On Spectrum of a Class of Jacobi Matrices on Graph-Trees and Multiple Orthogonal Polynomials

Alexander I. Aptekarev, Sergey A. Denisov, and Maxim L. Yattselev

Abstract We consider applications of the theory of Multiple Orthogonal Polynomials (MOPs) to the spectral theory of difference self-adjoint operators on rooted trees. We use the coefficients of the recurrence relations for the Angelesco systems of MOPs to generate potentials for general class of the corresponding operators. Here we present asymptotic behavior of the recurrence coefficients for the ray's sequences regime.

1 Introduction

We present the results that recently appeared in [1, 2], where it was shown that the theory of Multiple Orthogonal Polynomials (MOPs) is related to the spectral theory of difference self-adjoint operators on rooted trees (like the theory of orthogonal polynomials is related to the spectral theory of Jacobi matrices). We start this introduction by recalling the necessary definitions and main relations between self-adjoint Jacobi matrices on trees and MOPs. Then we state the main results of the paper.

In what follows, we let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_{\geq 0} := \{0, 1, 2 \ldots, \}$. We write $|\vec{n}| := n_1 + \cdots + n_d$ for $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geq 0}$, and let $\vec{e}_1 = (1, \ldots, 0), \ldots, \vec{e}_d = (0, \ldots, 1)$, $\vec{1} = (1, \ldots, 1) = \vec{e}_1 + \cdots + \vec{e}_d$.

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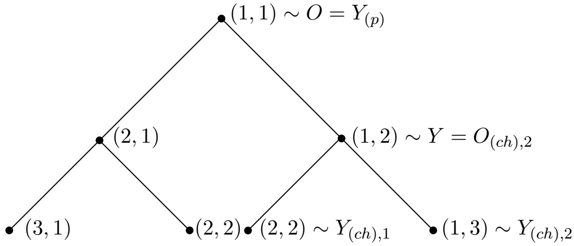

Fig. 1 Three generations of $\mathcal{T}$ (for $d = 2$)

1.1 Definition of Jacobi Operators

Denote by $\mathcal{T}$ an infinite d-homogeneous rooted tree (Cayley tree), and by $\mathcal{V}$ the set of its vertices with O being the root. By untwining increasing paths on $\mathbb{N}^d$ that originate at $\vec{1}$ onto $\mathcal{T}$ with $\vec{1}$ corresponding to O, see Fig. 1 (for $d = 2$), we can define the projection $\Pi : \mathcal{V} \to \mathbb{N}^d$. Under this projection, the increasing non-self-intersecting paths on $\mathbb{N}^d$ are in one-to-one correspondence with non-self-intersecting paths on $\mathcal{T}$. Every vertex $Y \in \mathcal{V}$, $Y \neq O$, has the unique parent, which we denote by $Y_{(p)}$. This allows us to define the following index function:

$$\imath : \mathcal{V} \to \{1, \ldots, d\}, \quad Y \mapsto \imath_Y \text{ such that } \Pi(Y) = \Pi(Y_{(p)}) + \vec{e}_{\imath_Y}, \tag{1}$$

and therefore to distinguish the "older" and the "younger" child of each vertex $Y \in \mathcal{V}$ by denoting $Z = Y_{(ch),\imath_Z}$ when $Y = Z_{(p)}$, see Fig. 1 (for $d = 2$).

Let $\mathsf{P} := \{a_{\vec{n},i}, b_{\vec{n},i}\}_{\vec{n} \in \mathbb{Z}^d_{\geq 0},\, i=1,\ldots,d}$ be a collection of real parameters such that

$$\begin{cases} 0 < a_{\vec{n},i} \text{ for all } \vec{n} \in \mathbb{N}^d, \quad i \in \{1, \ldots, d\}, \\ \displaystyle\sup_{\vec{n} \in \mathbb{N}^d, i \in \{1,\ldots,d\}} a_{\vec{n},i} < \infty, \quad \sup_{\vec{n} \in \mathbb{Z}^d_{\geq 0}, i \in \{1,\ldots,d\}} |b_{\vec{n},i}| < \infty. \end{cases} \tag{2}$$

