

**ORTHOGONAL POLYNOMIALS ON THE CIRCLE FOR THE WEIGHT w
SATISFYING CONDITIONS $w, w^{-1} \in \text{BMO}$**

S. DENISOV, K. RUSH

ABSTRACT. For the weight w satisfying $w, w^{-1} \in \text{BMO}(\mathbb{T})$, we prove the asymptotics of $\{\Phi_n(e^{i\theta}, w)\}$ in $L^p[-\pi, \pi]$, $2 \leq p < p_0$ where $\{\Phi_n(z, w)\}$ are monic polynomials orthogonal with respect to w on the unit circle \mathbb{T} . Immediate applications include the estimates on the uniform norm and asymptotics of the polynomial entropies. The estimates on higher order commutators between the Calderon-Zygmund operators and BMO functions play the key role in the proofs of main results.

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1. INTRODUCTION

Let σ be a probability measure on the unit circle. Define the monic orthogonal polynomials $\{\Phi_n(z, \sigma)\}$ by requiring

$$\deg \Phi_n = n, \text{coeff}(\Phi_n, n) = 1, \int_{-\pi}^{\pi} \Phi_n(e^{i\theta}, \sigma) \overline{\Phi_m(e^{i\theta}, \sigma)} d\sigma = 0, \quad m < n,$$

where $\text{coeff}(Q, j)$ denotes the coefficient in front of z^j in the polynomial Q . We can also define the orthonormal polynomials by the formula

$$\phi_n(z, \sigma) = \frac{\Phi_n(z, \sigma)}{\|\Phi_n(e^{i\theta}, \sigma)\|_{L^2_\sigma}}$$

Later, we will need to use the following notation: for every polynomial $Q_n(z) = q_n z^n + \dots + q_0$ of degree at most n , we introduce the $(*)$ -operation:

$$Q_n(z) \xrightarrow{(*)} Q_n^*(z) = \bar{q}_0 z^n + \dots + \bar{q}_n$$

This $(*)$ depends on n . In the paper, we use the shorthand $\|f\|_p = \|f\|_{L^p(\mathbb{T})}$, $\|f\|_{L^p_w} = (\int_{\mathbb{T}} |f(\theta)|^p w d\theta)^{1/p}$. L^p stands for $L^p(\mathbb{T})$ or $L^p[-\pi, \pi]$. The symbols C, C_1 are reserved for absolute constants which value can change from one formula to another.

The current paper is mainly motivated by two problems: the problem of Steklov in the theory of orthogonal polynomials [2] and the problem of the asymptotical behavior of the polynomial entropy [9].

The problem of Steklov [22] consists in obtaining the sharp estimates for $\|\phi_n(e^{i\theta}, \sigma)\|_{L^\infty[-\pi, \pi]}$ assuming that the probability measure σ satisfies $\sigma' \geq \delta/(2\pi)$ a.e. on $[-\pi, \pi]$ and $\delta \in (0, 1)$. This question attracted a lot of attention [1, 11, 12, 13, 14, 18, 19, 23] and was recently resolved in [2]. In particular, the following stronger result was proved

Theorem 1.1 ([2]). *Assume that the measure is given by the weight w : $d\sigma = w d\theta$. Let $p \in [1, \infty)$ and $C > C_0(p, \delta)$, then*

$$C_1(p, \delta) \sqrt{n} \leq \sup_{w \geq \delta/(2\pi), \|w\|_1=1, \|w\|_p \leq C} \|\phi_n(e^{i\theta}, w)\|_\infty \leq C_2(p, \delta) \sqrt{n}$$

Remark. If the measure σ satisfies the Szegő condition [24]

$$\int_{-\pi}^{\pi} \log \sigma'(\theta) d\theta > -\infty \quad (1)$$

then $\|\Phi_n\|_{L^2_\sigma} \sim 1$ and the polynomials ϕ_n and Φ_n are of the same size. In particular, ϕ_n can be replaced by Φ_n in the previous Theorem.

Remark. In the formulation of the Steklov problem, the normalization that σ is a probability measure, i.e.,

$$\int_{-\pi}^{\pi} d\sigma = 1$$

is not restrictive because of the following scalings: $\phi_n(z, \sigma) = \alpha^{1/2} \phi_n(z, \alpha\sigma)$ and $\Phi_n(z, \alpha\sigma) = \Phi_n(z, \sigma)$, $\alpha > 0$.

The previous Theorem handles all $p < \infty$ but not the case $p = \infty$. That turns out to be essential: if the weight w is bounded, we get an improvement in the exponent.

Theorem 1.2. ([8], Denisov-Nazarov) *If $T \gg 1$, we have*

$$\sup_{1 \leq w \leq T} \|\Phi_n(e^{i\theta}, w)\|_{p_0} \leq C(T), \quad p_0(T) = 2 + \frac{C_1}{T}, \quad \sup_{1 \leq w \leq T} \|\Phi_n(e^{i\theta}, w)\|_{\infty} \leq C(T) n^{\frac{1}{2} - \frac{C}{T}}$$

and, if $0 < \epsilon \ll 1$,

$$\sup_{1 \leq w \leq 1+\epsilon} \|\Phi_n(e^{i\theta}, w)\|_{p_0} \leq C(\epsilon), \quad p_0(\epsilon) = \frac{C_2}{\epsilon}, \quad \sup_{1 \leq w \leq 1+\epsilon} \|\Phi_n(e^{i\theta}, w)\|_{\infty} \leq C(\epsilon) n^{C\epsilon}$$

The uniform bound on the L^p norm suggests that maybe a stronger result on the asymptotical behavior is true. It is well-known that for σ in the Szegő class (i.e., (1) holds), the following asymptotics is valid [10]

$$\phi_n^*(e^{i\theta}, \sigma) \xrightarrow{(*)} S(e^{i\theta}, \sigma), \quad \int_{-\pi}^{\pi} \left| \frac{\phi_n^*(e^{i\theta}, \sigma)}{S(e^{i\theta}, \sigma)} - 1 \right|^2 d\theta \rightarrow 0, \quad n \rightarrow \infty \quad (2)$$

where $\xrightarrow{(*)}$ refers to weak-star convergence and $S(z, \sigma)$ is the Szegő function, i.e., the outer function in \mathbb{D} which satisfies $|S(e^{i\theta}, \sigma)|^{-2} = 2\pi\sigma'(\theta)$, $S(0, \sigma) > 0$. In particular, if $\sigma' \geq (2\pi)^{-1}\delta$, then $\|\phi_n^* - S\|_2 \rightarrow 0$. Recall that $\phi_n(z, \sigma) = z^n \overline{\phi_n^*(z, \sigma)}$, $z \in \mathbb{T}$.

