CONTINUITY OF WEIGHTED OPERATORS, MUCKENHOUPT A_p WEIGHTS, AND STEKLOV PROBLEM FOR ORTHOGONAL POLYNOMIALS

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Abstract. We consider weighted operators acting on $L^p(\mathbb{R}^d)$ and show that they depend continuously on the weight $w \in A_p(\mathbb{R}^d)$ in the operator topology. Then, we use this result to estimate $L^p_w(\mathbb{T})$ norm of polynomials orthogonal on the unit circle when the weight w belongs to Muckenhoupt class $A_2(\mathbb{T})$ and $p > 2$. The asymptotics of the polynomial entropy is obtained as an application.

To Peter Yuditskii on the occasion of his 65-th birthday

Contents

1. Introduction

Suppose μ is a probability measure on the unit circle $\mathbb{T}$ and $\{\varphi_n(z,\mu)\}$ is the sequence of polynomials orthonormal with respect to μ, i.e.

$$\deg \varphi_n = n, \qquad k_n \stackrel{\text{def}}{=} \text{coeff}_n \varphi_n > 0, \qquad (\varphi_n, \varphi_k)_{L^2_\mu(\mathbb{T})} = \delta_{n,k}, \tag{1.1}$$

where $\delta_{n,k}$ is the Kronecker symbol and $\text{coeff}_j Q$ denotes the coefficient at the power z^j in polynomial Q. One version of Steklov's problem in the theory of orthogonal polynomials can be phrased as follows: given a Banach space X with norm $\|\cdot\|_X$, what regularity of μ is needed to have $\sup_{n\in\mathbb{N}} \|\varphi_n(z,\mu)\|_X < \infty$? This problem has a long history. It goes back to Steklov's conjecture which asked to prove that the sequence $\{p_n(x,\rho)\}$ is bounded for every $x \in (a,b)$, where $\{p_n\}$ are polynomials orthonormal on the interval $[a,b]$ with respect to a weight ρ that satisfies $\rho(x) \geqslant c > 0, x \in [a,b]$. The negative answer to this question was given by Rakhmanov [26, 27] and the sharp estimates on supremum norm were obtained only recently in [2]. If $X = L^2_\mu(\mathbb{T})$, we have $\|\varphi_n\|_X = 1$ by definition. In this paper, we will be concerned with the case when $X = L^p_\mu(\mathbb{T}), p > 2$ and absolutely continuous μ is given by its weight, i.e., $d\mu = \frac{w}{2\pi} d\theta$. It is the natural choice since the space $L^p_w(\mathbb{T})$ interpolates between the trivial case when $X = L^2_w(\mathbb{T})$ and the space $L^\infty_w(\mathbb{T})$, which was studied in [2, 10] for weights w that satisfy Steklov's condition: $w^{-1} \in L^\infty(\mathbb{T})$.

We recall the definition of Muckenhoupt class $A_p(\mathbb{T})$ (see [30], p.194).

Definition. The weight $w \in A_p(\mathbb{T}), p \in (1,\infty)$ if

$$[w]_{A_p(\mathbb{T})} \stackrel{\text{def}}{=} \sup_I \left(\langle w \rangle_I \left(\langle w^{\frac{1}{1-p}} \rangle_I \right)^{p-1} \right) < \infty, \quad \langle w \rangle_I \stackrel{\text{def}}{=} \frac{1}{|I|} \int_I w d\theta\,, \tag{1.2}$$

where I is an arc in $\mathbb{T}$.

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Given $w \in A_2(\mathbb{T})$, we define the following quantity

$$p_{\rm cr}(t) = \sup\{p : \sup_n \|\varphi_n(z,w)\|_{L^p_w(\mathbb{T})} < \infty,\ [w]_{A_2(\mathbb{T})} \leqslant t\}\,.$$

Clearly, $p_{\rm cr}(t)$ is non-increasing on $[1,\infty)$ as a function in t and $p_{\rm cr}(t) \geqslant 2$. The study of how $p_{\rm cr}(t)$ depends on t amounts to considering another more precise version of Steklov's problem. Our first main result is the following theorem.

Theorem 1.1. *We have*

$$p_{\rm cr}(t) > 2, \quad \lim_{t\to 1} p_{\rm cr}(t) = +\infty, \quad \lim_{t\to\infty} p_{\rm cr}(t) = 2\,.$$

Remark. In Appendix, we take w as Fisher-Hartwig weight and prove $p_{\rm cr}(t) < C(t-1)^{-1/2}$ for $t \in (1,2]$. For $t > 2$, the estimate $p_{\rm cr}(t) < 2 + Ct^{-1/6}$ will be obtained in the third section.

The proof of this theorem in the perturbative regime, i.e., when t is close to 1, requires the following general result in the theory of weighted L^p spaces. Consider spaces $L^p(\mathbb{R}^d)$ or $L^p(\mathbb{T}^d)$, $d \in \mathbb{N}$. If $\mathcal{H}$ is a linear bounded operator from $L^p(\mathbb{R}^d)$ to itself, its operator norm will be denoted by $\|\mathcal{H}\|_{p,p}$. Suppose $w \in A_p(\mathbb{R}^d)$ and H is a linear operator that satisfies weighted bound

$$\|w^{1/p} H w^{-1/p}\|_{p,p} \leqslant \mathcal{F}([w]_{A_p}, p), \quad p \in (1,\infty) \tag{1.3}$$

with some $p \in (1,\infty)$ and function $\mathcal{F}(t,p)$ which is continuous in t on $(1,\infty)$. In what follows, we do not need to know $\mathcal{F}$ explcitely. However, $\mathcal{F}$ is known in many applications. For example, the Hunt-Muckenhoupt-Wheeden theorem ([30], p.205) shows that H can be taken as a singular integral operator and recent breakthrough on domination of singular integrals by sparse operators provides the sharp dependence of $\mathcal{F}$ on $[w]_{A_p}$. In particular, for a large class of singular integral operators, one can take $\mathcal{F}(t,p) = C(p)t^{\max(1,(p-1)^{-1})}$, (see, e.g., [19], p.264).

Recall that $f \in {\rm BMO}(\mathbb{R}^d)$ if

$$\|f\|_{{\rm BMO}(\mathbb{R}^d)} \stackrel{\rm def}{=} \sup_B \langle |f - \langle f\rangle_B|\rangle_B < \infty\,,$$

where B denotes a ball in $\mathbb{R}^d$ (see, e.g., p.140 in [30]). The theorem that comes next is a slight improvement of a result by Pattakos and Volberg [24, 25], see also the paper [23] where the sublinear operators were treated.

Theorem 1.2. *Suppose $p \in (1,\infty)$, $[w]_{A_p(\mathbb{R}^d)} < \infty$, $\|f\|_{\rm BMO} < \infty$, and H satisfies (1.3). Consider $w_\delta = we^{\delta f}$. Then, there is $\delta_0(p, [w]_{A_p}, \|f\|_{\rm BMO}) > 0$ such that*

$$\|w_\delta^{1/p} H w_\delta^{-1/p} - w^{1/p} H w^{-1/p}\|_{p,p} < |\delta| C(p, [w]_{A_p}, \|f\|_{\rm BMO}, \mathcal{F})$$

for all $\delta : |\delta| < \delta_0$.

Two corollaries of theorem 1.1 are straightforward and we give their proofs in the end of section 3. To state them, we need a few definitions. Given a weight w, define

$$q_{\rm cr}(w) = \sup\{q : \|w^{-1}\|_{L^q(\mathbb{T})} < \infty\}\,. \tag{1.4}$$

Clearly, if $w \in A_2(\mathbb{T})$ then $q_{\rm cr}(w) > 1$ and $\lim_{[w]_{A_2}\to 1} q_{\rm cr}(w) = \infty$ as follows from the definition of $A_p(\mathbb{T})$ and inclusion of Muckenhoupt classes (see theorem 1 in [32] where the sharp bounds were obtained).

Definition. If $w \in L^1(\mathbb{T})$ and it has finite logarithmic integral, i.e., $\log w \in L^1(\mathbb{T})$, we define function D, the Szegő function, as an outer function in $\mathbb{D}$ that satisfies

$$|D|^2 = w\,. \tag{1.5}$$

The formula for D is

$$D(z) = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1+\bar{\xi}z}{1-\bar{\xi}z} \log\sqrt{w(\theta)}d\theta\right),\ \xi = e^{i\theta}\,, z \in \mathbb{D}\,. \tag{1.6}$$

Remark. If $w \in A_2(\mathbb{T})$, then $w^{-1} \in L^1(\mathbb{T})$. Thus, $\log w \in L^1(\mathbb{T})$ and D is well-defined.

Given a polynomial Q of degree at most n, its reversed polynomial Q^* is defined by $Q^* = z^n\overline{Q(1/\bar{z})}$. Notice that the map $Q \mapsto Q^*$ depends on n. Our first corollary establishes the asymptotics of $\{\varphi_n^*\}$ (and thus of $\{\varphi_n\}$ since $\varphi_n(\xi) = \xi^n\overline{\varphi_n^*(\xi)}$ if $\xi \in \mathbb{T}$).

Corollary 1.3. *Suppose* $[w]_{A_2} < \infty$ *and* $\|\frac{w}{2\pi}\|_1 = 1$*, then*

$$\lim_{n\to\infty} \|\varphi_n^* - D^{-1}\|_{L^p_w(\mathbb{T})} = 0$$

for every $p \in \Big[2, \min\Big(p_{\rm cr}([w]_{A^2}), 2(1 + q_{\rm cr}(w))\Big)\Big)$.

Another application of theorem 1.1 has to do with the asymptotics of polynomial entropy $E(n, \mu)$, which is defined by

$$E(n, \mu) = \int_{\mathbb{T}} |\varphi_n(\xi, \mu)|^2 \log |\varphi_n(\xi, \mu)| d\mu\,,$$

where $\xi = e^{i\theta}$, $\theta \in [-\pi, \pi)$.