For a function f on $\mathcal{V}$, we denote by f_Y its value at a vertex $Y \in \mathcal{V}$. Given P satisfying (2) and $\vec{\kappa} \in \mathbb{R}^d$ with $|\vec{\kappa}| = 1$, we define the corresponding Jacobi operator, say $\mathcal{J}_{\vec{\kappa}}$, by

$$\begin{cases} (\mathcal{J}_{\vec{\kappa}} f)_Y := a^{1/2}_{\Pi(Y_{(p)}),\imath_Y} f_{Y_{(p)}} + b_{\Pi(Y_{(p)}),\imath_Y} f_Y + \sum_{i=1}^d a^{1/2}_{\Pi(Y),i} f_{Y_{(ch),i}}, & Y \neq O, \\ (\mathcal{J}_{\vec{\kappa}} f)_O := \sum_{i=1}^d \kappa_i b_{\vec{1}-\vec{e}_i,i} f_O + \sum_{i=1}^d a^{1/2}_{\vec{1},i} f_{O_{(ch),i}}, & Y = O. \end{cases} \tag{3}$$

Thus defined operator $\mathcal{J}_{\vec{\kappa}}$ is bounded and self-adjoint on $\ell^2(\mathcal{V})$.

During the last decade there is an increasing interest to the spectral theory of the self-adjoint operators on the graph-trees.

1.2 Multiple Orthogonal Polynomials and Recurrence Relations

The authors started in [1] to investigate some properties of operator $\mathcal{J}_{\vec{\kappa}}$, defined by (3), (2), where potential $\mathsf{P} := \{a_{\vec{n},i}, b_{\vec{n},i}\}$ is defined by means of the coefficients of recurrence relations of MOPs. We recall the main definitions regarding these notions.

Let $\vec{\mu} := (\mu_1, \dots, \mu_d)$, $\quad d \in \mathbb{N}$ be a vector of positive finite Borel measures defined on $\mathbb{R}$ and given a multi-index $\vec{n} \in \mathbb{Z}^d_{\geq 0}$, $|\vec{n}| \geq 1$. Type I MOPs $\left\{A^{(j)}_{\vec{n}}\right\}^d_{j=1}$ are polynomial coefficients of the linear form

$$Q_{\vec{n}}(x) := \sum_{j=1}^{d} A^{(j)}_{\vec{n}}(x) d\mu_j(x), \quad \deg\left(A^{(i)}_{\vec{n}}\right) < n_i, \quad i \in \{1, \dots, d\}$$

defined by requiring that

$$\int x^l Q_{\vec{n}}(x) = 0, \quad l < |\vec{n}| - 1, \quad A^{(i)}_{\vec{1}-\vec{e}_i} \equiv 0. \tag{4}$$

Type II MOPs $P_{\vec{n}}(x)$, deg $\left(P_{\vec{n}}\right) \leq |\vec{n}|$, are defined by

$$\int P_{\vec{n}}(x) x^l \, \mathrm{d}\mu_i(x) = 0, \quad l < n_i, \quad i \in \{1, \dots, d\}. \tag{5}$$

The polynomials of the first and second type always exist. If $P_{\vec{n}}$ is defined uniquely up to a constant, then the multi-index $\vec{n}$ is called *normal* and we choose the normalization for the polynomial $P_{\vec{n}}$ to be monic: $P_{\vec{n}}(x) = x^{|\vec{n}|} + \cdots$. It turns out that $\vec{n}$ is normal if and only if the following linear form $Q_{\vec{n}}(x)$ is defined uniquely up to multiplication by a constant. In that case, we will normalize the polynomials of the first type by

$$\int_{\mathbb{R}} x^{|\vec{n}|-1} Q_{\vec{n}}(x) = 1\,. \tag{6}$$

We shall say that vector $\vec{\mu}$ is *perfect* if all the multi-indices $\vec{n} \in \mathbb{Z}^d_+$ are normal.