The results stated above give rise to three questions: (a) What upper estimate can we get assuming $w \in \text{BMO}(\mathbb{T})$ [21] instead of $w \in L^\infty(\mathbb{T})$? Recall that $L^\infty(\mathbb{T}) \subset \text{BMO}(\mathbb{T}) \subset \cap_{p < \infty} L^p(\mathbb{T})$. (b) Is it possible to relax the Steklov condition $w \geq 1$? (c) Can one obtain an asymptotics of $\{\phi_n^*\}$ in L^p classes with $p > 2$?

The partial answers to these questions are contained in the following Theorems which are the main results of the paper. We start with a comment about some notation. If α is a positive parameter, we write $\alpha \ll 1$ to indicate the following: there is an absolute constant α_0 (sufficiently small) such that $\alpha < \alpha_0$. Similarly, we write $\alpha \gg 1$ as a substitute for: there is a constant α_0 (sufficiently large) so that $\alpha > \alpha_0$. The symbol $\alpha_1 \ll \alpha_2$ ($\alpha_1 \gg \alpha_2$) will mean $\alpha_1/\alpha_2 \ll 1$ ($\alpha_1/\alpha_2 \gg 1$).

Theorem 1.3. *If $w : \|w^{-1}\|_{\text{BMO}} \leq s, \|w\|_{\text{BMO}} \leq t$, then there is $\Pi \in L^{p_0}[-\pi, \pi]$, $p_0 > 2$ such that*

$$\lim_{n \rightarrow \infty} \|\Phi_n^* - \Pi\|_{p_0} = 0$$

and we have for p_0 :

$$p_0 = \begin{cases} 2 + \frac{C_1}{(st) \log^2(st)}, & \text{if } st \gg 1 \\ \frac{C_2}{(st)^{1/4}}, & \text{if } 0 < st \ll 1 \end{cases} \quad (3)$$

We also have the bound for the uniform norm

$$\|\Phi_n^*\|_\infty \leq C_{(st)} n^{1/p_0} \quad (4)$$

where $C_{(u)}$ denotes a function of u .

In the case when an additional information is given, e.g., $w \in L^\infty$ or $w^{-1} \in L^\infty$, this result can be improved.

Theorem 1.4. *Under the conditions of the previous Theorem, we have*

- If $w \geq 1$, then p_0 can be taken as

$$p_0 = \begin{cases} 2 + \frac{C_1}{t \log t}, & \text{if } t \gg 1 \\ \frac{C_2}{\sqrt{t}}, & \text{if } 0 < t \ll 1 \end{cases}$$

- If $w \leq 1$, then we have

$$p_0 = \begin{cases} 2 + \frac{C_1}{s \log s}, & \text{if } s \gg 1 \\ \frac{C_2}{\sqrt{s}}, & \text{if } 0 < s \ll 1 \end{cases}$$

We also have the bound for the uniform norm

$$\|\Phi_n^*\|_\infty \leq C_{(t,s)} n^{1/p_0} \quad (5)$$

where $C_{(t,s)}$ depends on t or s .

Remark. It is clear that the allowed exponent p_0 is decaying in s and t so it can be chosen larger than 2 for all values of s and t .

Remark. As we have already mentioned, the following scaling invariance holds: $\Phi_n(z, \sigma) = \Phi_n(z, \alpha\sigma)$, $\alpha > 0$. The BMO norm is 1-homogeneous, e.g., $\|\alpha w\|_{BMO} = \alpha \|w\|_{BMO}$, so the estimates in the Theorem 1.3 are invariant under scaling $w \rightarrow \alpha w$.

In the case when $w = C$, we get $\|w\|_{BMO} = \|w^{-1}\|_{BMO} = 0$ and, although $\Phi_n^*(z, w) = 1$, we can not say anything about the size of $\phi_n^*(z, w)$. The next Lemma explains how our results can be generalized to $\{\phi_n^*\}$.

Lemma 1.1. *In the Theorem 1.3, if one makes an additional assumption that $\|w\|_1 = 1$, then $\|\phi_n^* - S\|_{p_0} \rightarrow 0$ with p_0 given by (3).*

Proof. Indeed, Lemma 3.3 from Appendix shows that

$$\int_{-\pi}^{\pi} \log w d\theta > -\infty$$

and thus the sequence $\{\|\Phi_n\|_{2,w}\}$ has a finite positive limit [10, 20]. Therefore, $\{\phi_n^*\} = \left\{ \frac{\Phi_n^*}{\|\Phi_n\|_{2,w}} \right\}$ has an L^{p_0} limit by Theorem 1.3. By (2), $\{\phi_n^*\}$ converges weakly to S and therefore we have the statement of the Lemma with Π being a multiple of S . \square

The polynomial entropy is defined as

$$E(n, \sigma) = \int_{\mathbb{T}} |\phi_n|^2 \log |\phi_n| d\sigma$$

where $\{\phi_n\}$ are orthonormal with respect to σ . In recent years, a lot of efforts were made to understand the asymptotics of $E(n, \sigma)$ [3, 4, 5] as $n \rightarrow \infty$. In [9], the sharp lower and upper bounds were obtained for σ in the Szegő class. In [2], it was shown that $E(n, w)$ can not exceed $C \log n$ if $w \geq 1$ and $w \in L^p[-\pi, \pi]$, $p < \infty$, and that this bound saturates. This leaves us with very natural question: what are regularity assumptions on w that guarantee boundedness of $E(n, w)$? The following corollary of Lemma 1.1 gives the partial answer.

