A new life of the classical Szegő formula

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Abstract: We collect several applications of a new version of the classical Szegö formula for orthogonal polynomials. The applications are concerned with the spectral theory of Krein strings, scattering theory for Dirac systems, and triangular factorizations of positive Wiener-Hopf operators.

1 Introduction

Let μ be a probability measure on the unit circle $\mathbb{T}$ of the complex plane $\mathbb{C}$, and let $\{\varphi_n\}$ be the family of orthonormal polynomials generated by μ. It is well-known that φ_n satisfy recurrence relations

$$\sqrt{1-|a_n|^2}\cdot\varphi_{n+1}^* = \varphi_n^* - z a_n \varphi_n, \qquad \varphi_0 = \varphi_0^* = 1, \qquad n \geq 0, \tag{1}$$

where $\varphi_n^*(z) = z^n\overline{\varphi_n(1/\bar{z})}$ and $\{a_n\}$ is a sequence of numbers in the unit disk $\mathbb{D} = \{z \in \mathbb{C}: \ |z| < 1\}$ depending only on μ. Assuming μ has the form $\mu = w\,dm + \mu_s$ for some density w with respect to the Lebesgue measure m on $\mathbb{T}$ and a singular part μ_s, let us introduce the function

$$\mathcal{K}(\mu, z) = \log \mathbf{P}(\mu, z) - \mathbf{P}(\log w, z), \qquad z \in \mathbb{D}. \tag{2}$$

As usual, we denote by $\mathbf{P}$ the operator of harmonic extension to $\mathbb{D}$:

$$\mathbf{P}(\mu, z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{\xi}z|^2}\, d\mu(\xi).$$

We also set $\mathbf{P}(v, z) = \mathbf{P}(v\,dm, z)$ for $v \in L^1(\mathbb{T})$. The classical Szegő formula can be written in the following way:

$$\mathcal{K}(\mu, 0) = -\int_{\mathbb{T}} \log w\, dm = -\log \prod_{n\geq 0}(1-|a_n|^2). \tag{3}$$

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It turns out that one can find expression for $\mathcal{K}(\mu, z)$ for all $z \in \mathbb{D}$:

$$\mathcal{K}(\mu, z) = \log \prod_{n=1}^{\infty} \frac{1 - |z f_k(z)|^2}{1 - |f_k(z)|^2},$$

where f_k are the Schur functions of μ, see [**4**]. More generally, the formula can be extended to the case where μ is a spectral measure of a self-adjoint differential operator and $\mathcal{K}(\mu, z)$ is defined for z in the upper half-plane $\mathbb{C}^+$. This approach turns out to be very fruitful: below we discuss several long standing problems that were solved by using the new version of formula (3).

2 Krein strings

The Krein string equation has the form

$$-y''(t, \lambda) = \lambda \rho(t) y(t, \lambda), \qquad t \in [0, L), \qquad \lambda \in \mathbb{C}.$$

Here $L > 0$ is the length of the string, and ρ denotes its density which is supposed to be an arbitrary σ-finite measure on the positive half-axis, $\mathbb{R}_+ = [0, +\infty)$. We do not exclude "wild" cases where the absolutely continuous part ρ_{ac} of ρ is zero. So, the piece of string $[0, x)$ has mass $M(x) = \rho([0, x))$ and M can be arbitrary positive non-decreasing left-continuous function on $\mathbb{R}_+$. The string is called long if

$$L + \lim_{x \to L} M(x) = \infty.$$

It is possible to associate a differential operator with each string $[M, L]$. Its main spectral measure, defined in terms of Weyl function, is called the spectral measure of a string. It turns out that the string is completely determined by its spectral measure. This makes interesting the problem of translating various properties of the spectral measure into the properties of the mass function M. The problem is usually very hard. It has strong connection to the problem of determining properties of the sequence of recurrence coefficients $\{a_n\}$ in (1) from properties of the orthogonality measure μ on $\mathbb{T}$. Note that (3) implies that

$$\log w \in L^1(\mathbb{T}) \quad \text{if and only if} \quad \sum_{n \geq 0} |a_n|^2 < \infty.$$

This explains the reason to expect a possibility of describing Krein strings whose spectral measures $\sigma = v\,dx + \sigma_s$ satisfy the following Szegő-type condition in the domain $\mathbb{C} \setminus \mathbb{R}_+$:

