha}\int_{|\widehat{x}|<l\tau^2}|\widehat{\theta}-\widehat{x}|^\alpha\cdot\frac{\sin^4(\widehat{x}/4)}{\widehat{x}^4}d\widehat{x} \sim l^{-\alpha}(1+|\widehat{\theta}|^\alpha)\,, \tag{7.13}$$

if $|\widehat{\theta}| < l\tau^2$. Combining the estimates and recalling that $l \sim n$, we get

$$\operatorname{Re}\Big((1-e^{i\theta})^\alpha * \mathcal{K}_l\Big)(\theta) \sim \tau^{-6}n^{-3} + \tau(n^{-\alpha}+|\theta|^\alpha)\,.$$

This proves (7.8). For the imaginary part, (7.11) gives

$$\operatorname{Im}\Big((1-z)^\alpha * \mathcal{K}_l\Big)(\theta) = -\sin(\alpha\pi/2)J_1 + J_2 + O(\delta^{-3}n^{-3})\,,$$

$$J_1 := \int_{|x|<\delta}|2\sin((\theta-x)/2)|^\alpha\cdot\operatorname{sgn}(\theta-x)\cdot\mathcal{K}_l(x)dx\,,$$

$$J_2 := O(\delta)\int_{|x|<\delta}|2\sin((\theta-x)/2)|^\alpha\cdot\mathcal{K}_l(e^{ix})dx\,.$$

For J_2,

$$J_2 = O(\tau^3)(n^{-\alpha}+|\theta|^\alpha) \tag{7.14}$$

due to (7.13) and the choice of δ.

Consider J_1. We write Taylor expansion for sin

$$\sin(\theta-x) = \theta - x + O((\theta-x)^2)\,.$$

For the Jackson kernel, we have the asymptotical representation

$$\mathcal{K}_l(\theta) = \frac{C}{l^3}\frac{\sin^4(l\theta/2)}{\theta^4} + O(l^{-1})\,.$$

Substituting these two expressions into the formula for J_1, we get

$$J_1 = J_{(1,1)} + \epsilon_{(n,\theta)}\,.$$

The expression for the main term, $J_{(1,1)}$, is

$$J_{(1,1)} = Cl^{-3}\int_{|x|<\delta}|\theta-x|^\alpha\cdot\operatorname{sgn}(\theta-x)\cdot\frac{\sin^4(lx/2)}{x^4}dx\,.$$

Carrying out the standard bounds, we get

$$|\epsilon_{(n,\theta)}| \lesssim n^{-1} \tag{7.15}$$

if $n > n_{(\tau)}$. In $J_{(1,1)}$, perform the change of variables $\widehat{x} = lx, \widehat{\theta} = l\theta$ to get

$$J_{(1,1)} = Cl^{-\alpha} S(\widehat{\theta}),\ S(\widehat{\theta}) := \int_{|\widehat{x}|<l\tau^2} \operatorname{sgn}(\widehat{\theta} - \widehat{x}) \cdot |\widehat{\theta} - \widehat{x}|^{\alpha} \cdot \frac{\sin^4(\widehat{x}/4)}{\widehat{x}^4} d\widehat{x}\,.$$

Notice that $S(0) = 0$ and

$$S'(\widehat{\theta}) = \alpha \int_{|\widehat{x}|<l\tau^2} |\widehat{x} - \widehat{\theta}|^{\alpha-1} \frac{\sin^4(\widehat{x}/4)}{\widehat{x}^4} d\widehat{x} \sim 1 + |\widehat{\theta}|^{\alpha-1}\,.$$

Since $S(\widehat{\theta}) = S(0) + \int_0^{\widehat{\theta}} S'(\xi)d\xi$, we have

$$|S(\widehat{\theta})| \sim \begin{cases} |\widehat{\theta}|, & |\widehat{\theta}| < 1\,, \\ |\widehat{\theta}|^{\alpha}, & |\widehat{\theta}| > 1\,. \end{cases}$$