Corollary 1.1. *If $w : w, w^{-1} \in \text{BMO}(\mathbb{T})$ and $\|w\|_1 = 1$, then*

$$\lim_{n \rightarrow \infty} E(n, w) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \log(2\pi w) d\theta$$

So far, the only classes in which the $E(n, w)$ was known to be bounded were the Baxter's class [20]: $d\sigma = wd\theta$, $w \in W(\mathbb{T})$, $w > 0$ ($W(\mathbb{T})$ denotes the Wiener algebra) or the class given by positive weights with a certain modulus of continuity [24]. Our conditions are obviously much weaker and, in a sense, sharp.

The structure of the paper is as follows: the main results are proved in the next section, the Appendix contains auxiliary results from harmonic analysis.

We use the following notation: H refers to the Hilbert transform, $P_{[i,j]}$ denotes the $L^2(d\theta)$ projection to the (i, \dots, j) Fourier modes. Given two non-negative functions $f_{1(2)}$ we write $f_1 \lesssim f_2$ if there is an absolute constant C such that

$$f_1 \leq C f_2$$

for all values of the arguments of $f_{1(2)}$. We define \gtrsim similarly and say that $f_1 \sim f_2$ if $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ simultaneously. Given two operators A and B , we write $[A, B] = AB - BA$ for their commutator. If w is a function, then in the expression like $[w, A]$, the symbol w is identified with the operator of multiplication by w . The Hunt-Muckenhoupt-Wheeden characteristic of the weight $w \in A_p$ will be denoted by $[w]_{A_p}$. For the basic facts about the BMO class, A_p and their relationship, we refer the reader to, e.g., the classical text [21]. If A is a linear operator from $L^p(\mathbb{T})$ space to $L^p(\mathbb{T})$, then $\|A\|_{p,p}$ denotes its operator norm.

2. PROOFS OF MAIN RESULTS

Before proving the main result, Theorem 1.3, we need some auxiliary Lemmas. We start with the following observation which goes back to S. Bernstein [6, 24].

Lemma 2.1. *For a monic polynomial Q of degree n , we have:*

$$Q(z) = \Phi_n(z, w) \quad \text{if and only if} \quad P_{[0, n-1]}(wQ) = 0. \quad (6)$$

Proof. It is sufficient to notice that (6) is equivalent to

$$\int_{-\pi}^{\pi} Q(e^{i\theta}) e^{-ij\theta} w(\theta) d\theta = 0, \quad j = 0, \dots, n-1$$

which is the orthogonality condition. □

Lemma 2.2. *If $f \in L^2(\mathbb{T})$ is real-valued function, $Q \in L^\infty(\mathbb{T})$, then*

$$z^n \overline{P_{[0,n-1]}(fz^n \overline{Q})} = P_{[1,n]}(fQ)$$

In particular, for a polynomial P of degree at most n with $P(0) = 1$, we have:

$$P(z) = \Phi_n^*(z, w) \quad \text{if and only if} \quad P_{[1,n]}(wP) = 0.$$

Proof. The first statement is immediate. The second one follows from the Lemma above and the formula $\Phi_n = z^n \overline{\Phi_n^*}$, $z \in \mathbb{T}$. \square

We have the following three identities for $\Phi_n^*(z, w)$; the first one was used in [8] recently. They are immediately implied by the Lemma above.

$$\Phi_n^* = 1 + P_{[1,n]} \left((1 - \alpha w) \Phi_n^* \right), \quad \alpha \in \mathbb{R} \quad (7)$$

$$\Phi_n^* = 1 + w^{-1} [w, P_{[1,n]}] \Phi_n^* \quad (8)$$

$$\Phi_n^* = 1 - [w^{-1}, P_{[1,n]}] (w \Phi_n^*) \quad (9)$$

Denote the higher order commutators recursively:

$$\mathbf{C}_0 = P_{[1,n]}, \quad \mathbf{C}_1 = [w, P_{[1,n]}], \quad \mathbf{C}_l = [w, \mathbf{C}_{l-1}], \quad l = 2, 3, \dots$$

Define the multiple commutators of w^{-1} and $P_{[1,n]}$ (in that order!) by $\tilde{\mathbf{C}}_j$.

Lemma 2.3. *The following representations hold*

$$w^j P_{[1,n]} \Phi_n^* = \sum_{l=1}^j \binom{j-1}{l-1} \mathbf{C}_l w^{j-l} \Phi_n^* \quad (10)$$

and

$$w^{-j} P_{[1,n]} \Phi_n^* = - \sum_{l=0}^j \binom{j}{l} \tilde{\mathbf{C}}_{l+1} w^{-(j-l)} (w \Phi_n^*) \quad (11)$$

where $j = 1, 2, \dots$

Proof. We will prove (10), the other formula can be obtained in the similar way. The case $j = 1$ of this expression is our familiar formula $w P_{[1,n]} \Phi_n^* = [w, P_{[1,n]}] \Phi_n^*$. Now the proof proceeds by induction. Suppose we have

$$w^{k-1} P_{[1,n]} \Phi_n^* = \sum_{l=1}^{k-1} \binom{k-2}{l-1} \mathbf{C}_l w^{k-1-l} \Phi_n^*$$

Multiply both sides by w and write

$$\begin{aligned} w^k P_{[1,n]} \Phi_n^* &= \sum_{l=1}^{k-1} \binom{k-2}{l-1} w \mathbf{C}_l w^{k-1-l} \Phi_n^* = \sum_{l=1}^{k-1} \binom{k-2}{l-1} \left(\mathbf{C}_{l+1} w^{k-1-l} \Phi_n^* + \mathbf{C}_l w^{k-l} \Phi_n^* \right) = \\ &= \sum_{l=1}^{k-1} \binom{k-2}{l-1} \mathbf{C}_l w^{k-l} \Phi_n^* + \sum_{l=2}^k \binom{k-2}{l-2} \mathbf{C}_l w^{k-l} \Phi_n^* = \sum_{l=1}^k \binom{k-1}{l-1} \mathbf{C}_l w^{k-l} \Phi_n^* \end{aligned}$$

because

$$\binom{k-1}{l-1} = \binom{k-2}{l-2} + \binom{k-2}{l-1}$$

.