Corollary 1.4. *If* $w \in A_2(\mathbb{T})$*, then*

$$\lim_{n\to\infty} E(n, w) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \log w d\theta\,.$$

Given a probability measure μ on $\mathbb{T}$, let F be defined by

$$F(z) = \int_{\mathbb{T}} \frac{1 + \bar{\xi} z}{1 - \bar{\xi} z} d\mu, \quad \xi = e^{i\theta}\,. \tag{1.7}$$

Notice that $\operatorname{Re} F > 0$ in $\mathbb{D}$ and $F(0) = 1$. For $\alpha \in \mathbb{T}$, consider the following one-parameter family (see, e.g., [28], p.36, formula (1.3.90))

$$F_\alpha(z) \overset{\text{def}}{=} \frac{\zeta + F(z)}{1 + \zeta F(z)}\,, \quad \zeta = \frac{1 - \alpha}{1 + \alpha} \in i(\mathbb{R} \cup \infty)\,.$$

Function F_α also has positive real part in $\mathbb{D}$ and $F_\alpha(0) = 1$, so

$$F_\alpha(z) = \int_{\mathbb{T}} \frac{1 + \bar{\xi} z}{1 - \bar{\xi} z} d\mu_\alpha\,,$$

which defines the family of Aleksandrov-Clark measures $\{\mu_\alpha\}$. Taking $z = 0$, we see that μ_α is a probability measure. If $\alpha = -1$, then $F_{-1} = 1/F$ and the resulting measure is called dual for μ, we will use notation $\mu_{\rm dual} (= \mu_{-1})$ for it. Measure $\mu_{\rm dual}$ plays an important role in the theory of polynomials orthogonal on the circle. In fact, the polynomials of the second kind $\{\psi_n\}$ defined by

$$\psi_n(z) = \int_{\mathbb{T}} \frac{1 + z\bar{\xi}}{1 - z\bar{\xi}} (\varphi_n(\xi, \mu) - \varphi_n(z, \mu)) d\mu, \quad \xi = e^{i\theta}$$

are orthonormal with respect to $\mu_{\rm dual}$ (see, e.g., [28], formulas (3.2.32) and (3.2.50) or section 1 in [13]). The Muckenhoupt class $A_2(\mathbb{T})$ turns out to be invariant with respect to taking dual. In fact, more general statement is true.

Theorem 1.5. *If* $w \in A_2(\mathbb{T})$ *and* $d\mu = \frac{w}{2\pi} d\theta$*, then* μ_α *is absolutely continuous and* $d\mu_\alpha = \frac{w_\alpha}{2\pi} d\theta$ *for every* $\alpha \in \mathbb{T}$*. Moreover,* $w_\alpha \in A_2(\mathbb{T})$*.*

This has an immediate implication for regularity of ψ_n. Indeed, if $w \in A_2(\mathbb{T})$, then $d\mu_{\rm dual} = \frac{w_{\rm dual}}{2\pi} d\theta$ with $w_{\rm dual} \in A_2(\mathbb{T})$ so theorem 1.1 can be applied and we get

$$\sup_n \|\psi_n\|_{L^p_{w_{\rm dual}}(\mathbb{T})} < \infty$$

with $p \in [2, p_{\rm cr}([w_{\rm dual}]_{A_2}))$.

The proofs of the main results in this paper involve complex interpolation, a suitable choice of the algebraic formulas, and a few facts from the general spectral theory.

Previous results. In [2], it was proved that, given every $q \in [1, \infty)$ and $n \in \mathbb{N}$, there is w_* that satisfies $\|w_*\|_{L^q(\mathbb{T})} < c_1, \|w_*^{-1}\|_{L^\infty(\mathbb{T})} < c_2$ and nonetheless $\|\varphi_n(\xi, w_*)\|_{L^\infty(\mathbb{T})} \geqslant C(c_1, c_2, q)\sqrt{n}$ with parameters c_1 and c_2 being n-independent. By Nikolskii inequality (see p.102, theorem 2.6, [11]), we see that $\|\varphi_n(\xi, w_*)\|_{L^p(\mathbb{T})} > C(c_1, c_2, p, q) n^{1/2 - 1/p}$ for every $p \in [2, \infty)$. Since the weight w_* is bounded below by c_2^{-1}, one also gets $\|\varphi_n(\xi, w_*)\|_{L^p_{w_*}(\mathbb{T})} > C(c_1, c_2, p, q) n^{1/2 - 1/p}$. Therefore, the stated conditions on w, i.e.,

$$\|w\|_{L^q(\mathbb{T})} < c_1, \quad \|w^{-1}\|_{L^\infty(\mathbb{T})} < c_2, \quad q \in [1, \infty)$$

do not provide the uniform in n weighted L^p estimates for polynomials if $p > 2$ is fixed. The question what regularity of w is enough to have $\sup_n \|\varphi_n\|_{L^p(\mathbb{T})} < \infty$ or $\sup_n E(n, w) < \infty$ has been addressed in [3–5, 9, 10, 22]. The following theorem was proved in [9].

Theorem 1.6 (**Denisov-Rush**, [9]). *Let* $s \stackrel{\text{def}}{=} \|w\|_{\text{BMO}(\mathbb{T})} < \infty$ *and* $t \stackrel{\text{def}}{=} \|w^{-1}\|_{\text{BMO}(\mathbb{T})} < \infty$. *Then, there is* $p(s,t) > 2$ *such that* $\sup_n \|\varphi_n(\xi, w)\|_{L^p(\mathbb{T})} < \infty$.

We will see later that theorem 1.1 implies theorem 1.6 and, in fact, gives a qualitatively stronger statement. It appears that A_2 regularity of w is, to the best of our knowledge, the weakest general condition that provides weighted L^p estimates on $\{\varphi_n\}$.

As far as theorem 1.2 is concerned, the continuity of operators in the weighted spaces with respect to a weight has been addressed previously. In [24,25], Pattakos and Volberg show that $A_\infty(\mathbb{R}^d)$ is a metric space with metric defined by

$$d_*(w_1, w_2) \stackrel{\text{def}}{=} \|\log w_1 - \log w_2\|_{\text{BMO}} .$$

These two authors studied other properties of $A_\infty(\mathbb{R}^d)$ as a metric space and established, among other things, the Lipschitz continuity of $\|H\|_{L^p_w, L^p_w}$ in $w \in A_p(\mathbb{R}^d)$ for H that satisfies (1.3).

The structure of our paper is as follows. The second section contains the proof of theorem 1.2 along with related information about the Muckenhoupt class. Theorem 1.1 and its corollaries are proved in the third section. The analysis of the Christoffel-Darboux kernel for the case when $w \in A_2(\mathbb{T})$ is done in section four. In section five, we discuss Alexandrov-Clark measures and give proof of theorem 1.5. The appendix contains an example of weight in the Fisher-Hartwig class for which the asymptotics of the polynomials is known. This provides an upper estimate for $p_{\text{cr}}(t)$ in the regime when t is close to 1.

1.1. Notation.

- If $p \in [1, \infty]$, the dual exponent is denoted by $p' = p/(p-1)$.
- Given a set $A \subseteq \mathbb{R}^d$ (or $A \subseteq \mathbb{T}$), we will use notation A^c for its complement, i.e., $A^c = \mathbb{R}^d \backslash A$ (or $A^c = \mathbb{T} \backslash A$).
- Given two Banach spaces $L^p(X, \mu)$, $L^q(Y, \nu)$, and a linear bounded operator $T : L^p(X, \mu) \to L^q(Y, \nu)$, its norm is denoted by $\|T\|_{p,q}$.
- By $L^p_w(\mathbb{T})$ we mean the space $L^p_\mu(\mathbb{T})$ where $d\mu = w\frac{d\theta}{2\pi}$.
- If f is locally integrable in $\mathbb{R}^d$ and B is a ball, then

$$\langle f \rangle_B \stackrel{\text{def}}{=} \frac{1}{|B|} \int_B f dx .$$

- Given function $f \in L^1(\mathbb{T})$, we will write $\mathfrak{h}(f)$ to denote the operator of harmonic conjugation [17], i.e.,

$$\mathfrak{h}(f) = \widetilde{f}(\xi) = \lim_{r \to 1} \frac{1}{2\pi} \int_{\mathbb{T}} f(\zeta) Q_r(\zeta, \xi)\, d\theta, \quad Q_r(\zeta, \xi) = \operatorname{Im} \frac{1 + r\bar{\zeta}\xi}{1 - r\bar{\zeta}\xi}, \quad \zeta = e^{i\theta}, \quad \xi \in \mathbb{T}. \tag{1.8}$$

- Given a function $f \in L^1(\mathbb{T})$, the Poisson integral is defined by (see [17], pp.2–3)

$$\mathcal{P}(f, z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar{\zeta} z|^2} f(\zeta) d\theta, \quad z \in \mathbb{D}, \quad \zeta = e^{i\theta} . \tag{1.9}$$

  The Cauchy integral over $\mathbb{T}$ is defined by (see [17], p.35)

$$\mathcal{C}(f, z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} d\theta, \quad z \in \mathbb{D}, \quad \zeta = e^{i\theta} . \tag{1.10}$$

- For two non-negative functions f_1 and f_2, we write $f_1 \lesssim f_2$ if there is an absolute constant C such that

$$f_1 \leqslant C f_2$$

  for all values of the arguments of f_1 and f_2. If the constant depends on a parameter α, we will write $f_1 \leqslant_\alpha f_2$. We define $\gtrsim$ similarly and say that $f_1 \sim f_2$ if $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ simultaneously.
- The symbol $C^\infty_c(\mathbb{R}^d)$ denotes the space of infinitely smooth function with compact support in $\mathbb{R}^d$.
- Given two operators, A and B, we use the symbol $[A, B] = AB - BA$ for their commutator.