Going back to the original variable, we get

$$|J_{(1,1)}(\theta)| \sim \begin{cases} n^{\tau}|\theta|, & |\theta| < 1/n\,, \\ |\theta|^{\alpha}, & 1/n < |\theta|\,. \end{cases}$$

Now, we can combine these bounds to control $|H_n|$. Notice that $|H_n| \sim |\operatorname{Re} H_n| + |\operatorname{Im} H_n|$. For $|\theta| > 1/n$, $J_{(1,1)}$ gives the main contribution and $|H_n| \sim |\theta|^{\alpha}$. When $\theta \in [-1/n, 1/n]$, we have $\operatorname{Re} H_n \sim \tau n^{-\alpha}$. Then, $|\epsilon_{(n,\theta)}|$ is negligible compared to $\operatorname{Re} H_n$, as can be seen from (7.15). Thus,

$$|H_n| \sim n^{\tau}|\theta| + \tau n^{-\alpha}\,.$$

Note carefully that $n^{\tau}|\theta| \sim \tau n^{-\alpha}$ when $|\theta| \sim \tau/n$.

Thus, we have (7.9) for $|\theta| < \tau^2$. To prove (7.9) for all θ, it is sufficient to notice that outside the interval $[-\tau^2, \tau^2]$, (7.6) gives (7.9) immediately due to the properties of the limiting function $(1 - e^{i\theta})^{\alpha}$.

In conclusion, we want to mention that τ_0 has been chosen "small enough" to make the asymptotical calculation (7.12) valid. □

Consider Q_n given by (3.23), i.e.,

$$Q_n = (1 - e^{i\theta})^{-\alpha/2} * \mathcal{F}_n\,. \tag{7.16}$$

The following Lemma can be proved in the standard way.

Lemma 7.2. *If $\alpha \in (1/2, 1)$, then there is $n_0 \in \mathbb{N}$ so that*

$$\operatorname{Re} Q_n \sim \begin{cases} n^{\alpha/2}, & |\theta| < n^{-1}, \\ |\theta|^{-\alpha/2}, & n^{-1} < |\theta| \end{cases}$$

and

$$|Q_n| \sim \begin{cases} n^{\alpha/2}, & |\theta| < n^{-1}, \\ |\theta|^{-\alpha/2}, & n^{-1} < |\theta| \end{cases}$$

for all $n > n_0$.

Proof. Let $\gamma := \alpha/2$. We have $\gamma \in (\frac{1}{4}, \frac{1}{2})$. The proof is similar to that of Lemma 1, except it is easier since we are not concerned with how the estimates depend on small parameter. We start by writing

$$\mathrm{Re}(1-e^{i\theta})^{-\gamma} \sim |\theta|^{-\gamma},\ |\,\mathrm{Im}(1-e^{i\theta})^{-\gamma}| \lesssim |\theta|^{-\gamma}\,. \tag{7.17}$$

The last inequality implies

$$|\,\mathrm{Im}\,Q_n| \lesssim \mathrm{Re}\,Q_n\,.$$

Therefore, we only need to focus on $\mathrm{Re}\,Q_n$. Substitute the formula (6.5) into the convolution and use (7.17) to get

$$\mathrm{Re}\,Q_n(\theta) \sim n^{-1}\int_{-\pi}^{\pi} |\theta - x|^{-\gamma}\frac{\sin^2(nx)}{x^2}dx + O(n^{-1})\,.$$

Consider the integral and change variables $\widehat{\theta} := n\theta, \widehat{x} := nx$. This gives

$$n^\gamma \int_{-\pi n}^{\pi n} |\widehat{x} - \widehat{\theta}|^{-\gamma}\frac{\sin^2(\widehat{x})}{\widehat{x}^2}d\widehat{x} \sim \frac{n^\gamma}{1+|\widehat{\theta}|^\gamma} \sim \left\{ \begin{array}{ll} n^\gamma, & |\theta| < 1/n, \\ |\theta|^{-\gamma}, & |\theta| > 1/n \end{array} \right. .$$

Comparing the last quantity to $O(n^{-1})$, we finish the proof. ☐

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