\square

Motivated by the previous Lemma, we introduce certain operators. Given $f \in L^p$, define $\{y_j\}$ recursively by

$$y_0 = f, y_j = w^j + \sum_{l=0}^{j-1} \binom{j-1}{l} \mathbf{C}_{l+1} y_{j-1-l}$$

Then, we let

$$z_j = w^{-j} - \sum_{l=0}^j \binom{j}{l} \tilde{\mathbf{C}}_{l+1} z_{j-l-1}$$

where $z_{-1} = y_1, z_0 = y_0 = f$. Notice that for fixed j both y_j and z_j are affine linear transformations in f . We can write

$$y_j = y'_j + y''_j$$

where

$$y'_0 = f, \quad y''_0 = 0$$

and, recursively,

$$y'_j = \sum_{l=0}^{j-1} \binom{j-1}{l} \mathbf{C}_{l+1} y'_{j-1-l}, \quad y''_j = w^j + \sum_{l=0}^{j-1} \binom{j-1}{l} \mathbf{C}_{l+1} y''_{j-1-l}$$

Similarly, we write $z_j = z'_j + z''_j$ where

$$z'_{-1} = y'_1, \quad z''_{-1} = y''_1, \quad z'_0 = f, \quad z''_0 = 0$$

and

$$z'_j = - \sum_{l=0}^j \binom{j}{l} \tilde{\mathbf{C}}_{l+1} z'_{j-l-1}, \quad z''_j = w^{-j} - \sum_{l=0}^j \binom{j}{l} \tilde{\mathbf{C}}_{l+1} z''_{j-l-1},$$

Let us introduce linear operators: $B_j f = y'_j, D_j f = z'_j$. We need an important Lemma.

Lemma 2.4.

$$w^j \Phi_n^* = y''_j + B_j \Phi_n^*, \quad w^{-j} \Phi_n^* = z''_j + D_j \Phi_n^*$$

Proof. This follows from

$$w^j \Phi_n^* = w^j + w^j P_{[1,n]} \Phi_n^*, \quad w^{-j} \Phi_n^* = w^{-j} + w^{-j} P_{[1,n]} \Phi_n^*$$

and the previous Lemma. □

The next Lemma, in particular, provides the bounds for B_j and D_j .

Lemma 2.5. *Assume $w \geq 0, \|w\|_{BMO} = t, \|w^{-1}\|_{BMO} = s, \|w\|_1 = 1$, and $p \in [2, 3]$. Then,*

$$\|B_j\|_{p,p} \leq (Ctj)^j, \quad \|D_j\|_{p,p} \leq (1+st)(Cs_j)^j$$

Moreover,

$$\|y''_j\|_p \leq (C\tilde{t}j)^j, \quad \|z''_j\|_p \leq \tilde{s}\tilde{t}(C\tilde{s}j)^j$$

with

$$\tilde{t} = \max\{t, 1\}, \quad \tilde{s} = \max\{s, 1\}$$

Proof. We will prove the estimates for $\|B_j\|_{p,p}$ and $\|y_j''\|_p$ only, the bounds for $\|D_j\|_{p,p}, \|z_j''\|_p$ are shown similarly. By John-Nirenberg inequality ([21], p.144), we get

$$\int_{-\pi}^{\pi} |w - (2\pi)^{-1}|^{jp} d\theta \lesssim j \int_0^{\infty} x^{jp-1} \exp(-Cx/t) dx = j(Ct)^{jp} \Gamma(jp) \leq (C_1 t j)^{jp}$$

where Stirling's formula was used for the gamma function Γ .

Since

$$|w|^{jp} \leq (|w - (2\pi)^{-1}| + (2\pi)^{-1})^{jp} \leq C^{jp} (|w - (2\pi)^{-1}|^{jp} + 1)$$

we have

$$\int_{-\pi}^{\pi} |w|^{jp} d\theta \leq C^{jp} (1 + (tj)^{jp}) \leq (C_1 \tilde{t} j)^{jp}$$

Lemma 3.2 yields

$$\|y_j'\|_p \leq \sum_{l=0}^{j-1} \frac{(j-1)!}{l!(j-1-l)!} (\tilde{C}(l+1)t)^{l+1} \|y_{j-1-l}'\|_p \leq (Ct)^j j! \sum_{k=0}^{j-1} \frac{(Ct)^{-k}}{k!} \|y_k'\|_p$$

Divide both sides by $(Ct)^j j!$ and denote $\beta_j = \frac{\|y_j'\|_p}{(Ct)^j j!}$. Then,

$$\beta_j \leq \sum_{l=0}^{j-1} \beta_l$$

Since $\beta_0 = \|f\|_p$, we have $\beta_j \leq 3^j \|f\|_p$ by induction and thus $\|y_j'\|_p \leq (Ctj)^j \|f\|_p$. The estimates for $\|y_j''\|_p, \|z_j'\|_p, \|z_j''\|_p$ can be obtained similarly. \square

Lemma 2.6. *If $\|w\|_1 = 1, \|w\|_{BMO} = t, \|w^{-1}\|_{BMO} = s$, and $p \in [2, 3]$, then*

$$\min_{l \in \mathbb{N}} \left(\Lambda^{-l} \|B_l\|_{p,p} \right) \leq \exp \left(-\frac{C\Lambda}{t} \right)$$

and

$$\min_{j \in \mathbb{N}} \left(\epsilon^j \|D_j\|_{p,p} \right) \leq (1 + st) \exp \left(-\frac{C}{\epsilon s} \right)$$

provided that $\Lambda \gg t$ and $\epsilon \ll s^{-1}$.

Proof. By the previous Lemma, we have

$$\left(\Lambda^{-l} \|B_l\|_{p,p} \right) \leq \left(\frac{Ct\Lambda}{\Lambda} \right)^l$$

Optimizing in l we get $l^* \sim C\Lambda/(te)$ and it gives the first estimate. The proof for the second one is identical. \square

Now we are ready to prove the main results of the paper.

Proof. (Theorem 1.3). Notice first that (4) follows from the Nikolskii inequality

$$\|Q\|_{\infty} < Cn^{1/p_0} \|Q\|_{p_0}, \quad \deg Q = n, \quad p_0 \geq 2$$

as long as the L^{p_0} norms are estimated.

By scaling invariance, we can assume that $\|w\|_1 = 1$. We consider two cases separately: $st \gg 1$ and $st \ll 1$. The proofs will be different.