2. Weighted operators are continuous in $w \in A_p(\mathbb{R}^d)$

We start by recalling a few basic facts from the theory of $A_p(\mathbb{R}^d)$ weights (see, e.g., [18] and [30]). Given the definition (1.2), the limiting case when $p \to \infty$ leads to $A_\infty(\mathbb{R}^d)$ which is characterized by (see, e.g., [15])

$$[w]_{A_\infty(\mathbb{R}^d)} \stackrel{\text{def}}{=} \sup_B \left(\langle w \rangle_B \exp\Big(-\langle \log w \rangle_B \Big) \right) . \tag{2.1}$$

The following results are well-known.

Lemma 2.1 (see, e.g., [30], p.218). *If $\|f\|_{\mathrm{BMO}} < \infty$, then there is $\delta_1(\|f\|_{\mathrm{BMO}}) > 0$ such that*

$$[e^{\delta f}]_{A_\infty(\mathbb{R}^d)} \lesssim 1$$

for all $\delta : |\delta| < \delta_1(\|f\|_{\mathrm{BMO}})$.

Proof. From John-Nirenberg theorem ([30], pp.145-146), we have

$$\sup_B \left(\langle e^{\delta |f - \langle f \rangle_B|} \rangle_B \right) \lesssim 1 \tag{2.2}$$

provided $|\delta| < \delta_1(\|f\|_{\mathrm{BMO}})$. In (2.1), take $w = e^{\delta f}$, to get

$$[w]_{A_\infty} = \sup_B \left(\langle e^{\delta (f - \langle f \rangle_B)} \rangle_B \right) \lesssim 1$$

by (2.2). □

The proofs for the next two lemmas are immediate corollaries from theorem 1_∞ and theorem 1 in [32].

Lemma 2.2. *Suppose $w \in A_\infty(\mathbb{R}^d)$. For every $p \in (1, \infty)$, there is $\delta_2(p, [w]_{A_\infty(\mathbb{R}^d)}) > 0$ such that*

$$[w^\delta]_{A_p(\mathbb{R}^d)} < C(p, [w]_{A_\infty(\mathbb{R}^d)}) \tag{2.3}$$

for every $\delta : |\delta| < \delta_2(p, [w]_{A_\infty(\mathbb{R}^d)})$.

Remark. The exact dependence of the right-hand side in (2.3) on the parameters will not be needed in this paper so we are only using the symbol C.

Lemma 2.3. *Given $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^d)$, there is $\delta_3(p, [w]_{A_p(\mathbb{R}^d)}) > 0$ such that $[w^{1+\delta}]_{A_p(\mathbb{R}^d)} \leqslant C(p, [w]_{A_p(\mathbb{R}^d)})$ for $\delta \in [0, \delta_3)$.*

Given these lemmas, we claim that

Lemma 2.4. *For every $p \in (1, \infty)$, $f \in \mathrm{BMO}(\mathbb{R}^d)$, and $w \in A_p(\mathbb{R}^d)$, we have*

$$[w e^{\delta f}]_{A_p(\mathbb{R}^d)} \leqslant C(p, [w]_{A_p(\mathbb{R}^d)}, \|f\|_{\mathrm{BMO}}) , \tag{2.4}$$

if $\delta : |\delta| < \delta_4(p, [w]_{A_p(\mathbb{R}^d)}, \|f\|_{\mathrm{BMO}})$.

Proof. Consider (1.2). Given w and some nonnegative w_0 we use Hölder's inequality

$$\left(\int_B w w_0 dx \right) \left(\int_B (w w_0)^{1/(1-p)} dx \right)^{p-1} \leqslant$$
$$\left(\int_B w^\alpha dx \right)^{1/\alpha} \left(\int_B w_0^{\alpha'} dx \right)^{1/\alpha'} \left(\int_B w^{\alpha/(1-p)} dx \right)^{(p-1)/\alpha} \left(\int_B w_0^{\alpha'/(1-p)} dx \right)^{(p-1)/\alpha'} ,$$

where α' is dual to α and $\alpha > 1$ is chosen such that $w^\alpha \in A_p(\mathbb{R}^d)$ (this choice is warranted by lemma 2.3). Now, if we let $w_0 = e^{\delta f}$, then $w_0^{\alpha'} \in A_p(\mathbb{R}^d)$ for small δ thanks to lemma 2.1 and lemma 2.2. This yields (2.4). □

Lemma 2.5. *If $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^d)$, $f \in \mathrm{BMO}(\mathbb{R}^d)$, and H satisfies (1.3), then*

$$\|w^{1/p} [H, f] w^{-1/p}\|_{p,p} \leqslant C(p, [w]_{A_p(\mathbb{R}^d)}, \|f\|_{\mathrm{BMO}}, \mathcal{F}) \tag{2.5}$$

and

$$\|w^{1/p} [f, [H, f]] w^{-1/p}\|_{p,p} \leqslant C(p, [w]_{A_p(\mathbb{R}^d)}, \|f\|_{\mathrm{BMO}}, \mathcal{F}). \tag{2.6}$$

Proof. Given two test functions $u, v \in C_c^\infty(\mathbb{R}^d)$, define operator-valued function

$$G(z) \stackrel{\text{def}}{=} w^{1/p} e^{zf} H e^{-zf} w^{-1/p}$$

and consider $\widehat{G}(z) = (G(z)u, v)$, where the inner product is in $L_2(\mathbb{R}^d)$. $\widehat{G}(z)$ is analytic in z around the origin and we can write Cauchy integral formula with $|z| < \epsilon$, when ϵ is small enough (and depends only on $p, [w]_{A_p(\mathbb{R}^d)}$, and $\|f\|_{\mathrm{BMO}}$):

$$\widehat{G}(z) = \frac{1}{2\pi i}\int_{|\xi|=\epsilon} \frac{\widehat{G}(\xi)}{\xi - z} d\xi, \quad \widehat{G}'(0) = (w^{1/p}[f, H]w^{-1/p}u, v) = \frac{1}{2\pi i}\int_{|\xi|=\epsilon} \frac{\widehat{G}(\xi)}{\xi^2} d\xi,$$

so

$$|(w^{1/p}[H, f]w^{-1/p}u, v)| \lesssim \epsilon^{-1} \max_{|\xi|=\epsilon} |\widehat{G}(\xi)|\,.$$

For any point $z : |z| = \epsilon$ on the circle, we can apply lemma 2.4 and (1.3) to choose $\epsilon(p, [w]_{A_p}, \|f\|_{\mathrm{BMO}})$ such that $\max_{|\xi|=\epsilon} |\widehat{G}(\xi)| < C(p, [w]_{A_p}, \|f\|_{\mathrm{BMO}}, \mathcal{F})\|u\|_p\|v\|_{p'}$ (here p' is dual to p). This implies (2.5) by the standard duality argument, i.e., by employing an identity

$$\|O\|_{p,p} = \sup_{u,v \in C_c^\infty(\mathbb{R}^d), \|u\|_p \leqslant 1, |v\|_{p'} \leqslant 1} |(Ou, v)|\,,$$

which holds for every linear bounded operator O and $p \in (1, \infty)$.

The estimate (2.6) follows from (2.5) by taking H in (2.5) as a commutator $[H, f]$ itself and using (2.5). ☐

Proof of theorem 1.2. Consider analytic operator-valued function defined for $z : \operatorname{Re} z \in [0, 1]$,

$$F(z) = w^{1/p} \exp(\alpha z f/p)\, H \exp(-\alpha z f/p)\, w^{-1/p} - w^{1/p} H w^{-1/p} - z\frac{\alpha}{p} w^{1/p}[f, H]w^{-1/p}\,,$$

where the parameter α will be chosen later, it will depend on $p, \|f\|_{\mathrm{BMO}}$, and $[w]_{A_p(\mathbb{R}^d)}$ only. Consider rectangle $\Pi = \{z : |\operatorname{Im} z| < 1, 0 < \operatorname{Re} z < 1\}$. We will estimate the operator norm of F on $\partial\Pi$ as follows. If $z \in \{z : |\operatorname{Im} z| = 1, \operatorname{Re} z \in [0, 1]\} \cup \{z : \operatorname{Re} z = 1, \operatorname{Im} z \in [-1, 1]\}$, the estimate is straightforward:

$$\|F(z)\|_{p,p} \leqslant C(p, [w]_{A_p}, \mathcal{F}) + C(p, [we^{\alpha f}]_{A_p}, \mathcal{F}) \leqslant C(p, [w]_{A_p}, \|f\|_{\mathrm{BMO}}, \mathcal{F}), \quad \alpha : |\alpha| < \alpha_4(p, [w]_{A_p}, \|f\|_{\mathrm{BMO}})\,,$$

where we first used (2.5), (1.3), and then lemma 2.4. Now, we take test functions $u, v \in C_c^\infty(\mathbb{R}^d)$ and consider $\widehat{F}(z) = (F(z)u, v)$. It is anaytic in Π and continuous on $\overline{\Pi}$. On the interval $z = i\xi, |\xi| < 1$, have

$$\widehat{F}(0) = 0, \quad \widehat{F}'(0) = 0, \quad \partial_\xi \widehat{F}(i\xi) = \frac{i\alpha}{p}(w^{1/p} e^{i\alpha\xi f/p}[f, H]e^{-i\alpha\xi f/p} w^{-1/p}u, v) - \frac{i\alpha}{p} w^{1/p}[f, H]w^{-1/p}\,,$$

and

$$\partial^2_{\xi\xi} \widehat{F}(i\xi) = \left(\frac{i\alpha}{p}\right)^2 (w^{1/p} e^{i\alpha\xi f/p}[f, [f, H]]e^{-i\alpha\xi f/p} w^{-1/p}u, v)\,,$$

$$|\partial^2_{\xi\xi} \widehat{F}(i\xi)| \leqslant C(p, [w]_{A_p}, \|f\|_{\mathrm{BMO}}, \mathcal{F})\|u\|_p\|v\|_{p'}$$

by lemma 2.5. The Fundamental Theorem of Calculus gives

$$\widehat{F}(i\xi) = \int_0^\xi \left(\int_0^\tau \partial^2_{\tau\tau} \widehat{F}(i\tau)d\tau\right) d\xi, \quad |\widehat{F}(i\xi)| \leqslant \xi^2 C(p, [w]_{A_p}, \|f\|_{\mathrm{BMO}}, \mathcal{F})\|u\|_p\|v\|_{p'}.$$