$$\int_0^{\infty} \frac{\log v(x)}{(1+x)\sqrt{x}}\, dx > -\infty. \tag{4}$$

Note, however, that the point 0 is an inner point of the open unit disk $\mathbb{D}$, while it is a boundary point for the domain $\mathbb{C} \setminus \mathbb{R}_+$. In particular, we cannot hope to use a Szegő formula at 0 for measures on the boundary of $\mathbb{C} \setminus \mathbb{R}_+$. Instead, we need to generalize (3) to other points of $\mathbb{D}$ (therefore avoiding usage of properties of orthogonal polinomials/solutions of Krein system related to the special value $\lambda = 0$ of the spectral parameter) and then find its analogue for $\mathbb{C} \setminus \mathbb{R}_+$. In fact, the quantity in (4) is nothing but $\mathcal{K}(\sigma, -1)$ for a properly defined entropy function $\mathcal{K}$ for $\mathbb{C} \setminus \mathbb{R}_+$ and a normalized measure σ. The idea to prove a formula for $\mathcal{K}(\mu, z)$ for orthogonal polynomials on $\mathbb{T}$ and then search for its variant for Krein strings looks very natural, but the real history is somewhat opposite: such a formula was first established for Krein strings and then translated to the unit disk (where its proof is significantly easier). Here is the result for Krein strings [**1**].

Theorem 1 *Let $[M, L]$ be a long string and let $\sigma = v\,dx + \sigma_s$ be the spectral measure of $[M, L]$. Then σ satisfies (4) if and only if $\sqrt{\rho_{ac}} \notin L^1(\mathbb{R}_+)$ and*

$$\sum_{n=0}^{+\infty}\Big(L_n M_n - 4\Big) < \infty, \quad L_n = t_{n+2} - t_n, \quad M_n = M(t_{n+2}) - M(t_n), \tag{5}$$

where $t_n = \min\big\{x \ge 0 :\ n = \int_0^t \sqrt{\rho_{ac}(t)}\,dt\big\}$.

It is interesting to note that quantities L_n, M_n have physical meaning: L_n is the length of a piece of the string which is covered by a traveling wave during the period $[n, n+2)$ of time, M_n is the mass of this piece. We have $M_n L_n = 4$ for any homogeneous string.

3 Scattering theory for Dirac operators

The one-dimensional Dirac operator on $\mathbb{R}_+$ is defined by

$$\mathcal{D}_Q : X \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) X' + QX, \qquad Q = \left(\begin{smallmatrix} q_1 & q_2 \\ q_2 & -q_1 \end{smallmatrix}\right) \tag{6}$$

on a dense subset of Lebesgue space $L^2(\mathbb{R}_+, \mathbb{C}^2)$ of squared summable functions on $\mathbb{R}_+$ with values in $\mathbb{C}^2$. This is one of the simplest self-adjoint differential operators, and many of analytic tools can be applied to investigation of its properties. As an example, let us consider the problem of the existence of wave operators

$$W_\pm(\mathcal{D}_Q, \mathcal{D}_0) = \lim_{t \to \mp\infty} e^{it\mathcal{D}_Q} e^{-it\mathcal{D}_0}. \tag{7}$$

Wave operators are basic objects of the scattering theory, their existence (that is, existence of the limits above in the strong operator topology) and completeness (that is, unitarity of $W_\pm$ as operators between the absolutely continuous subspaces of $\mathcal{D}_0$, $\mathcal{D}_Q$) are the main questions of interest.

It is known that the limit in (7) for potentials Q with entries in $L^p(\mathbb{R}_+)$, $1 \le p \le 2$, can be expressed in terms of the Szegő function of the spectral measure μ of $\mathcal{D}_Q$. The Szegő function of a Poisson summable measure $\mu = w dx + \mu_s$ on the real line $\mathbb{R}$ such that

$$\mathcal{K}(\mu, i) = \log\left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2}\right) - \frac{1}{\pi}\int_{\mathbb{R}} \frac{\log w(x)}{1+x^2}\,dx < +\infty \tag{8}$$

is the outer function in $\mathbb{C}^+$ satisfying $|D_\mu|^2 = w$ on $\mathbb{R}$. Let χ_E is the indicator function of a set of full Lebesgue measure on $\mathbb{R}$ such that $\mu_s(E) = 0$. Denote by$\mathcal{F}_Q$, $\mathcal{F}_0$ the Fourier transforms generated by generalized eigensolutions of $\mathcal{D}_Q$, $\mathcal{D}_0$, correspondingly. We have

$$W_-(\mathcal{D}_Q, \mathcal{D}_0) = \gamma \mathcal{F}_Q^{-1}\chi_E D_\mu^{-1}\mathcal{F}_0, \qquad W_+(\mathcal{D}_Q, \mathcal{D}_0) = \bar{\gamma}\mathcal{F}_Q^{-1}\chi_E \overline{D_\mu^{-1}}\mathcal{F}_0 \tag{9}$$

for every Q with entries in $L^p(\mathbb{R}_+)$, $1 \le p \le 2$. This formula, well-known for specialists in scattering theory, immediately rises two questions:

(a) does it hold for any Dirac operator with spectral measure such that $\mathcal{K}(\mu, i) < \infty$?

(b) how to describe potentials Q that generate measures μ such that $\mathcal{K}(\mu, i) < \infty$?