1. The case $st \gg 1$.

Let $p = 2 + \delta$ with $\delta < 1$. Take two n -independent parameters ϵ and Λ such that $\epsilon s \ll 1$ and $\Lambda t^{-1} \gg 1$. Consider the following sets $\Omega_1 = \{x : w \leq \epsilon\}$, $\Omega_2 = \{x : \epsilon < w < \Lambda\}$, $\Omega_3 = \{x : w \geq \Lambda\}$. Notice that

$$\epsilon s \ll 1, t\Lambda^{-1} \ll 1 \implies (\epsilon s)(t\Lambda^{-1}) \ll 1 \implies \epsilon\Lambda^{-1} \ll (st)^{-1} \ll 1 \implies \epsilon \ll \Lambda$$

From (7), we have

$$\Phi_n^* = 1 + P_{[1,n]}(1 - w/\Lambda)\Phi_n^*$$

The idea of our proof is to rewrite this identity in the form

$$\Phi_n^* = f_n + \mathcal{O}(n)\Phi_n^*$$

where $\|f_n\|_p < C(s, t)$ and $\mathcal{O}(n)$ is a contraction in L^p for the suitable choice of p . To this end, we consider operators

$$\begin{aligned}\mathcal{O}_1(n)f &= \epsilon^j P_{[1,n]}(1 - w/\Lambda)\chi_{\Omega_1} \left(\frac{w}{\epsilon}\right)^j D_j f \\ \mathcal{O}_2(n)f &= P_{[1,n]}(1 - w/\Lambda)\chi_{\Omega_2} f \\ \mathcal{O}_3(n)f &= \Lambda^{-l} P_{[1,n]}(1 - w/\Lambda)(\Lambda/w)^l \chi_{\Omega_3} B_l f\end{aligned}$$

where j and l will be fixed later, they will be n -independent. Let us estimate the (L^p, L^p) norms of these operators. Since $\|P_{[1,n]}\|_{p,p} \leq 1 + C\delta$ (see Lemma 3.1), we choose j and l as in Lemma 2.6 to ensure

$$\begin{aligned}\|\mathcal{O}_1(n)\|_{p,p} &\leq st \exp\left(-\frac{\widehat{C}}{\epsilon s}\right) \\ \|\mathcal{O}_2(n)\|_{p,p} &\leq (1 + C\delta)(1 - \epsilon\Lambda^{-1}) \\ \|\mathcal{O}_3(n)\|_{p,p} &\leq \exp\left(-\frac{\widehat{C}\Lambda}{t}\right)\end{aligned}$$

Lemma 2.4 now yields

$$\Phi_n^* = 1 + f_1(n) + f_3(n) + (\mathcal{O}_1(n) + \mathcal{O}_2(n) + \mathcal{O}_3(n))\Phi_n^*$$

where

$$f_1(n) = \epsilon^j P_{[1,n]}(1 - w/\Lambda)\chi_{\Omega_1} \left(\frac{w}{\epsilon}\right)^j z_j'', \quad f_3(n) = \Lambda^{-l} P_{[1,n]}(1 - w/\Lambda)(\Lambda/w)^l \chi_{\Omega_3} y_l''$$

$$\text{Let } f(n) = 1 + f_1(n) + f_3(n)$$

Then Lemma 2.5 provides the bound

$$\|f(n)\|_p \leq C(s, t) \tag{12}$$

uniform in n . Denote $\mathcal{O}(n) = \mathcal{O}_1(n) + \mathcal{O}_2(n) + \mathcal{O}_3(n)$ and select parameters $\epsilon, \Lambda, \delta$ such that $\|\mathcal{O}(n)\|_{p,p} < 1 - C\delta$. To do so, we first let $\delta = c\epsilon\Lambda^{-1}$ with small positive absolute constant c . Then, we consider

$$st \exp\left(-\frac{\widehat{C}}{\epsilon s}\right) + \exp\left(-\frac{\widehat{C}\Lambda}{t}\right) = \frac{c_1 \epsilon}{\Lambda}$$

with c_1 again being a small constant. Now, solving equations

$$st \exp(-\widehat{C}/(\epsilon s)) = \exp(-\widehat{C}\Lambda/t), \quad c_1 \epsilon/\Lambda = 2 \exp(-\widehat{C}\Lambda/t)$$

we get the statement of the Theorem. Indeed, we have two equations:

$$\epsilon = \frac{\widehat{C}t}{s(\widehat{C}\Lambda + t \log(st))}$$

and

$$\frac{\Lambda}{t} = \frac{1}{\widehat{C}} \left(C + \log(st\Lambda) + \log\left(\frac{\Lambda}{t} + \frac{\log(st)}{\widehat{C}}\right) \right)$$

Denote

$$u = \widehat{C}\Lambda/t$$

and then

$$u = C + \log(st) + 2 \log u + \log\left(1 + \frac{\log(st)}{u}\right)$$

To find the required root, we restrict the range of u to $c_1 \log(st) < u < c_2 \log(st)$ for $c_1 \ll 1$, $c_2 \gg 1$. Rewrite the equation above as

$$u - 2 \log u - \log\left(1 + \frac{\log(st)}{u}\right) = \log(st) + C$$

Differentiating the left hand side in u , we see that l.h.s.' ~ 1 within the given range. Therefore, there is exactly one solution u and $u \sim \log(st)$. Then, since

$$\log\left(1 + \frac{\log(st)}{u}\right)$$

is $O(1)$, we get

$$u = \log(st) + 2 \log u + O(1) = \log(st) + 2 \log \log(st) + O(1)$$

by iteration. Thus,

$$\frac{\epsilon}{\Lambda} = C e^{-u} \sim \frac{1}{st \log^2(st)}$$

and $\delta \sim \frac{1}{st \log^2(st)}$. Now that we proved that $\|\mathcal{O}(n)\|_{p,p} \leq 1 - C\delta < 1$, we can rewrite

$$\Phi_n^* = f(n) + \sum_{j=1}^{\infty} \mathcal{O}^j(n) f(n)$$

and the series converges geometrically in L^p with tail being uniformly small in n due to (12).