The last bound implies

$$\|F(i\xi)\|_{p,p} \leqslant \xi^2 C(p, [w]_{A_p}, \|f\|_{\mathrm{BMO}}, \mathcal{F})$$

after we use duality argument. Notice that the function $|\widehat{F}|$ is subharmonic in Π. Thus, by mean-value inequality, one has

$$|\widehat{F}(\delta)| \leqslant \left(\int_{\partial\Pi} |\widehat{F}(\xi)| d\omega_\delta(\xi)\right)\,,$$

where $\omega_z(\xi)$ denotes the harmonic measure at point z (see, e.g., [12] p.13, formula (3.4)). By duality again,

$$\|F(\delta)\|_{p,p} \leqslant \left(\int_{\partial\Pi} \|F(\xi)\|_{p,p} d\omega_\delta(\xi)\right)\,.$$

When $\delta \to 0$, measure $\omega_\delta(\xi)$ concentrates on the left side of $\partial\Pi$ around point 0 and we have $\lim_{\delta\to 0}\|F(\delta)\|_{p,p} = 0$. Putting the estimates together, we can make it more precise. Recall that the harmonic measure on the upper half-plane $\mathbb{C}^+$ with the reference point z is given by

$$\frac{1}{\pi}\frac{\operatorname{Im} z}{\operatorname{Im}^2 z + (\operatorname{Re} z - t)^2}\,, z \in \mathbb{C}^+,\ t \in \mathbb{R}\,.$$

Consider a conformal map φ from $\mathbb{C}^+$ to Π. For example, we can take φ as the following Schwarz-Christoffel integral (see [31], p.181 and pp.188-189, formula (6-76)):

$$\varphi(z) = C\int_0^z \frac{d\eta}{\sqrt{(1-\eta^2)(1-k^2\eta^2)}}, \quad z \in \mathbb{C}^+,$$

where C and k are constants that can be found explicitly and $k \in (0,1)$. Under the inverse map φ^{-1}, the left side $\{i\xi, |\xi| < 1\}$ of Π goes to the interval $[-1,1]$ and its right side $\{1+i\xi, |\xi| < 1\}$ goes to $[k^{-1},\infty) \cup (-\infty, -k^{-1}]$. Clearly, $\varphi(0) = 0$. Now, we obtain

$$\int_{\partial\Pi}\|F(\xi)\|_{p,p}d\omega_\delta(\xi) \lesssim \int_{\mathbb{R}}\frac{\delta}{\delta^2+t^2}\|F(\varphi(t))\|_{p,p}dt$$

where $\varphi(t): \mathbb{R} \to \partial\Pi$. Substituting the estimates for $\|F\|_{p,p}$ and using $|\varphi(z)/z| \sim 1$, $|z| < 0.5$, we get

$$\int_{\mathbb{R}}\frac{\delta}{\delta^2+t^2}\|F(\varphi(t))\|_{p,p}dt \leqslant$$
$$C(p,[w]_{A_p},\|f\|_{\mathrm{BMO}},\mathcal{F})\left(\int_{-0.5}^{0.5}\frac{\delta t^2}{\delta^2+t^2}dt + \int_{|t|>0.5}\frac{\delta}{\delta^2+t^2}dt\right) \leqslant$$
$$C(p,[w]_{A_p},\|f\|_{\mathrm{BMO}},\mathcal{F})\delta\,.$$

Finally, we get the statement of the theorem since

$$w^{1/p}\exp(\alpha\delta f/p)\,H\exp(-\alpha\delta f/p)\,w^{-1/p} - w^{1/p}Hw^{-1/p} = F(\delta) + \delta\frac{\alpha}{p}w^{1/p}[f,H]w^{-1/p}\,,$$

and

$$\|F(\delta)\|_{p,p} \leqslant C(p,[w]_{A_p},\|f\|_{\mathrm{BMO}},\mathcal{F})\delta\,,$$
$$\|\frac{\alpha}{p}w^{1/p}[f,H]w^{-1/p}\|_{p,p} \leqslant C(p,[w]_{A_p},\|f\|_{\mathrm{BMO}},\mathcal{F})\,.$$

□

Remark. Clearly, the theorem holds if $A_p(\mathbb{R}^d)$ is replaced by $A_p(\mathbb{T})$.

3. Steklov problem in the theory of orthogonal polynomials: $w \in A_2(\mathbb{T})$ and bounds for $\|\varphi_n(z,w)\|_{L^p_w(\mathbb{T})}$

This section contains the proofs of theorem 1.1 and its two corollaries. In the proof of theorem 1.1, we will consider separately two cases: when $[w]_{A^2(\mathbb{T})} \in [1,2)$ and when $[w]_{A^2(\mathbb{T})} \geqslant 2$. It will be more convenient for us to work with monic orthogonal polynomials, which are defined as

$$\Phi_n(z,\mu) = \frac{\varphi_n(z,\mu)}{k_n}\,.$$

If $w \in A_2(\mathbb{T})$, then $w^{-1} \in L^1(\mathbb{T})$ by definition. Thus, $\log w \in L^1(\mathbb{T})$ as well. This means that $\mu : d\mu = \frac{w}{2\pi}d\theta$ belongs to Szegő class of measures and, consequently, the sequence $\{k_n\}$ has a finite and positive limit (see [13], section 2). More precisely, we have an estimate:

$$\exp\left(\frac{1}{4\pi}\int_{\mathbb{T}}\log w d\theta\right) \leqslant \left|\frac{\Phi_n(z,w)}{\varphi_n(z,w)}\right| \leqslant 1, \quad \forall z \in \mathbb{C}, \tag{3.1}$$

(see, e.g., [10]). This bound shows that we can focus on estimating $\|\Phi_n(\xi,w)\|_{L^p_w(\mathbb{T})}$.

Later in the text, we will need to use the second resolvent identity which is contained in the following proposition.

Proposition 3.1. *Suppose X is an Banach space and H, V are linear bounded operators from X to X. Then,*

$$(I+H+V)^{-1} = (I+H)^{-1} - (I+H+V)^{-1}V(I+H)^{-1},$$
$$(I+H+V)^{-1} = (I+H)^{-1}(I+V(I+H)^{-1})^{-1},$$

provided the operators involved are well-defined and bounded in X. Moreover, assuming $\|V\|\cdot\|(I+H)^{-1}\|<1$, we get

$$\|(I+H+V)^{-1}\| \leqslant \frac{\|(I+H)^{-1}\|}{1-\|V\|\cdot\|(I+H)^{-1}\|}\,. \tag{3.2}$$

Finally, if $\|V\|<1$, then

$$\|(I+V)^{-1}\| \leqslant \frac{1}{1-\|V\|}\,. \tag{3.3}$$

The proof of this proposition is a straightforward calculation. The following well-known lemma (see, e.g., [18], corollary 6) will be important later on.

Lemma 3.2. *If $[w]_{A_2(\mathbb{T})} = 1+\tau, \tau\in[0,1]$, then*

$$\|\log w\|_{\rm BMO} \lesssim \sqrt{\tau}\,.$$

Let $\mathcal{P}_n$ denote the orthogonal $L^2(\mathbb{T})$ projection to the frequencies $\{1,\ldots,e^{in\theta}\}$. Consider the perturbative regime, i.e., the case when $[w]_{A_2(\mathbb{T})} = 1+\tau$ and $\tau\in[0,1]$.

Lemma 3.3. *We have $\lim_{\tau\to 0} p_{\rm cr}(1+\tau) = \infty$.*

Proof. Fix any $p\geqslant 2$. We need to show that there is $\tau>0$ small enough so that $[\upsilon]_{A_2} < 1+\tau$ implies

$$\sup_n \|\Phi_n(z,\upsilon)\|_{L^p_\upsilon(\mathbb{T})} < \infty\,.$$

Our argument is based on a representation (see, e.g., [9], formula (8) for Φ_n^*):

$$\Phi_n = z^n - \upsilon^{-1}[\mathcal{P}_{n-1},\upsilon]\Phi_n\,. \tag{3.4}$$

This formula can be obtained by combining trivial identity $\Phi_n = z^n + \mathcal{P}_{n-1}\Phi_n$, which holds for all monic polynomials of degree n, with $\mathcal{P}_{n-1}(\upsilon\Phi_n) = 0$, which follows from that fact that Φ_n is orthogonal to $\{1,z,\ldots,z^{n-1}\}$ in $L^2_\upsilon(\mathbb{T})$. Thus, we infer from (3.4) that

$$\left(\upsilon^{1/p}\Phi_n\right) = \upsilon^{1/p}z^n - \upsilon^{-1/p'}\mathcal{P}_{n-1}\upsilon^{1/p'}\left(\upsilon^{1/p}\Phi_n\right) + \upsilon^{1/p}\mathcal{P}_{n-1}\upsilon^{-1/p}\left(\upsilon^{1/p}\Phi_n\right).$$

Denoting $\zeta_n \stackrel{\rm def}{=} \upsilon^{1/p}\Phi_n$, $O_{1,n} \stackrel{\rm def}{=} \upsilon^{-1/p'}\mathcal{P}_{n-1}\upsilon^{1/p'} - \mathcal{P}_{n-1}$, $O_{2,n} \stackrel{\rm def}{=} \upsilon^{1/p}\mathcal{P}_{n-1}\upsilon^{-1/p} - \mathcal{P}_{n-1}$, we rewrite it as

$$\zeta_n = \upsilon^{1/p}z^n - O_{1,n}\zeta_n + O_{2,n}\zeta_n\,. \tag{3.5}$$

If $\mathcal{P}^+$ denotes the orthogonal $L^2(\mathbb{T})$ projection onto Hardy space $H^2(\mathbb{T})$ (Riesz projection), then we can write an identity

$$\mathcal{P}_n = \mathcal{P}^+ - z^{n+1}\mathcal{P}^+z^{-(n+1)} = z^{n+1}[z^{-(n+1)},\mathcal{P}^+]\,. \tag{3.6}$$

