De Branges canonical systems with finite logarithmic integral

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Abstract: Krein – de Branges spectral theory provides a correspondence between canonical Hamiltonian systems and measures on the real line with finite Poisson integral. We revisit this area by giving a description of canonical Hamiltonian systems whose spectral measures have logarithmic integral converging over the real line. Our result can be viewed as a spectral version of the classical Szegő theorem in the theory of polynomials orthogonal on the unit circle. It extends Krein–Wiener completeness theorem, a key fact in the prediction of stationary Gaussian processes.

1 Main results

In this note, which is based on two publications [**1, 2**] and contains the formulation of two main results from aforementioned papers, we revisit the spectral theory of de Branges' canonical system, which is defined by the system of differential equations of the form

(1) $\quad J\frac{d}{dt}M(t,z) = z\mathcal{H}(t)M(t,z),\ M(0,z) = I_{2\times 2} = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\ J = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right),\ t\ge 0,\ z\in\mathbb{C}.$

The 2×2 matrix-function $\mathcal{H}$ on $\mathbb{R}_+ = [0,+\infty)$ is called the Hamiltonian of canonical system (1). Our assumptions of $\mathcal{H}$ are:

(a) $\mathcal{H}(t)\ge 0$ and $\operatorname{trace}\mathcal{H}(t) > 0$ for Lebesgue almost every $t\in\mathbb{R}_+$,

(b) the entries of $\mathcal{H}$ are real measurable functions absolutely summable on compact subsets of $\mathbb{R}_+$.

In 1960's, L. de Branges developed his theory of Hilbert spaces of entire functions. One result in this area is the theorem that establishes a bijection between Hamiltonians $\mathcal{H}$ in (1) and nonconstant analytic functions in $\mathbb{C}_+ = \{z\in\mathbb{C}:\ \operatorname{Im} z > 0\}$ with nonnegative imaginary part. Every such function is generated by a nonnegative measure on the real line. Below, we make a another step in de Branges' theory by identifying Hamiltonians that correspond to measures in the Szegő class, i.e., the measures whose logarithmic integral converges over $\mathbb{R}$.

We start with some definitions. A Hamiltonian $\mathcal{H}$ on $\mathbb{R}_+$ is called singular if

$$\int_0^{+\infty}\operatorname{trace}\mathcal{H}(t)\,dt = +\infty.$$

Two Hamiltonians $\mathcal{H}_1$, $\mathcal{H}_2$ on $\mathbb{R}_+$ are called equivalent if there exists an increasing absolutely continuous function η defined on $\mathbb{R}_+$ such that $\eta(0)=0$, $\lim_{t\to+\infty}\eta(t) = +\infty$, and $\mathcal{H}_2(t) = \eta'(t)\mathcal{H}_1(\eta(t))$ for Lebesgue almost every $t\in\mathbb{R}_+$. Clearly, $\eta(t)$ rescales the variable t. We say that Hamiltonian $\mathcal{H}$ is trivial if there is a non-negative matrix A with $\operatorname{rank} A = 1$, such that $\mathcal{H}$ is equivalent to A, i.e., $\mathcal{H}(t) = \eta'(t)A$ for a.e. $t\in\mathbb{R}_+$, where

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η is an increasing absolutely continuous function on $\mathbb{R}_+$, which satisfies $\eta(0) = 0$ and $\lim_{t\to+\infty} \eta(t) = +\infty$. If Hamiltonian is not trivial, it is called nontrivial.