To show that Φ_n^* converges in L^p as $n \rightarrow \infty$, it is sufficient to prove that $\mathcal{O}^j(n)f(n)$ converges for each j . This, however, is immediate. Indeed,

$$P_{[1,n]}f \rightarrow P_{[1,\infty]}f, \quad \text{as } n \rightarrow \infty$$

in L^q for all $f \in L^q$, $1 < q < \infty$. Since $w, w^{-1} \in \text{BMO} \subset \cap_{p \geq 1} L^p$ ([21], this again follows from John-Nirenberg estimate), we see that multiplication by $w^{\pm j}$ maps L^{p_1} to L^{p_2} continuously by Hölder's inequality provided that $p_2 < p_1$ and $j \in \mathbb{Z}$. Therefore, if $\mu_j \in L^\infty$, $j = 1, \dots, k$, then

$$\mu_1 w^{\pm j_1} P_{[1,n]} \mu_2 w^{\pm j_2} \dots \mu_{k-1} w^{\pm j_{k-1}} P_{[1,n]} \mu_k w^{\pm j_k} \quad (13)$$

has the limit in each L^p , $p < \infty$ when $n \rightarrow \infty$. Notice that each $f(n)$ and $\mathcal{O}^j(n)f(n)$ can be written as a linear combination of expressions of type (13) ($\{\mu_j\}$ taken as the characteristic functions). Now that δ is chosen, we define p_0 in the statement of the Theorem as $p_0 = 2 + \delta$.

2. The case $st \ll 1$.

The proof in this case is much easier. Let us start with two identities

$$\Phi_n^* = 1 + w^{-1}[w, P_{[1,n]}]\Phi_n^*, \quad \Phi_n^* = 1 + [P_{[1,n]}, w^{-1}]w\Phi_n^*$$

which can be recast as

$$w\Phi_n^* = w + [w, P_{[1,n]}\Phi_n^*, \quad \Phi_n^* = 1 + [P_{[1,n]}, w^{-1}]w\Phi_n^*$$

Substitution of the first formula into the second one gives

$$\Phi_n^* = 1 + [P_{[1,n]}, w^{-1}]w + G_n \Phi_n^*$$

where

$$G_n = [P_{[1,n]}, w^{-1}][w, P_{[1,n]}]$$

We have

$$\|1 + [P_{[1,n]}, w^{-1}]w\|_p \leq C(s, t, p)$$

and

$$\|G_n\|_{p,p} \lesssim stp^4$$

by Lemma 3.4. Taking $p < p_0 \sim (st)^{-1/4}$ we have that G_n is a contraction. Now, the convergence of all terms in the geometric series can be proved as before. \square

Let us give a sketch of how the arguments can be modified to prove Theorem 1.4.

Proof. (Theorem 1.4). Consider the case $w \geq 1$ first.

1. The case $t \gg 1$.

The proof is identical except that we can chose $\epsilon = 1/2$ so that $\Omega_1 = \emptyset$. We get an equation for Λ

$$\frac{C}{\Lambda} = \exp\left(-\frac{\widehat{C}\Lambda}{t}\right), \quad \Lambda = \widehat{C}^{-1}t(\log \Lambda - \log C)$$

Denote $\widehat{C}\Lambda/t = u$, then

$$u = \log t + \log u + O(1), \quad u = \log t + \log \log t + O(1)$$

and $\delta \sim (t \log t)^{-1}$.

2. The case $t \ll 1$.

We have

$$\Phi_n^* = 1 + L_n \Phi_n^*, \quad L_n f = w^{-1}[w, P_{[1,n]}]f$$

and Lemma 3.4 yields

$$\|L_n\|_{p,p} \lesssim p^2 t < 0.5$$

for $p < p_0 = O(t^{-1/2})$.

The case $w \leq 1$ can be handled similarly. When s is large, we take $\Lambda = 1$ in the proof of the previous Theorem and get an equation for ϵ :

$$C\epsilon = s \exp\left(-\frac{\widehat{C}}{\epsilon s}\right)$$

Its solution for large s gives the required asymptotics for ϵ and, correspondingly, for δ and p_0 . If s is small, it is enough to consider the equation

$$\Phi_n^* = 1 - [w^{-1}, P_{[1,n]}]w\Phi_n^*$$

where the operator $[w^{-1}, P_{[1,n]}]w$ is contraction in L^{p_0} for the specified p_0 . \square

Now we are ready to prove Corollary 1.1.

Proof. (of Corollary 1.1). The following inequality follows from the Mean Value Formula

$$|x^2 \log x - y^2 \log y| \leq C(1 + x|\log x| + y|\log y|)|x - y|, \quad x, y \geq 0$$

Since $w \in \cap_{p < \infty} L^p$, the Theorem 1.3 yields

$$\int_{-\pi}^{\pi} \|\phi_n\|^2 \log |\phi_n| - |S|^2 \log |S| w d\theta \lesssim \int_{-\pi}^{\pi} (1 + |\phi_n \log \phi_n| + |S \log S|) \|\phi_n^* - |S|\| w d\theta \rightarrow 0, \quad n \rightarrow \infty$$

by applying the trivial bound: $u|\log u| \leq C(\delta)(1 + u^{1+\delta})$, $\delta > 0$ and the generalized Hölder's inequality to $|\phi_n|^{1+\delta}$ (or $|S|^{1+\delta}$), $\|\phi_n^* - |S|\|$, and w . To conclude the proof, it is sufficient to notice that

$$\int_{-\pi}^{\pi} |S|^2 \log |S| w d\theta = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \log(2\pi w) d\theta$$

because $|S|^{-2} = 2\pi w$. □

3. APPENDIX

In this Appendix, we collect some auxiliary results used in the main text.