We now apply theorem 1.2 with $H=\mathcal{P}^+$, $w=1$, and $w_\delta = e^{\delta f} = \upsilon$. Then, $f = \delta^{-1}\log\upsilon$ and lemma 3.2 gives

$$\|f\|_{\rm BMO} \lesssim \delta^{-1}\sqrt{\tau} \leqslant 1\,,$$

when $\tau<\delta^2$. Since $\|w^{1/p}\mathcal{P}^+w^{-1/p}\|_{p,p} \leqslant \mathcal{F}([w]_{A^p},p)$ by Hunt-Muckenhoupt-Wheeden theorem, the theorem 1.2 then yields

$$\lim_{\tau\to 0}\|\upsilon^{1/p}\mathcal{P}^+\upsilon^{-1/p} - \mathcal{P}^+\|_{p,p} = 0$$

for every $p\in(1,\infty)$. In particular, it also holds for p':

$$\lim_{\tau\to 0}\|\upsilon^{1/p'}\mathcal{P}^+\upsilon^{-1/p'} - \mathcal{P}^+\|_{p',p'} = 0\,.$$

Indeed, we use the standard identity in the operator theory, which follows from duality considerations:

$$\|\mathcal{O}\|_{p,p} = \|\mathcal{O}^*\|_{p',p'}\,,$$

where $\mathcal{O}^*$ is adjoint operator to $\mathcal{O}$ with respect to L^2 inner product and $\mathcal{O}$ is linear bounded operator in L^p space. Since $\mathcal{P}^+$ is self-adjoint in $L^2(\mathbb{T})$, we get

$$\|\upsilon^{1/p'}\mathcal{P}^+\upsilon^{-1/p'} - \mathcal{P}^+\|_{p',p'} = \|\upsilon^{-1/p'}\mathcal{P}^+\upsilon^{1/p'} - \mathcal{P}^+\|_{p,p}$$

and hence

$$\lim_{\tau\to 0}\|\upsilon^{-1/p'}\mathcal{P}^+\upsilon^{1/p'}-\mathcal{P}^+\|_{p,p}=0\,.$$

Summarizing, (3.6) gives two bounds

$$\|O_{1,n}\|_{p,p}\leqslant 2\|\upsilon^{-1/p'}\mathcal{P}^+\upsilon^{1/p'}-\mathcal{P}^+\|_{p,p},\ \|O_{2,n}\|_{p,p}\leqslant 2\|\upsilon^{1/p}\mathcal{P}^+\upsilon^{-1/p}-\mathcal{P}^+\|_{p,p}$$

that hold uniformly in n. Therefore,

$$\lim_{\tau\to 0}\|O_{2,n}\|_{p,p}=0,\quad \lim_{\tau\to 0}\|O_{1,n}\|_{p,p}=0\,.$$

Now, we apply (3.3) with $V=O_{1,n}$ to (3.5) in the space $L^p(\mathbb{T})$. This gives the statement of the lemma. Here, we notice that $\sup_n\|z^n\upsilon^{1/p}\|_p<\infty$ because $\upsilon\in A_2(\mathbb{T})\subset L_1(\mathbb{T})$. □

Next, we consider more complicated case when $[w]_{A_2(\mathbb{T})}\geqslant 2$.

Remark. We have $w^{-1/p'}=(w^{-p/p'})^{1/p}$ and

$$[w^{-p/p'}]_{A_p(\mathbb{T})}=[w]^{p/p'}_{A_{p'}(\mathbb{T})}\tag{3.7}$$

as can be directly verified.

Lemma 3.4. *For every $w\in A_2(\mathbb{T})$ and $l\in\mathbb{N}$, define a simple function w_l as follows: let $w_l=\langle w\rangle_{I_j}$ on each interval $I_j=2^{-l}(2\pi)[j,j+1), j=0,\ldots,2^l-1$. Then, $\lim_{l\to\infty}\Phi_n(z,w_l)=\Phi_n(z,w)$ uniformly in z over compacts in $\mathbb{C}$ and*

$$[w_l]_{A_2(\mathbb{T})}\leqslant C([w]_{A_2(\mathbb{T})}).$$

Proof. From the construction, we immediately get $\{w_l\}\to w$ in the weak–$(*)$ sense when $l\to\infty$. Since the coeffcients of $\Phi_n(z,\mu)$ depend continuously on the moments of measure μ, we have the first statement of the lemma. The second one can be verified directly using the definition of $A_2(\mathbb{T})$ characteristic.

□

Next, we need the following interpolation result. Given $w\in A_2(\mathbb{T})$ and $p_*\geqslant 2$, define

$$Q_{w,p(z)}\overset{\text{def}}{=}w^{-1/p'(z)}\mathcal{P}_{n-1}w^{1/p'(z)}-w^{1/p(z)}\mathcal{P}_{n-1}w^{-1/p(z)}\,,\tag{3.8}$$

where

$$\frac{1}{p(z)}=\frac{z}{p_*}+\frac{1-z}{2},\quad \frac{1}{p'(z)}=1-\frac{1}{p(z)}=\frac{1+z}{2}-\frac{z}{p_*},\quad \operatorname{Re}z\in[0,1]\,,\tag{3.9}$$

so that $1/p(z)+1/p'(z)=1$.

Proposition 3.5. *Suppose $w,w^{-1}\in L^\infty(\mathbb{T})$, parameter κ is real, and*

$$\sup_{0\leqslant\operatorname{Re}z\leqslant 1}\|Q_{w,p(z)}\|_{p(t),p(t)}<\infty\,,\tag{3.10}$$

where $t\overset{\text{def}}{=}\operatorname{Re}z\in[0,1]$. If there is a positive number $\mathbf{\Lambda}$ such that

$$\|(I-\kappa Q_{w,p(t+iy)})^{-1}\|_{p(t),p(t)}\leqslant 2\mathbf{\Lambda}$$

for all $t\in[0,1]$ and $y\in\mathbb{R}$, then there is an $t_*(\mathbf{\Lambda})\in(0,1]$, so that

$$\|(I-\kappa Q_{w,p(t+iy)})^{-1}\|_{p(t),p(t)}\leqslant \mathbf{\Lambda}$$

for all $y\in\mathbb{R}$ and $t\in[0,t_*]$.

Proof. We notice that $Q_{w,p(iy)}$ is bounded and antisymmetric operator in Hilbert space $L^2(\mathbb{T})$. Therefore, $\|(I-\kappa Q_{w,p(iy)})^{-1}\|_{2,2}\leqslant 1$. Given conditions $w,w^{-1}\in L^\infty(\mathbb{T})$, it is easy to check that the operator-valued function $(I-\kappa Q_{w,p(z)})^{-1}$ is analytic and continuous in the sense of Stein (p.209, [6]). Applying Stein's interpolation theorem, we get

$$\|(I-\kappa Q_{w,p(t+iy)})^{-1}\|_{p(t),p(t)}\leqslant\exp\left(\frac{\sin(\pi t)}{2}\int_{\mathbb{R}}\frac{\log(2\mathbf{\Lambda})}{\cosh(\pi y)+\cos(\pi t)}dy\right)=1+O(t),\quad t\to 0\,,$$

which proves the proposition. □

Remark. We emphasize here that positive t_* does not depend on n or w.

Now, we are ready to prove the following lemma.

Lemma 3.6. *For every $t\geqslant 2$, we have $p_{\rm cr}(t)>2$.*

Proof. Consider $w \in A_2(\mathbb{T})$. It will be more convenient later on to work with weights which are bounded above and below. With fixed n, we can use lemma 3.4 to approximate w by w_n which satisfies

$$\|w_n\|_{L^\infty(\mathbb{T})} < C(n,w), \quad \|w_n^{-1}\|_{L^\infty(\mathbb{T})} < C(n,w)\,,$$

$$[w_n]_{A_2(\mathbb{T})} \leqslant \gamma \stackrel{\text{def}}{=} C([w]_{A_2}),\, n \in \mathbb{N}$$

and

$$|\Phi_n(z,w)| \leqslant 2|\Phi_n(z,w_n)|$$

for each $z \in \mathbb{T}$. In what follows, we suppress the dependence of w_n in n and do the proof understanding that w depends on n and satisfies

$$\|w\|_{L^\infty(\mathbb{T})} < \infty, \quad \|w^{-1}\|_{L^\infty(\mathbb{T})} < \infty, \quad [w]_{A_2(\mathbb{T})} \leqslant \gamma < \infty\,,$$

where γ does not depend on n.