Both questions were open for a long time and got their answer only recently. To simplify the presentation, we assume $q_1 = 0$ in theorems below, while the general case can be covered as well.

Theorem 2 *Let q be a real-valued function on $\mathbb{R}_+$ such that $q \in L^1[0,r]$ for every $r > 0$, and let $Q = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}$. Assume that the spectral measure μ of $\mathcal{D}_Q$ satisfy (8). Then the wave operators $W_\pm(\mathcal{D}_Q, \mathcal{D}_0)$ exist, complete, and are given by (9).*

Theorem 3 *For $n \geq 0$, define the functions $g_n(t) = \exp\left(2\int_n^t q(s)\,ds\right)$, $t \in [n, n+2)$. The spectral measure of $\mathcal{D}_Q$ satisfies (8) if and only if*

$$\sum_{n\geq 0}\left(\int_n^{n+2} g_n(t)\,dt \cdot \int_n^{n+2}\frac{dt}{g_n(t)} - 4\right) < \infty. \tag{10}$$

Both theorems were obtained [**3**] by systematically using the entropy function $\mathcal{K}(\mu, z)$. In fact, we get a family of such functions $\mathcal{K}(\mu_r, z)$, $r \geq 0$ if we denote by μ_r the spectral measure of the operator $\mathcal{D}_Q$ on $[r, +\infty)$. We would like to mention that $\mathcal{K}(\mu_r, z)$ satisfies a nonlinear differential equation as a function in $r \in \mathbb{R}_+$.

4 Triangular factorization of Wiener-Hopf operators

Another classical problem that can be solved by using the entropy function $\mathcal{K}(\mu, z)$ concerns triangular factorization of Wiener-Hopf operators. In the abstract setting, given a bounded invertible positive operator T on a separable Hilbert space H, one may ask about existence of a bounded invertible operator A that is upper triangular with respect to a given chain of subspaces $\mathcal{L}$ and factorizes T into the product $T = A^*A$. The words "upper triangular with respect to $\mathcal{L}$" simply mean that $AE \subset E$ for every $E \in \mathcal{L}$. The theory of triangular factorization of positive operators was developed by Gohberg and Krein in 60's. It was fairly nontrivial problem if there are nonfactorable bounded invertible positive oprators. The affirmative answer to this question was given by Larson in 1985. He showed that every uncountable chain $\mathcal{L}$ gives rise to a nonfactorable operator. The approach of Larson is highly nonconstructive, and it is desirable to find a concrete example of a nonfactorable operator to understood better this phenomenon. The simplest uncountable chain one can imagine is the chain of subspaces $\{L^2[0,r]\}_{r>0}$ in $L^2(\mathbb{R}_+)$. It turns out that the problem of triangular factorization of positive bounded Wiener-Hopf operators

$$W_\psi : f \mapsto \int_{\mathbb{R}_+} \psi(t-s) f(s)\,ds, \qquad f \in L^2(\mathbb{R}_+)$$

with respect to this chain can be reformulated in terms of the spectral theory of Krein strings and more general objects - canonical Hamiltonian systems. This fact was observed by L.Sakhnovich who posed this problem for Wiener-Hopf operators in 1994. More precisely, a real distribution ψ generates factorable with respect to the chain $\{L^2[0,r]\}_{r>0}$ bounded positive operator W_ψ if and only if the Krein string corresponding to the spectral measure $\sigma = \check{\psi}(\sqrt{x})\,dx$ on $\mathbb{R}$ has absolutely continuous mass distribution M. Here $\check{\psi}$ is the inverse Fourier transform of ψ. Since W_ψ is bounded, positive and invertible, we have $c_1 \leq \check{\psi}(x) \leq c_2$ for almost all $x \in \mathbb{R}$ and some constants c_1, c_2. So, the spectral measure σ has a density separated from zero and infinity. Asymptotic analysis of the entropy function $\mathcal{K}(\sigma, -x)$ as $x \to +\infty$ then gives us information about quantities

$$\sum_{n\geq 0}(M_{n,\varepsilon}L_{n,\varepsilon} - 4\varepsilon^2),$$

where, as before, $t_s = \min\bigl\{x \geq 0 :\ s = \int_0^t \sqrt{\rho_{ac}(t)}\,dt\bigr\}$, and we set $L_{n,\varepsilon} = t_{(n+2)\varepsilon} - t_{n\varepsilon}$, $M_{n,\varepsilon} = M(t_{(n+2)\varepsilon}) - M(t_{n\varepsilon})$. Analysing these quantities when ε tends to zero, we get the following conclusion [**2**].

Theorem 4 *Every bounded invertible positive Wiener-Hopf operator W_ψ on $L^2(\mathbb{R}_+)$ admits triangular factorization with respect to the chain $\{L^2[0,r]\}_{r>0}$.*

The reader can find more information in papers listed below.

References

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