It is known that for the perfect $\vec{\mu}$ the polynomials $P_{\vec{n}}(x)$ and the forms $Q_{\vec{n}}(s)$ satisfy the following Nearest-Neighbor Recurrence Relations (NNRRs):

$$\begin{cases} zP_{\vec{n}}(z) = P_{\vec{n}+\vec{e}_j}(z) + b_{\vec{n},j} P_{\vec{n}}(z) + \sum_{i=1}^d a_{\vec{n},i} P_{\vec{n}-\vec{e}_i}(z), \\ zQ_{\vec{n}}(z) = Q_{\vec{n}-\vec{e}_j}(z) + b_{\vec{n}-\vec{e}_j,j} Q_{\vec{n}}(z) + \sum_{i=1}^d a_{\vec{n},i} Q_{\vec{n}+\vec{e}_i}(z), \end{cases} \quad \text{for each} \quad j \in \{1, \dots, d\}. \tag{7}$$

Here the coefficients $\{a_{\vec{n},i}, b_{\vec{n},i}\}$ have representations

$$a_{\vec{n},j} = \frac{\int_{\mathbb{R}} P_{\vec{n}}\, x^{n_j} d\mu_j}{\int_{\mathbb{R}} P_{\vec{n}-\vec{e}_j}\, x^{n_j-1} d\mu_j}, \qquad b_{\vec{n}-\vec{e}_j,j} = \int_{\mathbb{R}} x^{|\vec{n}|} Q_{\vec{n}} - \int_{\mathbb{R}} x^{|\vec{n}|-1} Q_{\vec{n}-\vec{e}_j}\,. \tag{8}$$

If $d > 1$, unlike in one-dimensional case, we can not prescribe $\{a_{\vec{n},j}\}$ and $\{b_{\vec{n},j}\}$ arbitrarily. In fact, coefficients in (7) satisfy the so-called "consistency conditions" which is a system of nonlinear difference equations. This discrete integrable system and associated Lax pair were studied before.

2 Angelesco Systems and Main Results

2.1 Angelesco Systems and Ray's Limits of NNRR Coefficients

We recall that $\vec{\mu}$ is an *Angelesco* system of measures if

$$\operatorname{supp}\mu_j = \Delta_j := [\alpha_j, \beta_j] : \qquad \Delta_i \cap \Delta_j = \varnothing, \quad i \neq j, \quad i, j = 1, \ldots, d\,, \tag{9}$$

i.e. supports $\{\Delta_i\}$ is the system of d closed segments separated by $d-1$ nonempty open intervals.

The Angelesco systems are important general class of the perfect systems.[1] MOPs with respect to this system were studied by Angelesco in 1919. Perfectness of $\vec{\mu}$ guaranties that the corresponding MOPS satisfy to NNRRs (7). It is not so difficult to see[2] that the corresponding NNRRs coefficients $\mathsf{P} := \{a_{\vec{n},i}, b_{\vec{n},i}\}_{\vec{n}\in\mathbb{Z}^d_{\geq 0},\, i=1,\ldots,d}$ satisfy conditions (2) (detailed proof see in [1]). Thus P generated by Angelesco systems can be used as potentials for general class of bounded and self-adjoint on $\ell^2(\mathcal{V})$ operators $\mathcal{J}_{\vec{\kappa}}$ defined by (3).

Moreover, asymptotic behavior of the recurrence coefficients $\{a_{\vec{n},j}, b_{\vec{n},j}\}$ for the *ray's sequences regime*, namely

$$\mathcal{N}_{\vec{c}} = \{\vec{n}\} \; : \qquad n_i = c_i|\vec{n}| + o\big(\vec{n}\big), \quad i \in \{1, \ldots, d\}, \qquad |\,\vec{c}\,| := \sum_{i=1}^{d} c_i = 1\,, \tag{10}$$

was studied in [1] for $\vec{c} = (c_1, \ldots, c_d) \in (0,1)^d$. We have

Theorem 1 ([1, Theorem 3.5]) *Let $\vec{\mu}$ be Angelesco system* (9)*, such that for each $i \in \{1, \ldots, d\}$ the measure μ_i is absolutely continuous with respect to the Lebesgue measure on Δ_i and that the density $w_i := \mathrm{d}\mu_i(x)/\mathrm{d}x$ extends to a holomorphic and non-vanishing function in some neighborhood of Δ_i .*

Then the ray's limits (10) *of coefficients $\big\{a_{\vec{n},i}, b_{\vec{n},i}\big\}$ from* (7) *exist for any $\vec{c} \in (0,1)^d$.*

$$\lim_{\mathcal{N}_{\vec{c}}} a_{\vec{n},i} = A_{\vec{c},i} \quad \textit{and} \quad \lim_{\mathcal{N}_{\vec{c}}} b_{\vec{n},i} = B_{\vec{c},i}\,, \quad i \in \{1, \ldots, d\}, \tag{11}$$

[1] Perfectness of Angelesco system easily follows directly from (5).