We recall that function m belongs to the Herglotz-Nevanlinna class $\mathcal{N}(\mathbb{C}_+)$ if it is analytic in $\mathbb{C}_+$ and $\operatorname{Im} m(z) \ge 0$ for $z \in \mathbb{C}_+$. It is well-known, that $m \in \mathcal{N}(\mathbb{C}_+)$ if and only if it admits the following integral representation

$$m(z) = \frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{x-z} - \frac{x}{x^2+1}\right) d\mu(x) + bz + a, \quad z \in \mathbb{C}_+, \tag{2}$$

where $b \ge 0$, $a \in \mathbb{R}$, and μ is a Radon measure on $\mathbb{R}$, which satisfies

$$\int_{\mathbb{R}} \frac{d\mu}{1+x^2} < \infty. \tag{3}$$

Those measures on $\mathbb{R}$ that satisfy (3) are called Poisson-finite. The class $\mathcal{N}(\mathbb{C}_+)$ appears naturally in the theory of canonical Hamiltonian systems. Let $\mathcal{H}$ be a nontrivial and singular Hamiltonian. Given condition (b) on $\mathcal{H}$, there exists unique matrix-valued function M that solves (1). Denote by $\Theta^\pm$, $\Phi^\pm$ its entries so that

$$M(t,z) = (\Theta(t,z), \Phi(t,z)) = \begin{pmatrix} \Theta^+(t,z) & \Phi^+(t,z) \\ \Theta^-(t,z) & \Phi^-(t,z) \end{pmatrix}. \tag{4}$$

Fix a parameter $\omega \in \mathbb{R}\cup\{\infty\}$. The Titchmarsh-Weyl function of $\mathcal{H}$ is defined by

$$m(z) = \lim_{t\to+\infty} \frac{\omega\Phi^+(t,z) + \Phi^-(t,z)}{\omega\Theta^+(t,z) + \Theta^-(t,z)}, \qquad z \in \mathbb{C}_+, \tag{5}$$

where the fraction $\frac{\infty c_1 + c_2}{\infty c_3 + c_4}$ for non-zero numbers c_1, c_3 is interpreted as c_1/c_3. In Titchmarsh-Weyl's theory for canonical systems, it is proved that the expression under the limit in (5) is well-defined for large $t > 0$ (i.e., the denominator is non-zero) for every given singular nontrivial Hamiltonian $\mathcal{H}$. Moreover, the limit $m(z)$ exists, does not depend on ω, m is analytic in $z \in \mathbb{C}_+$ and has non-negative imaginary part, i.e., $m \in \mathcal{N}(\mathbb{C}_+)$. In particular, m admits representation (2). The measure μ in (2) is called the spectral measure for the Hamiltonian $\mathcal{H}$. It is obvious that equivalent Hamiltonians have equal Titchmarsh-Weyl functions.

Now we can state the result of de Branges that establishes a bijection between Hamiltonians and Herglotz-Nevanlinna functions.

Theorem 1 (de Branges) *For every nonconstant function $m \in \mathcal{N}(\mathbb{C}_+)$, there exists a singular nontrivial Hamiltonian $\mathcal{H}$ on $\mathbb{R}_+$ such that m is the Titchmarsh-Weyl function* (5) *for $\mathcal{H}$. Moreover, any two singular nontrivial Hamiltonians $\mathcal{H}_1$, $\mathcal{H}_2$ on $\mathbb{R}_+$ generated by m are equivalent.*

For trivial Hamiltonians, function m is a real constant. In fact, in that case, one can solve (1) explicitly and this calculation shows that $m(z) = const \in \mathbb{R}\cup\infty$. For example, $\mathcal{H} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ gives

$$\Theta^+ = 1, \quad \Theta^- = -zt, \quad \Phi^+ = 0, \quad \Phi^- = 1, \tag{6}$$

so $m = 0$. Similarly, if $\mathcal{H} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then $\Theta^+ = 1, \Theta^- = 0, \Phi^+ = zt, \Phi^- = 0$ and we let $m = \infty$.