As in the proof of lemma 3.3, we can write

$$\zeta_n = w^{1/p} z^n + Q_{w,p}\zeta_n\,,$$

where $\zeta_n \stackrel{\text{def}}{=} w^{1/p}\Phi_n$ and $Q_{w,p} \stackrel{\text{def}}{=} -B_n + C_n, B_n \stackrel{\text{def}}{=} w^{-1/p'}\mathcal{P}_{n-1}w^{1/p'}, C_n \stackrel{\text{def}}{=} w^{1/p}\mathcal{P}_{n-1}w^{-1/p}$ and all operators are considered in Banach space $L^p(\mathbb{T})$. It is sufficient to prove that

$$\sup_n \|(I - Q_{w,\widetilde{p}_\gamma})^{-1}\|_{\widetilde{p}_\gamma,\widetilde{p}_\gamma} < \infty \tag{3.11}$$

with some $\widetilde{p}_\gamma > 2$ because $\sup_n \|w^{1/p}z^n\|_p < \infty$ and

$$\zeta_n = (I - Q_{w,p})^{-1}(w^{1/p}z^n)\,.$$

By open inclusion of Muckenhoupt classes (see [30], corollary on p.202 or theorem 1 in [32]), there is $\widehat{p}_\gamma > 2$ such that $\widehat{p}_\gamma' < 2$ and $\widehat{\gamma} \stackrel{\text{def}}{=} [w]_{A_{\widehat{p}_\gamma'}} < \infty$. Thus, by (3.7),

$$[w^{-p/p'}]_{A_p} = [w]^{p/p'}_{A_{p'}} \leqslant \widehat{\gamma}^{\widehat{p}/\widehat{p}'} \tag{3.12}$$

for all $p \in [2, \widehat{p}_\gamma]$. We need this bound to control B_n through writing it as

$$B_n = (w^{-p/p'})^{1/p}\mathcal{P}_{n-1}(w^{-p/p'})^{-1/p}$$

and viewing $w_1 \stackrel{\text{def}}{=} w^{-p/p'}$ as element of $A_p(\mathbb{T})$. Now, we use Hunt-Muckenhoupt-Wheeden theorem, which implies that

$$\sup_n \|B_n\|_{p,p} = \sup_n \|w_1^{1/p}\mathcal{P}_{n-1}w_1^{-1/p}\|_{p,p} < \mathcal{F}_1(p,\gamma)\,, \tag{3.13}$$

where $\mathcal{F}_1$ is defined for $p \in [2, \widehat{p}_\gamma]$. Analogous bound for C_n is obvious:

$$\sup_n \|C_n\|_{p,p} < \mathcal{F}_2(p,\gamma) \tag{3.14}$$

for all $p \in (2,\infty)$ since $w \in A_2(\mathbb{T}) \subset A_p(\mathbb{T})$. Define $Q_{w,p(z)}$ by (3.8) and take $p_* \in [2, \widehat{p}_\gamma]$. The bounds (3.13) and (3.14) imply that

$$\sup_n \|Q_{w,p(z)}\|_{p(t),p(t)} < \infty$$

for $t = \operatorname{Re} z \in [0,1]$.

Now, we proceed as follows. Recall, see (3.11), that our goal is to show that $(I - Q_{w,\widetilde{p}_\gamma})^{-1}$ is bounded in $L^{\widetilde{p}}(\mathbb{T})$ for some $\widetilde{p}_\gamma > 2$ with bound on the operator norm independent in n. In (3.8), we take parameter p_* as follows: $p_*^{(1)} = \widehat{p}_\gamma$ and define $p_1(z) \stackrel{\text{def}}{=} p(z)$ where $p(z)$ is from (3.9). Consider $Q^{(j)}_{w,p(z)} \stackrel{\text{def}}{=} jQ_{w,p(z)}/N, j = 1, \ldots, N$ where N is large and will be fixed later (it will depend on γ only). Notice that, by (3.13) and (3.14), we get

$$\|Q_{w,p(t+iy)}\|_{p(t),p(t)} \leqslant \|w^{-1/p'(t)}\mathcal{P}_{n-1}w^{1/p'(t)}\|_{p(t),p(t)} + \|w^{1/p(t)}\mathcal{P}_{n-1}w^{-1/p(t)}\|_{p(t),p(t)} < C_\gamma\,.$$

Let $\mathbf{\Lambda}$ be an absolute constant larger than one. We take N to satisfy

$$1 - C_\gamma\mathbf{\Lambda}/N > 1/2\,. \tag{3.15}$$

Next, we use (3.3) to get

$$\|(I - Q^{(1)}_{w,p(t+iy)})^{-1}\|_{p(t),p(t)} \leqslant \frac{1}{1 - C_\gamma/N} \leqslant \frac{1}{1 - C_\gamma\mathbf{\Lambda}/N} \leqslant 2 \leqslant 2\mathbf{\Lambda}$$

since $\mathbf{\Lambda} > 1$ by our choice. We continue with an inductive argument in which the bound for $\{Q^{(j)}_{w,p(z)}\}$ provides the bound for $\{Q^{(j+1)}_{w,p(z)}\}$ when $j = 1, \dots, N-1$.

• **Base of induction: handling** $Q^{(1)}_{w,p(z)}$**.** Apply proposition 3.5 with $\kappa = 1/N$ to get an absolute constant t_* so that

$$\|(I - Q^{(1)}_{w,p(t+iy)})^{-1}\|_{p(t),p(t)} \leqslant \mathbf{\Lambda}$$

for $t \in [0, t_*]$ and $y \in \mathbb{R}$. Next, we use (3.2) with $H = -Q^{(1)}_{w,p(t+iy)}$ and $V = -N^{-1}Q_{w,p(t+iy)}$. This gives

$$\|(I - Q^{(2)}_{w,p(t+iy)})^{-1}\|_{p(t),p(t)} \leqslant \frac{\mathbf{\Lambda}}{1 - C_\gamma \mathbf{\Lambda}/N} \leqslant 2\mathbf{\Lambda}, \quad t \in [0, t_*] \tag{3.16}$$

by (3.15).

That finishes the first step. Next, we will explain how estimates on $Q^{(2)}_{w,p(z)}$ give bounds for $Q^{(3)}_{w,p(z)}$.

• **Handling** $Q^{(2)}_{w,p(z)}$**.** In proposition 3.5, we now take $\kappa = \kappa_2 \overset{\text{def}}{=} 2/N, p^{(2)}_* \overset{\text{def}}{=} p_1(t_*) = p(t_*)$ (here $p(t_*)$ is obtained at the previous step) and compute new $p_2(z), p_2'(z)$ by (3.9):

$$\frac{1}{p_2(z)} = \frac{z}{p(t_*)} + \frac{1-z}{2} = \frac{zt_*}{p_*} + \frac{1-zt_*}{2} = \frac{1}{p_1(zt_*)} = \frac{1}{p(zt_*)} \,. \tag{3.17}$$

Therefore, when z belongs to $0 < \operatorname{Re} z < 1$, zt^* belongs to $0 < \operatorname{Re} z < t_*$ and $p_2(z) = p(zt_*)$. In this domain, we have an estimate (3.16) which can be rewritten as

$$\|(I - Q^{(2)}_{w,p_2(t+iy)})^{-1}\|_{p_2(t),p_2(t)} \leqslant 2\mathbf{\Lambda}, \quad t \in [0,1], \quad y \in \mathbb{R}\,,$$

where $p_2(z)$ is different from $p_1(z) = p(z)$ only by the choice of parameter p_* in (3.9) and is in fact a rescaling of the original $p(z)$ as follows from (3.17). Thus, from proposition 3.5, we have

$$\|(I - Q^{(2)}_{w,p_2(t+iy)})^{-1}\|_{p_2(t),p_2(t)} \leqslant \mathbf{\Lambda}$$

for $t \in [0, t_*], y \in \mathbb{R}$. We use the perturbative bound (3.2) one more time with $H = -Q^{(2)}_{w,p_2(t+iy)}$ and $V = -N^{-1}Q_{w,p_2(t+iy)}$ to get

$$\|(I - Q^{(3)}_{w,p_2(t+iy)})^{-1}\|_{p_2(t),p_2(t)} \leqslant 2\mathbf{\Lambda}$$

for $t \in [0, t_*], y \in \mathbb{R}$.

• **Induction in** j **and the bound for** $Q^{(N)}_{w,p(z)}$**.** Next, we take $p^{(3)}_* \overset{\text{def}}{=} p^{(2)}_*(t_*)$ and repeat the process in which the bound

$$\|(I - Q^{(j)}_{w,p_j(t+iy)})^{-1}\|_{p_j(t),p_j(t)} \leqslant 2\mathbf{\Lambda}, \quad t \in [0,1], \quad y \in \mathbb{R}\,,$$

implies

$$\|(I - Q^{(j+1)}_{w,p_{j+1}(t+iy)})^{-1}\|_{p_{j+1}(t),p_{j+1}(t)} \leqslant 2\mathbf{\Lambda}$$

for $t \in [0,1]$ and $y \in \mathbb{R}$. Notice that each time the new $p_j(z)$ is in fact a rescaling of the original $p(z)$ by t_*^{j-1} as can be seen from a calculation analogous to (3.17). In $N-1$ steps, we get

$$\|(I - Q^{(N)}_{w,p_{N-1}(t+iy)})^{-1}\|_{p_{N-1}(t),p_{N-1}(t)} \leqslant 2\mathbf{\Lambda}\,, \quad t \in [0, t_*], \quad y \in \mathbb{R}.$$

Thus, taking $y = 0$ and $t = t_*$, and recalling that $p_{N-1}(z) = p(t_*^{N-2}z)$, one has

$$\|(I - Q^{(N)}_{w,p(t_*^{N-1})})^{-1}\|_{p(t_*^{N-1}),p(t_*^{N-1})} \leqslant 2\mathbf{\Lambda}\,.$$

Since $Q^{(N)}_{w,p(t_*^N)} = Q_{w,p(t_*^N)}$, we get (3.11) with

$$\widetilde{p}_\gamma = \frac{2\widehat{p}_\gamma}{2t_*^{N-1} + \widehat{p}_\gamma(1 - t_*^{N-1})}.$$

The estimates (3.15) implies that we can take $N \sim C_\gamma$. □

Proof of theorem 1.1. From lemma 3.3 and lemma 3.6, we get that $p_{\rm cr}(t) > 2$ and $\lim_{t\to 1} p_{\rm cr}(t) = \infty$. To show that $p_{\rm cr}(t) \to 2$ when $t \to \infty$, it is enough to start with arbitrarily large t and present a weight $\widehat{w}$ such that $[\widehat{w}]_{A_2(\mathbb{T})} \leqslant t$ and $\sup_n \|\varphi_n(\xi, \widehat{w})\|_{L^{p(t)}_{\widehat{w}}(\mathbb{T})} = +\infty$ with some $p(t)$ which depends on t and