[2] In fact the condition $0 < a_{\vec{n},i}$ for all $\vec{n} \in \mathbb{N}^d$, $\quad i \in \{1, \ldots, d\}$ follows directly from (8).

We remark that expressions for limits $A_{\vec{c},i}$, $B_{\vec{c},i}$ were obtained in [1] as well and we recall them in Sect. 2. The validity of Theorem 1 was deduced in [1] from the obtained there results on the strong asymptotics of the Angelesco MOPs for the ray's regimes with $\vec{c} \in (0,1)^d$.

2.2 Main Results

Here we restrict ourselves by the case $d = 2$. For this case in [2] we extend the results of [1] on the strong asymptotics of the Angelesco MOPs for the ray's regimes with $\vec{c} \in [0,1]^2$. From the obtained in [2] results on the strong asymptotics of the Angelesco MOPs we deduce for the ray's regimes with $\vec{c} \in [0,1]^2$ the following extension of Theorem 1.

Theorem 2 *Let $\vec{\mu}$ be Angelesco system (9) for $d = 2$, satisfying conditions of Theorem 1.*

Then the ray's limits

$$\lim_{\mathcal{N}_c} a_{\vec{n},i} = A_{c,i} \quad \textit{and} \quad \lim_{\mathcal{N}_c} b_{\vec{n},i} = B_{c,i} \tag{12}$$

exist for any $c \in [0,1]$ and $i \in \{1,2\}$, where $\mathcal{N}_c$ is any subsequence of $\mathbb{Z}^2_{\geq 0}$ such that $n_1/|\vec{n}| \to c$ as $|\vec{n}| \to \infty$ along this subsequence.

Of course this very technical study in [2] of the Angelesco MOP's asymptotics in the boundary layers of multi-indices $\vec{n} \in \mathbb{Z}^2_{\geq 0}$ should have a serious motivation. Indeed, it is a spectral theory of Jacobi-matrix operators (3) defined on the graph-tree from Fig. 1.

Now, let $\mathsf{P} := \{a_{\vec{n},i}, b_{\vec{n},i}\}_{\vec{n} \in \mathbb{Z}^2_{\geq 0},\, i=1,2}$ be a collection satisfying (2) for $d = 2$ and the constants $\{A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}\}_{c \in [0,1]}$ are the limits of the NNRRs coefficients for an Angelesco MOPs defined on intervals (Δ_1, Δ_2). We say that $\mathsf{P} \in \mathsf{P}_{Ang}(\Delta_1, \Delta_2)$ if P satisfies (12). In accordance with Theorem 2 class $\mathsf{P}_{Ang}(\Delta_1, \Delta_2)$ is not empty. For the bounded and self-adjoint on $\ell^2(\mathcal{V})$ operators $\mathcal{J}_{\vec{\kappa}}$ defined by (3) with potentials from this class we have the following characterization of the essential spectrum.

Theorem 3 *Let $\mathcal{J}_{\vec{\kappa}}$ be the Jacobi operator defined by (3) corresponding to a collection of parameters $\mathsf{P} \in \mathsf{P}_{Ang}(\Delta_1, \Delta_2)$, then $\sigma_{\text{ess}}(\mathcal{J}_{\vec{\kappa}}) = \Delta_1 \cup \Delta_2$.*

2.3 Expressions for the Ray's Limits

In this subsection we define values of the limits standing at the right-hand sides in (12). For $c \in (0,1)$ these limits were obtained and proven in [1, Theorem 3.5]. To define these limits for $c \in \{0,1\}$ is rather easy because on the marginal rays stand usual orthogonal polynomials, see (5). However to prove these limits when one approaches to the marginal ray from the inside of the lattice is the main technical subject of this paper.