Given a Poisson-finite measure μ on $\mathbb{R}$, we will denote by w the density of μ with respect to the Lebesgue measure dx on $\mathbb{R}$, and by $\mu_{\mathbf{s}}$ the singular part of μ, so that

$\mu = w\,dx + \mu_{\mathbf{s}}$. Our goal is to characterize singular nontrivial Hamiltonians whose spectral measures have finite logarithmic integral, i.e., the integral

$$\int_{\mathbb{R}} \frac{\log w(x)}{1+x^2}\,dx$$

converges. The trivial bound $\log w \le w$ shows that logarithmic integral of a Poisson-finite measure can diverge only to $-\infty$. It will be convenient to call the set of all measures with finite logarithmic integral the Szegő class $\mathrm{Sz}(\mathbb{R})$, i.e.,

$$\mathrm{Sz}(\mathbb{R}) = \Big\{\mu : \int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} + \int_{\mathbb{R}} \frac{|\log w(x)|}{1+x^2}\,dx < +\infty\Big\}.$$

If $m \in \mathcal{N}(\mathbb{C}_+)$ and measure μ in (2) is in Szegő class, we can define

$$(7)\quad \mathcal{K}_m = \log \operatorname{Im} m(i) - \frac{1}{\pi}\int_{\mathbb{R}} \frac{\log w(x)}{1+x^2}\,dx = \log\left(b + \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\mu}{1+x^2}\right) - \frac{1}{\pi}\int_{\mathbb{R}} \frac{\log w(x)}{1+x^2}\,dx\,.$$

One can use $b \ge 0$ and Jensen's inequality to show that $\mathcal{K}_m \ge 0$. Notice that $\mathcal{K}_m = 0$ if and only if m is a constant with positive imaginary part.

Next, we introduce the class of Hamiltonians that characterizes measures in Szegő class. If $\mathcal{H}$ is such that $\sqrt{\det \mathcal{H}} \notin L^1(\mathbb{R}_+)$, define

$$(8)\quad \widetilde{\mathcal{K}}(\mathcal{H}) = \sum_{n=0}^{\infty}\left(\det \int_{\eta_n}^{\eta_{n+2}} \mathcal{H}(t)dt - 4\right), \qquad \eta_n = \min\Big\{t \ge 0 : \int_0^t \sqrt{\det \mathcal{H}(s)}\,ds = n\Big\}.$$

Since the entries of $\mathcal{H}$ are locally summable functions, the function $t \mapsto \sqrt{\det \mathcal{H}(t)}$ is also locally summable on $\mathbb{R}_+$ and $\{\eta_n\}$ make sense. One can check the following bound:

$$\det \int_{\eta_n}^{\eta_{n+2}} \mathcal{H}(t)\,dt \ge \left(\int_{\eta_n}^{\eta_{n+2}} \sqrt{\det \mathcal{H}(t)}\,dt\right)^2 = 4, \quad n \ge 0\,.$$

This inequality shows that the series in (8) contains only non-negative terms and hence its sum $\widetilde{\mathcal{K}}(\mathcal{H}) \in \mathbb{R}_+ \cup \{+\infty\}$ is well-defined but could be $+\infty$, in general. Actually, $\widetilde{\mathcal{K}}(\mathcal{H})$ can be rewritten in the form reminiscent of matrix A_2 Muckenhoupt condition. Roughly speaking, $\widetilde{\mathcal{K}}(\mathcal{H})$ measures how fast the entries of $\mathcal{H}$ oscillate. In fact, we have $\widetilde{\mathcal{K}}(\mathcal{H}) = 0$ if and only if the Hamiltonian $\mathcal{H}$ is equivalent to a constant positive matrix. Notice that if the Hamiltonian is trivial then its determinant is zero and $\widetilde{\mathcal{K}}$ is undefined. Define the class $\mathbf{H}$ of Hamiltonians by

$$\mathbf{H} = \Big\{\text{singular nontrivial } \mathcal{H} : \sqrt{\det \mathcal{H}} \notin L^1(\mathbb{R}_+),\ \widetilde{\mathcal{K}}(\mathcal{H}) < +\infty\Big\}.$$

Here is our main result:

Theorem 2 *The spectral measure of a singular nontrivial Hamiltonian $\mathcal{H}$ on $\mathbb{R}_+$ belongs to the Szegő class $\mathrm{Sz}(\mathbb{R})$ if and only if $\mathcal{H} \in \mathbf{H}$. Moreover, we have*

$$(9)\qquad c_1\mathcal{K}_m \le \widetilde{\mathcal{K}}(\mathcal{H}) \le c_2\mathcal{K}_m e^{c_2\mathcal{K}_m},$$

for some absolute positive constants c_1, c_2.