Lemma 3.1. *For every $p \in [2, \infty)$,*

$$\|P_{[1,n]}\|_{p,p} \leq 1 + C(p-2). \quad (14)$$

Proof. If \mathcal{P}^+ is the projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ (analytic Hardy space), then

$$\mathcal{P}^+ = 0.5(1 + iH) + P_0,$$

where H is the Hilbert transform on the circle and P_0 denotes the Fourier projection to the constants, i.e.,

$$P_0 f = (2\pi)^{-1} \int_{\mathbb{T}} f(x) dx \quad (15)$$

We therefore have a representation

$$P_{[1,n]} = z\mathcal{P}^+ z^{-1} - z^{n+1}\mathcal{P}^+ z^{-(n+1)} = 0.5i(zHz^{-1} - z^{n+1}Hz^{-(n+1)}) + zP_0 z^{-1} - z^{n+1}P_0 z^{-(n+1)}. \quad (16)$$

Since $\|H\|_{p,p} = \cot(\pi/(2p))$ [17], we have

$$\|P_{[1,n]}\|_{p,p} \leq \cot\left(\frac{\pi}{2p}\right) + 2 \quad (17)$$

by triangle inequality. On the other hand, $\|P_{[1,n]}\|_{2,2} = 1$ so by Riesz-Thorin theorem, we can interpolate between $p = 2$ and, e.g., $p = 3$ to get

$$\|P_{[1,n]}\|_{p,p} \leq 1 + C(p-2), \quad p \in [2, 3].$$

Noticing that $\cot(\pi/(2p)) \sim p$, $p > 3$, we get the statement of the Lemma. □

Remark. In the proof above, we could have used the expression for the norm $\|\mathcal{P}^+\|_{p,p}$ obtained in [15].

The proof of the following Lemma uses some standard results of Harmonic Analysis.

Lemma 3.2. *If $\|w\|_{BMO} = t$ and $p \in [2, 3]$, then we have*

$$\|\mathbf{C}_j\|_{p,p} \leq (Cjt)^j$$

Proof. Consider the following operator-valued function

$$F(z) = e^{zw} P_{[1,n]} e^{-zw}$$

If we can prove that $F(z)$ is weakly analytic around the origin (i.e., analyticity of the scalar function $\langle F(z)f_1, f_2 \rangle$ with fixed $f_{1(2)} \in C^\infty$), then

$$F(z) = \frac{1}{2\pi i} \int_{|\xi|=\epsilon} \frac{F(\xi)}{\xi - z} d\xi, \quad z \in B_\epsilon(0)$$

understood in a weak sense. By induction, one can then easily show the well-known formula

$$\mathbf{C}_j = \partial^j F(0) = \frac{j!}{2\pi i} \int_{|\xi|=\epsilon} \frac{F(\xi)}{\xi^{j+1}} d\xi$$

which explains that we can control $\|\mathbf{C}_j\|_{p,p}$ by the size of $\|F(\xi)\|_{p,p}$ on the circle of radius ϵ . Indeed,

$$\begin{aligned} \|\mathbf{C}_j\|_{p,p} &= \sup_{f_{1(2)} \in C^\infty, \|f_1\|_p \leq 1, \|f_2\|_{p'} \leq 1} |\langle \mathbf{C}_j f_1, f_2 \rangle| \leq \\ & \frac{j!}{2\pi} \sup_{f_{1(2)} \in C^\infty, \|f_1\|_p \leq 1, \|f_2\|_{p'} \leq 1} \left| \int_{|\xi|=\epsilon} \frac{\langle F(\xi) f_1, f_2 \rangle}{\xi^{j+1}} d\xi \right| \leq \frac{j!}{\epsilon^j} \max_{|\xi|=\epsilon} \|F(\xi)\|_{p,p} \end{aligned}$$

The weak analyticity of $F(z)$ around the origin follows immediately from, e.g., the John-Nirenberg estimate ([21], p.144). To bound $\|F\|_{p,p}$, we use the following well-known result (which is again an immediate corollary from John-Nirenberg inequality, see, e.g., [21], p.218).

There is ϵ_0 such that

$$\|\tilde{w}\|_{BMO} < \epsilon_0 \implies [e^{\tilde{w}}]_{A_p} \leq [e^{\tilde{w}}]_{A_2} < C, \quad p > 2$$

The Hunt-Muckenhoupt-Wheeden Theorem ([21], p.205), asserts that

$$\sup_{[\hat{w}]_{A_p} \leq C} \|H\|_{(L^p_{\hat{w}}(\mathbb{T}), L^p_{\hat{w}}(\mathbb{T}))} = \sup_{[\hat{w}]_{A_p} \leq C} \|\hat{w}^{1/p} H \hat{w}^{-1/p}\|_{p,p} = C(p) < \infty, \quad p \in [2, \infty). \quad (18)$$

We also have

$$\sup_{[\hat{w}]_{A_p} \leq C} \|P_0\|_{(L^p_{\hat{w}}(\mathbb{T}), L^p_{\hat{w}}(\mathbb{T}))} = \sup_{[\hat{w}]_{A_p} \leq C, \|f\|_p \leq 1} \|\hat{w}^{1/p} P_0(\hat{w}^{-1/p} f)\|_p \leq (2\pi)^{-1} \sup_{[\hat{w}]_{A_p} \leq C, \|f\|_p \leq 1} \left(\|f\|_p \|\hat{w}\|_1^{1/p} \|\hat{w}^{-1/p}\|_{p'} \right)$$

by Hölder's inequality. The last expression is bounded by a constant since

$$[\hat{w}]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q \hat{w} dx \cdot \left(\frac{1}{|Q|} \int_Q \hat{w}^{-p'/p} dx \right)^{p/p'} \right) \leq C,$$

where Q is any subarc of \mathbb{T} . Finally, taking $\epsilon \ll t^{-1}$, we get the statement. \square

The following Lemma provides an estimate which is not optimal but it is good enough for our purposes.