$\lim_{t\to\infty} p(t) = 2$. To this end, we use the following result established in [10], theorem 3.2: *given any $t > 2$, there is a weight w that satisfies $1 \leqslant w \leqslant t$ and a subsequence $\{k_n\}$ such that*

$$\|\varphi_{k_n}(\xi, w)\|_{L^\infty(\mathbb{T})} \geqslant C(t) k_n^{1/2 - ct^{-1/6}}\,.$$

The weight w in the statement does not satisfy condition $\|\frac{w}{2\pi}\|_{L^1(\mathbb{T})} = 1$. However, for $\widehat{w} = 2\pi w/\|w\|_{L^1(\mathbb{T})}$, we will have

$$\Big\|\frac{\widehat{w}}{2\pi}\Big\|_{L^1(\mathbb{T})} = 1, \quad \frac{\sup_{\mathbb{T}} \widehat{w}}{\inf_{\mathbb{T}} \widehat{w}} \leqslant t \tag{3.18}$$

and

$$\|\varphi_{k_n}(\xi, \widehat{w})\|_{L^\infty(\mathbb{T})} \geqslant C(t) k_n^{1/2 - ct^{-1/6}}\,.$$

Nikolskii inequality (see p.102, theorem 2.6, [11]) gives $\|\varphi_{k_n}(\xi, \widehat{w})\|_{L^p(\mathbb{T})} \geqslant C(t,p) k_n^{1/2-1/p-ct^{-1/6}}$ and thus

$$\|\varphi_{k_n}(\xi, \widehat{w})\|_{L^p_{\widehat{w}}(\mathbb{T})} \geqslant C(t,p) k_n^{1/2-1/p-ct^{-1/6}}\,.$$

The weight $\widehat{w}$ satisfies the trivial bound $[\widehat{w}]_{A_2(\mathbb{T})} \leqslant t$. Therefore,

$$p_{\rm cr}(t) \leqslant \frac{2t^{1/6}}{t^{1/6} - 2c} = 2 + O(t^{-1/6}),\ t \to \infty\,.$$

□

Remark. Some lower bounds on $p_{\rm cr}(t)$ when $t \to 1$ and $t \to \infty$ can be traced through the proof. We do not include these calculations here.

Proof of corollary 1.3. We have (see [16], formula (5.37) or [13], section 2)

$$\lim_{n\to\infty} \|\varphi_n^* - D^{-1}\|_{L^2_w(\mathbb{T})} = 0\,. \tag{3.19}$$

Recall that $q_{\rm cr}(w)$ was defined in (1.4). Take $\widetilde{p} \in [2, \min(p_{\rm cr}([w]_{A^2}), 2(1 + q_{\rm cr}(w))))$. For $p \in [2, \widetilde{p})$, we use Hölder's inequality

$$\int_{\mathbb{T}} |\varphi_n^* - D^{-1}|^p w d\theta \leqslant \left(\int_{\mathbb{T}} |\varphi_n^* - D^{-1}|^{p_1\alpha} w d\theta\right)^{1/\alpha} \cdot \left(\int_{\mathbb{T}} |\varphi_n^* - D^{-1}|^{p_2\alpha'} w d\theta\right)^{1/\alpha'}\,, \tag{3.20}$$

where $p_1 + p_2 = p, p_1\alpha = \widetilde{p}, p_2\alpha' = 2, \alpha^{-1} + \alpha'^{-1} = 1, \alpha \in (1, \infty)$. In fact, solving these equations gives $\alpha = (\widetilde{p} - 2)/(p - 2)$, $p_1 = \widetilde{p}(p - 2)/(\widetilde{p} - 2)$, $p_2 = 2(\widetilde{p} - p)/(\widetilde{p} - 2)$. The second factor in the right hand side of (3.20) converges to zero due to (3.19). For the first one, we apply the triangle inequality to write

$$\sup_n \left(\int_{\mathbb{T}} |\varphi_n^* - D^{-1}|^{\widetilde{p}} w d\theta\right)^{1/\widetilde{p}} \leqslant \sup_n \|\varphi_n^*\|_{\widetilde{p},w} + \|D^{-1}\|_{\widetilde{p},w}\,.$$

The first term is finite thanks to theorem 1.1. For the second one, we use $w = |D|^2$ to write

$$\|D^{-1}\|_{\widetilde{p},w}^{\widetilde{p}} = \int_{\mathbb{T}} |D^{-1}|^{\widetilde{p}} w d\theta = \int_{\mathbb{T}} w^{1-\widetilde{p}/2} d\theta < \infty\,,$$

because $\widetilde{p}/2 - 1 < q_{\rm cr}(w)$. □

Proof of corollary 1.4. Let $S \stackrel{\rm def}{=} D^{-1}$ for shorthand. Recall that $|\varphi_n| = |\varphi_n^*|$ on $\mathbb{T}$. The following inequality follows from the Mean Value Formula

$$|x^2 \log x - y^2 \log y| \lesssim (1 + x|\log x| + y|\log y|)|x - y|, \quad x, y \geqslant 0\,.$$

Hence,

$$\int_{-\pi}^{\pi} ||\varphi_n^*|^2 \log|\varphi_n^*| - |S|^2 \log|S|| w d\theta \lesssim \int_{-\pi}^{\pi} (1 + |\varphi_n^* \log|\varphi_n^*|| + |S \log|S||)||\varphi_n^*| - |S|| w d\theta\,.$$

Then, one can write

$$\int_{-\pi}^{\pi} (1 + |\varphi_n^* \log|\varphi_n|| + |S \log|S||)||\varphi_n^*| - |S|| w d\theta \leqslant$$

$$C(\delta) \left(\int_{-\pi}^{\pi} (1 + |\varphi_n^*|^{2+\delta} + |S|^{2+\delta}) w d\theta\right)^{1/2} \left(\int_{\mathbb{T}} |\varphi_n^* - S|^2 w d\theta\right)^{1/2}$$

by applying Cauchy-Schwarz inequality and the trivial bound: $(1+u|\log u|)^2 \leqslant C(\delta)(1+u^{2+\delta}),\ \delta>0$. The second factor converges to zero when $n\to\infty$ due to (3.19). For the first one, theorem 1.1 and identity $|S| = w^{-1/2}$ allow us to find $\delta>0$ such that

$$\sup_n \int_{-\pi}^{\pi} (|\varphi_n^*|^{2+\delta} + |S|^{2+\delta}) w d\theta < \infty \,.$$

□

In the rest of this section, we will show that theorem 1.1 implies theorem 1.6. We start with the following lemma.

Lemma 3.7. *If $w, w^{-1} \in \mathrm{BMO}(\mathbb{T})$, then $w \in A_2(\mathbb{T})$.*

Proof. Let $s \stackrel{\text{def}}{=} \|w\|_{\mathrm{BMO}(\mathbb{T})}, t \stackrel{\text{def}}{=} \|w^{-1}\|_{\mathrm{BMO}(\mathbb{T})}$ for shorthand. Consider any interval $I\subseteq \mathbb{T}$. We define $a \stackrel{\text{def}}{=} \langle w\rangle_I, b \stackrel{\text{def}}{=} \langle w^{-1}\rangle_I$. We have

$$\langle |w-a|\rangle_I \leqslant s, \quad \langle |w^{-1}-b|\rangle_I \leqslant t$$

by the definition of BMO space. To estimate $A_2(\mathbb{T})$ characteristic, we need to bound ab. We assume without loss of generality that $I=[0,1]$ and that $a\leqslant b$. Apply triangle's inequality and an estimate

$$\frac{1}{|I|}\|w-\langle w\rangle_I\|^2_{L^2(I)} \lesssim s^2$$

(see [30], p.144, formula (7)), to get

$$\|w\|_2 \leqslant \|w-a\|_2 + \|a\|_2 \lesssim s + a\,, \tag{3.21}$$

where here and in the rest of the proof all estimates are done with respect to $I=[0,1]$. Consider a set $\Omega \stackrel{\text{def}}{=} \{|w^{-1}-b| \leqslant 0.5b\}$. By John-Nirenberg inequality ([30], p.145, formula (8)), we can estimate the measure of its complement via

$$|\Omega^c| \lesssim \exp\left(-c_1 b t^{-1}\right)\,, \tag{3.22}$$

where c_1 is an absolute positive constant. We can rewrite Ω as follows $\Omega = \{0.5b \leqslant w^{-1} \leqslant 1.5b\} = \{2/(3b) \leqslant w \leqslant 2/b\}$ and this formula shows that

$$\int_{w>2/b} d\theta \leqslant |\Omega^c| \lesssim \exp(-c_1 b t^{-1})\,. \tag{3.23}$$

Then,

$$a = \int_{w\leqslant 2/b} w d\theta + \int_{w>2/b} w d\theta$$

and consequently

$$\int_{w>2/b} w d\theta = a - \int_{w\leqslant 2/b} w d\theta \geqslant a - 2/b\,.$$

On the other hand, by Cauchy-Schwarz inequality and (3.23),

$$\int_{w>2/b} w d\theta \leqslant \|w\|_2 \left(\int_{w>2/b} d\theta\right)^{1/2} \lesssim (s+a)\exp\left(-c_1 b t^{-1}/2\right)\,.$$

Putting these bounds together, we get

$$ab \lesssim 1 + (s+a)b\exp(-c_1 b t^{-1}/2)\,.$$

Since $\sup_{t>0} bt^{-1}\exp(-c_1 bt^{-1}/2) \lesssim 1$, the following estimate holds

$$ab \lesssim 1 + st + ab\exp(-c_1 bt^{-1}/2)\,.$$

Recall that $a\leqslant b$. Thus, an elementary bound $\sup_{t>0} b^2 t^{-2}\exp(-c_1 bt^{-1}/2) < \infty$ yields

$$ab\exp(-c_1 bt^{-1}/2) \leqslant b^2\exp(-c_1 bt^{-1}/2) \lesssim t^2\,.$$

We finally get

$$ab \lesssim 1 + st + t^2 \lesssim 1 + s^2 + t^2$$

and that proves the lemma. □

Now, given this lemma, we can argue in the following way. If $w, w^{-1} \in \mathrm{BMO}(\mathbb{T})$, then $w \in A_2(\mathbb{T})$ and theorem 1.1 yields

$$\sup_n \int_{\mathbb{T}} |\varphi_n|^p w d\theta < \infty\,, \quad 2 \leqslant p < p_{\mathrm{cr}}([w]_{A^2})\,. \tag{3.24}$$