Let $\Delta_1 = [\alpha_1, \beta_1]$ and $\Delta_2 = [\alpha_2, \beta_2]$ be two intervals on the real line such that $\beta_1 < \alpha_2$. Denote by ω_1 and ω_2 the arcsine distributions on Δ_1 and Δ_2, respectively. Then it is known that

$$E(\omega_i, \omega_i) \leq E(\nu, \nu), \quad E(\mu, \nu) := -\int \log|x-y|\, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y),$$

for any probability Borel ν measure on Δ_i. The logarithmic potentials of these measures satisfy

$$\ell_i - V^{\omega_i} \equiv 0 \quad \text{on} \quad \Delta_i,$$

for some constants ℓ_1 and ℓ_2, where $V^\nu(z) := -\int \log|z-x|\, \mathrm{d}\nu(x)$. Now, given $c \in (0, 1)$, define

$$M_c := \big\{(\nu_1, \nu_2) : \ \mathrm{supp}(\nu_i) \subseteq \Delta_i, \ \|\nu_1\| = c, \ \|\nu_2\| = 1-c\big\}. \tag{13}$$

Then, as it was proven by A. Gonchar and E. Rakhmanov, there exists the unique pair of measures $(\omega_{c,1}, \omega_{c,2}) \in M_c$ such that

$$I(\omega_{c,1}, \omega_{c,2}) \leq I(\nu_1, \nu_2), \quad I(\nu_1, \nu_2) := 2E(\nu_1, \nu_1) + 2E(\nu_2, \nu_2) + E(\nu_1, \nu_2) + E(\nu_2, \nu_1), \tag{14}$$

for all pairs $(\nu_1, \nu_2) \in M_c$. It is also known that there exist constants $\ell_{c,i}$, $i \in \{1, 2\}$, such that

$$\begin{cases} \ell_{c,1} - V^{2\omega_{c,1}+\omega_{c,2}} \equiv 0 \text{ on } \mathrm{supp}(\omega_{c,1}), \\ \ell_{c,2} - V^{\omega_{c,1}+2\omega_{c,2}} \equiv 0 \text{ on } \mathrm{supp}(\omega_{c,2}). \end{cases} \tag{15}$$

It is further known from A. Gonchar and E. Rakhmanov that $\mathrm{supp}(\omega_{c,1}) = [\alpha_1, \beta_{c,1}] =: \Delta_{c,1}$ and $\mathrm{supp}(\omega_{c,2}) = [\alpha_{c,2}, \beta_2] =: \Delta_{c,2}$. Thus the intervals $\Delta_{c,i}$ dependent on the parameter c.

Let $\mathfrak{R}_c$, $c \in (0, 1)$, be a 3-sheeted Riemann surface realized as follows: cut a copy of $\overline{\mathbb{C}}$ along $\Delta_{c,1} \cup \Delta_{c,2}$, which henceforth is denoted by $\mathfrak{R}_c^{(0)}$, the second copy of $\overline{\mathbb{C}}$ is cut along $\Delta_{c,1}$ and is denoted by $\mathfrak{R}_c^{(1)}$, while the third copy is cut along $\Delta_{c,2}$ and is denoted by $\mathfrak{R}_c^{(2)}$. These copies are then glued to each other crosswise along the corresponding cuts, see Fig. 2. It can be easily verified that thus constructed Riemann surface has genus 0. We denote by π the natural projection from $\mathfrak{R}_c$ to $\overline{\mathbb{C}}$ and employ the notation $\boldsymbol{z}$ for a generic point on $\mathfrak{R}_c$ with $\pi(\boldsymbol{z}) = z$ as well as $z^{(i)}$ for a point on $\mathfrak{R}_c^{(i)}$ with $\pi(z^{(i)}) = z$. Since $\mathfrak{R}_c$ has genus zero, one can arbitrarily prescribe zero/pole divisors of rational functions on $\mathfrak{R}_c$ as long as the degree of the divisor is zero. Clearly, a rational function with a given divisor is unique up to multiplication by a constant.