The bound (9) is essentially sharp up to numerical values of c_1 and c_2. For $\mathcal{H}$ such that $\widetilde{\mathcal{K}}(\mathcal{H}) \le 1$, (9) gives $\mathcal{K}_m \sim \widetilde{\mathcal{K}}(\mathcal{H})$. Moreover, we can present two examples for both of which $\widetilde{\mathcal{K}}(\mathcal{H}) > 1$. In the first example, we have $\mathcal{K}_m \sim \log(1+L)$ and $\widetilde{\mathcal{K}}(\mathcal{H}) \sim L$, where L is arbitrarily large parameter. This shows that the exponent in the right hand side of

(9) can not be dropped. In the second example, we have $\mathcal{K}_m \sim L$ and $\widetilde{\mathcal{K}}(\mathcal{H}) \sim L$, where L is again arbitrarily large parameter. Thus, the left bound in (9) can not be improved.

Diagonal canonical Hamiltonian systems are related to the equation of a vibrating string:

$$-\frac{d}{dM(t)}\frac{d}{dt}\Big(y(t,z)\Big) = zy(t,z), \qquad t \in [0,L), \qquad z \in \mathbb{C}. \tag{10}$$

Here $0 < L \le +\infty$ is the length of the string, $M : (-\infty, L) \to \mathbb{R}_+$ is an arbitrary non-decreasing and right-continuous function (mass distribution) that satisfies $M(t) = 0$ for $t < 0$. If M is smooth and strictly increasing on $\mathbb{R}_+$, then equation (10) takes the form $-y'' = zM'y$.

We consider those L and M that satisfy the following conditions:

$$L + \lim_{t\to L} M(t) = \infty \quad \text{and} \quad \lim_{t\to L} M(t) > 0\,, \tag{11}$$

where the last bound means that M is not identically equal to zero. If (11) holds, we say that M and L form $[M,L]$ pair. To every $[M,L]$ pair one can relate a string and Weyl-Titchmarsh function q with spectral measure σ supported on the positive half-axis $\mathbb{R}_+$. Theorem 2 can be applied to Krein strings as follows.

Theorem 3 *Let $[M,L]$ satisfy (11) and $\sigma = v\,dx + \sigma_s$ be the spectral measure of the corresponding string. Then, we have $\int_0^\infty \frac{\log v(x)}{(1+x)\sqrt{x}}\,dx > -\infty$ if and only if $\sqrt{M'} \notin L^1(\mathbb{R}_+)$ and*

$$\widetilde{\mathcal{K}}[M,L] = \sum_{n=0}^{+\infty}\Big((t_{n+2}-t_n)(M(t_{n+2})-M(t_n)) - 4\Big) < \infty, \tag{12}$$

where $t_n = \min\big\{t \ge 0 :\ n = \int_0^t \sqrt{M'(s)}\,ds\big\}$.

Condition (11) ensures that the string $[M,L]$ has a unique spectral measure and it does not restrict the generality of Theorem 3: if (11) is violated, then either $M = 0$ and $\int_0^\infty \frac{\log v(x)}{(1+x)\sqrt{x}}\,dx = -\infty$.

Our main result has applications to scattering theory of Dirac and wave equations. It provides the necessary framework to prove existence of wave/modified wave operators under optimal assumptions on the decay of coefficients. See, e.g., [**3**].

References

[1] R. Bessonov, S. Denisov, De Branges canonical systems with finite logarithmic integral, submitted, available on arXiv.

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