Lemma 3.3. *Suppose $w \geq 0$, $\|w\|_{BMO} = t$, $\|w^{-1}\|_{BMO} = s$, and $\|w\|_1 = 1$. Then,*

$$(2\pi)^2 \leq \|w^{-1}\|_1 \lesssim 1 + (1+t)s$$

Proof. Denote $\|w^{-1}\|_1 = M$. Then, by Cauchy-Schwarz inequality,

$$2\pi \leq \|w\|_1^{1/2} \|w^{-1}\|_1^{1/2} = M^{1/2}$$

On the other hand, by John-Nirenberg estimate for w^{-1} ,

$$|\{\theta : |w^{-1} - (2\pi)^{-1}M| > \lambda\}| \lesssim \exp\left(-\frac{C\lambda}{s}\right)$$

Choosing $\lambda = (4\pi)^{-1}M$, we get

$$|\Omega^c| \lesssim \exp\left(-\frac{CM}{s}\right) \lesssim \left(\frac{s}{M}\right)^2, \quad \text{where } \Omega = \left\{\theta : \frac{4\pi}{3M} \leq w \leq \frac{4\pi}{M}\right\} \quad (19)$$

Then, $\|w\|_1 = 1$ and therefore

$$\begin{aligned} 1 &= \int_{w \leq (4\pi)/M} w d\theta + \int_{w > (4\pi)/M} w d\theta \\ &\int_{w > (4\pi)/M} w d\theta \geq 1 - 8\pi^2 M^{-1} \end{aligned} \quad (20)$$

By John-Nirenberg inequality, we have

$$\|w - (2\pi)^{-1}\|_p < Ctp, \quad p < \infty \quad (21)$$

We choose $p = 2$ in the last estimate and use Cauchy-Schwarz inequality in (20) to get

$$1 - 8\pi^2 M^{-1} \leq \int_{w > (4\pi)/M} w d\theta \leq \|w\|_2 \cdot |\{\theta : w > 4\pi/M\}|^{1/2} \leq \|w\|_2 \cdot |\Omega_c|^{1/2} \lesssim \frac{(1+t)s}{M}$$

where we used (19) and (21) for the last bound. So, $M \lesssim (1+t)s + 1$. \square

Lemma 3.4. For $p \in [2, \infty)$, we have

$$\|[w, P_{[1,n]}]\|_{p,p} \lesssim p^2 \|w\|_{BMO}$$

Proof. The proof is standard but we give it here for completeness. Assume $\|w\|_{BMO} = 1$. By duality and formula (16), it is sufficient to show that

$$\|[w, P_0]\|_{p,p} \leq C(p-1)^{-1}, \quad p \in (1, 2] \quad (22)$$

and

$$\|[w, H]\|_{p,p} \leq C(p-1)^{-2}, \quad p \in (1, 2]. \quad (23)$$

For (22), we write

$$\|w \int_{\mathbb{T}} f dx - \int_{\mathbb{T}} w f dx\|_p \leq \|f\|_1 \|w - \langle w \rangle_{\mathbb{T}}\|_p + \|f\|_p \|w - \langle w \rangle_{\mathbb{T}}\|_{p'}$$

where

$$\langle w \rangle_{\mathbb{T}} = \frac{1}{2\pi} \int_{\mathbb{T}} w dx.$$

From John-Nirenberg theorem, we have

$$\|w - \langle w \rangle_{\mathbb{T}}\|_{p'} \lesssim p' \|w\|_{BMO}, \quad p' > 2,$$

which proves (22). To prove (23), we will interpolate between two bounds: the standard Coifman-Rochberg-Weiss theorem for $p = 2$ ([7],[21])

$$\|[H, w]\|_{2,2} \leq C \quad (24)$$

and the following estimate

$$|\{x : |([H, w]f)(x)| > \alpha\}| \leq C \int_{\mathbb{T}} \frac{|f(t)|}{\alpha} \left(1 + \log^+ \left(\frac{|f(t)|}{\alpha}\right)\right) dt \quad (25)$$

(See [16], the estimate was obtained on \mathbb{R} for smooth f with compact support. The proof, however, is valid for \mathbb{T} as well and, e.g., piece-wise smooth continuous f). Assume a smooth f is given and denote $\lambda_f(t) = |\{x : |f(x)| > t\}|$, $t \geq 0$. Take $A > 0$ and consider $f_A = f \cdot \chi_{|f| \leq A} + A \cdot \text{sgn} f \cdot \chi_{|f| > A}$, $g_A = f - f_A$. Let $T = [H, w]$. Then,

$$\|Tf\|_p^p = p \int_0^\infty t^{p-1} \lambda_{Tf}(t) dt \leq p \int_0^\infty t^{p-1} \lambda_{Tf_A}(t/2) dt + p \int_0^\infty t^{p-1} \lambda_{Tg_A}(t/2) dt = I_1 + I_2$$

Let $A = t$. From Chebyshev inequality and (24), we get

$$I_1 \lesssim \int_0^\infty t^{p-3} \|f_A\|_2^2 dt = 2 \int_0^\infty t^{p-3} \int_0^A \xi \lambda_f(\xi) d\xi dt \lesssim (2-p)^{-1} \int_0^\infty \xi^{p-1} \lambda_f(\xi) d\xi \lesssim (2-p)^{-1} \|f\|_p^p$$

For I_2 , we use (25) (notice that g_A is continuous and piece-wise smooth)

$$I_2 \lesssim - \int_0^\infty t^{p-1} \int_0^\infty \frac{\xi}{t} \left(1 + \log^+ \frac{\xi}{t}\right) d\lambda_{g_A}(\xi) \lesssim \|f\|_p^p + \int_0^\infty t^{p-1} \int_{2t}^\infty t^{-1} \left(1 + \log^+((\tau - t)/t)\right) \lambda_f(\tau) d\tau \lesssim \|f\|_p^p \int_0^1 \xi^{p-2} \left(1 + \log^+ \frac{1-\xi}{\xi}\right) d\xi$$

We have

$$\int_0^{1/2} \xi^{p-2} \left(1 + \log^+ \frac{1-\xi}{\xi}\right) d\xi \lesssim \int_2^\infty u^{-p} \log u du \lesssim \int_0^\infty e^{-\delta t} t dt \lesssim \delta^{-2}$$

with $\delta = p - 1$. □

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SERGEY DENISOV: DENISOV@WISC.EDU

UNIVERSITY OF WISCONSIN–MADISON
 MATHEMATICS DEPARTMENT
 480 LINCOLN DR., MADISON, WI, 53706, USA

AND

KELDYSH INSTITUTE FOR APPLIED MATHEMATICS, RUSSIAN ACADEMY OF SCIENCES
 MIUSSKAYA PL. 4, 125047 MOSCOW, RUSSIA

KEITH RUSH: JKRUSH@WISC.EDU

UNIVERSITY OF WISCONSIN–MADISON
 MATHEMATICS DEPARTMENT
 480 LINCOLN DR., MADISON, WI, 53706, USA