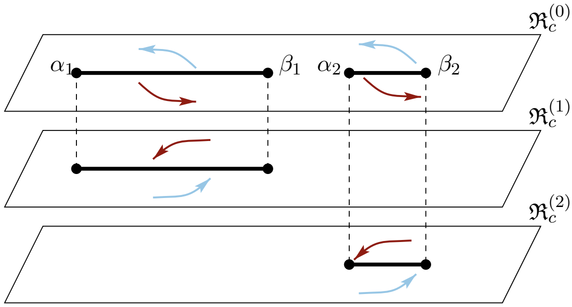

Fig. 2 Surface $\mathfrak{R}_c$ when $\beta_{c,1} = \beta_1$ and $\alpha_{c,2} = \alpha_2$

Proposition 1 *Let $\mathfrak{R}_c$, $c \in (0,1)$, be as above and $\chi_c(\boldsymbol{z})$ be the conformal map of $\mathfrak{R}_c$ onto $\overline{\mathbb{C}}$ such that*

$$\chi_c\big(z^{(0)}\big) = z + \mathcal{O}\big(z^{-1}\big) \quad \text{as} \quad z \to \infty.$$

Further, let numbers $A_{c,1}$, $A_{c,2}$, $B_{c,1}$, $B_{c,2}$, $c \in (0,1)$, be defined by

$$\chi_c\big(z^{(i)}\big) =: B_{c,i} + A_{c,i} z^{-1} + \mathcal{O}\big(z^{-2}\big) \quad \text{as} \quad z \to \infty, \; i \in \{1,2\}. \tag{16}$$

Finally, let $w_i(z) := \sqrt{(z-\alpha_i)(z-\beta_i)}$ be the branch of the corresponding algebraic function holomorphic outside of Δ_i and normalized so that $w_i(z)/z \to 1$ as $z \to \infty$; in that case

$$\varphi_i(z) := \frac{1}{2}\left(z - \frac{\beta_i + \alpha_i}{2} + w_i(z)\right) \tag{17}$$

is the conformal map of $\overline{\mathbb{C}} \setminus \Delta_i$ onto the complement of the disk of radius $(\beta_i - \alpha_i)/4$ satisfying $\varphi_i(z) = z + \mathcal{O}(1)$ as $z \to \infty$. Then it holds that

$$\lim_{c\to 0} \begin{cases} A_{c,2} = \big[(\beta_2 - \alpha_2)/4\big]^2 & =: A_{0,2}, \\ B_{c,2} = (\beta_2 + \alpha_2)/2 & =: B_{0,2}, \\ A_{c,1} = 0 & =: A_{0,1}, \\ B_{c,1} = B_{0,2} + \varphi_2(\alpha_1) & =: B_{0,1}, \end{cases} \tag{18}$$

and analogous limits hold when $c \to 1$. Moreover, all the constants $A_{c,1}$, $A_{c,2}$, $B_{c,1}$, $B_{c,2}$ are continuous functions of the parameter $c \in [0,1]$.

It is worth to notice that the constants $A_{c,1}$ and $A_{c,2}$ are always positive. Indeed, denote by $\boldsymbol{\alpha}_1, \boldsymbol{\beta}_{c,1}, \boldsymbol{\alpha}_{c,2}, \boldsymbol{\beta}_2$ the ramification points of $\mathfrak{R}_c$ with natural projections $\alpha_1, \beta_{c,1}, \alpha_2, \beta_{c,2}$, respectively. Then the symmetries of $\mathfrak{R}_c$ and $\chi_c(\boldsymbol{z})$ yield that $\chi_c(\boldsymbol{z})$ is real and changes from $-\infty$ to ∞ when $\boldsymbol{z}$ moves along the cycle

$$\infty^{(0)} \to \boldsymbol{\alpha}_1 \to \infty^{(1)} \to \boldsymbol{\beta}_{c,1} \to \boldsymbol{\alpha}_{c,2} \to \infty^{(2)} \to \boldsymbol{\beta}_2 \to \infty^{(1)}$$

whose natural projection is the extended real line. Thus, $\chi_c(z)$ is increasing when it moves past $\infty^{(1)}$ and $\infty^{(2)}$, which yields the claim (this argument also shows that $B_{c,1} < B_{c,2}$).

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