Therefore, for every $q \in [2, p)$, we can use Hölder's inequality

$$\int_{\mathbb{T}} |\varphi_n|^q d\theta = \int_{\mathbb{T}} |\varphi_n|^q w^\beta w^{-\beta} d\theta \leqslant \left(\int_{\mathbb{T}} |\varphi_n|^{q\alpha} w^{\beta\alpha} d\theta\right)^{1/\alpha} \left(\int_{\mathbb{T}} w^{-\beta\alpha'} d\theta\right)^{1/\alpha'} \tag{3.25}$$

and choose $\alpha \in (1, \infty)$ and $\beta > 0$ such that $\beta\alpha = 1, q\alpha = p$. The first factor in the right hand side of (3.25) is controlled by (3.24). Since $w^{-1} \in \mathrm{BMO}(\mathbb{T})$, the second factor is finite due to John-Nirenberg estimate and we get $\sup_n \|\varphi_n\|_{L^q(\mathbb{T})} < \infty$ as claimed in theorem 1.6. This argument shows that theorem 1.1 is qualitatively stronger than theorem 1.6.

4. The Christoffel-Darboux Kernel and bounds for the associated Projection Operator

In this section, we study the projection operators associated to $\{\varphi_n(z, w)\}_{n\geqslant 0}$. Recall the Christoffel-Darboux kernel is defined as (see [28], p.120)

$$K_n(z, \zeta, w) = \sum_{k=0}^{n} \varphi_k(z, w)\overline{\varphi_k(\zeta, w)}.$$

In particular, $K_n(z, \zeta, w)$ is integral kernel associated to the orthogonal projection operator $\mathcal{P}^w_{[0,n]}$ onto $\mathrm{Span}\{\varphi_0, \ldots, \varphi_n\}$ in $L^2_w(\mathbb{T})$; see [28] for more details. In this section, we prove that these projections are uniformly bounded:

Theorem 4.1. *Suppose $w \in A_2(\mathbb{T})$, with $\gamma \stackrel{\mathrm{def}}{=} [w]_{A_2(\mathbb{T})}$. Then, there exists $\epsilon_\gamma > 0$ such that*

$$\sup_n \|\mathcal{P}^w_{[0,n]}\|_{L^p_w(\mathbb{T}), L^p_w(\mathbb{T})} < \infty$$

for all $p \in [2 - \epsilon_\gamma, 2 + \epsilon_\gamma]$.

Recall (check (1.6)) that the Szegő function D can be introduced for any weight w that satisfies $\log w \in L^1(\mathbb{T})$. We define the subspace $H_{2,w}(\mathbb{T})$ as the closure of $\mathrm{Span}\{\varphi_n\}_{n\geqslant 0} = \mathrm{Span}\{z^n\}_{n\geqslant 0}$ in $L^2_w(\mathbb{T})$ metric. Denote by $\mathcal{P}^w_{[0,\infty]}$ the operator of orthogonal projection onto $H_{2,w}(\mathbb{T})$ in $L^2_w(\mathbb{T})$. By Beurling's theorem ([17], p.79), function f belongs to $H_{2,w}(\mathbb{T})$ if and only if $f = D^{-1}g$ where g is an element of the Hardy space $H_2(\mathbb{T})$, e.g., $H_{2,w}(\mathbb{T}) = D^{-1}H_2(\mathbb{T})$. Recall the standard notation that $H_2(\mathbb{T})$ denotes the restriction of functions in $H_2(\mathbb{D})$ onto $\mathbb{T}$. Since $w = |D|^2$, the map $g \to D^{-1}g$ is unitary isomorphism between $L^2(\mathbb{T})$ and $L^2_w(\mathbb{T})$. The restriction of the same map to $H^2(\mathbb{T})$ is unitary isomorphism between $H_2(\mathbb{T})$ and $H_{2,w}(\mathbb{T})$. Finally, the orthogonal projection of $f \in L^2(\mathbb{T})$ to $H_2(\mathbb{T})$ is given by $\lim_{r\to 1} \mathcal{C}(f, r\xi)$ (see (1.10) and [14], p.2) where the limit exists for a.e. $\xi \in \mathbb{T}$. Thus, we can write

$$\mathcal{P}^w_{[0,\infty]}(f)(\xi) \stackrel{\mathrm{def}}{=} \lim_{r\to 1} \frac{1}{D(\xi)} \mathcal{C}\Big(fD, r\xi\Big), \quad \xi \in \mathbb{T}, \tag{4.1}$$

where $\mathcal{C}$ is Cauchy integral.

Lemma 4.2. *If $p \in (1, \infty)$ and $w^{1-p/2} \in A_p(\mathbb{T})$, then $\mathcal{P}^w_{[0,\infty]}$ is bounded on $L^p_w(\mathbb{T})$.*

Proof. Let $\zeta \in \mathbb{T}$ and $z \in \mathbb{D}$. The Cauchy kernel in (1.10) can be written as

$$\frac{1}{1 - \bar{\zeta}z} = \frac{1}{2}\left(\frac{1 + \bar{\zeta}z}{1 - \bar{\zeta}z} + 1\right).$$

The first term inside the parenthesis

$$\frac{1 + \bar{\zeta}z}{1 - \bar{\zeta}z} = \frac{\zeta + z}{\zeta - z}$$

is the so-called Schwarz kernel. Two real parts of Schwarz kernel is Poisson kernel (1.9) and its imaginary part, when restricted to $\mathbb{T}$, defines $\mathfrak{h}$ in (1.8). Therefore, for $f \in L^p_w(\mathbb{T})$, we can use (4.1) and (1.5) to get

$$\begin{aligned} |\mathcal{P}^w_{[0,\infty]}(f)| &\lesssim \lim_{r\to 1} \frac{1}{|D|}\mathcal{P}(|fD|, r\xi) + \frac{1}{|D|}\int_{\mathbb{T}} |fD| d\theta + \left|\frac{1}{D}\mathfrak{h}\,(fD)\right| \\ &= |f| + \frac{1}{|D|}\int_{\mathbb{T}} |fD| d\theta + \left|\frac{1}{D}\mathfrak{h}\,(fD)\right| \end{aligned} \tag{4.2}$$

due to (see p.11, [17]) and the identity

$$\lim_{r\to 1}\mathcal{P}(g,r\xi)=g(\xi),\quad \text{a.e. } \xi\in\mathbb{T}$$

which holds for $g\in L^1(\mathbb{T})$. Since $f\in L^p_w(\mathbb{T})$ and $w=|D|^2$, we get

$$\left\|\frac{1}{|D|}\int_{\mathbb{T}}|fD|d\theta\right\|_{L^p_w(\mathbb{T})}=\left(\int_{\mathbb{T}}w^{1-p/2}d\theta\right)^{1/p}\cdot\left(\int_{\mathbb{T}}|f|\sqrt{w}d\theta\right).$$

Since $w^{1-p/2}\in A_p(\mathbb{T})$ and $A_p(\mathbb{T})\subset L^1(\mathbb{T})$, the first integral converges. For the second one, we use Hölder's inequality

$$\int_{\mathbb{T}}|f|\sqrt{w}d\theta=\int_{\mathbb{T}}(|f|w^{1/p})(w^{1/2-1/p})d\theta\leqslant\left(\int_{\mathbb{T}}|f|^pwd\theta\right)^{1/p}\left(\int_{\mathbb{T}}w^{(1/2-1/p)p'}d\theta\right)^{1/p'}.$$

To show that the integral

$$\int_{\mathbb{T}}w^{(1/2-1/p)p'}d\theta=\int_{\mathbb{T}}w^{\frac{(p-2)}{2(p-1)}}d\theta$$

converges, we recall that $w^{1-p/2}\in A_p(\mathbb{T})$ implies that $w^{\frac{(p-2)}{2(p-1)}}\in L^1(\mathbb{T})$ as follows from the definition of $A_p(\mathbb{T})$ given in (1.2). We are left with estimating $L^p_w(\mathbb{T})$ norm of the third term in (4.2). The operator of harmonic conjugation $\mathfrak{h}$ is one of the basic singular integral operators and the Hunt-Muckenhoupt-Wheeden theorem claims (see, e.g., [30], p.205) that $\upsilon^{1/p}\mathfrak{h}\upsilon^{-1/p}$ is a bounded operator on $L^p(\mathbb{T})$ if $\upsilon\in A_p(\mathbb{T})$ and $p\in(1,\infty)$. Since $w=|D|^2$ and $w^{1-p/2}\in A_p(\mathbb{T})$, we get statement of the lemma thanks to the formula

$$\|w^{-1/2}\mathfrak{h}(w^{1/2}f)\|_{L^p_w(\mathbb{T})}=\|w^{-1/2+1/p}\mathfrak{h}(w^{1/2-1/p}(w^{1/p}f))\|_{L^p(\mathbb{T})}$$

after one takes $\upsilon=w^{1-p/2}$ and notices that $\|w^{1/p}f\|_{L^p(\mathbb{T})}=\|f\|_{L^p_w(\mathbb{T})}$. □

This yields the following corollary.

Corollary 4.3. *Let $w\in A_2(\mathbb{T})$. Then, $\mathcal{P}^w_{[0,\infty]}$ is bounded on $L^p_w(\mathbb{T})$ for all $p\in[4/3,4]$.*

Proof. The projection is self-adjoint operator in $L^2_w(\mathbb{T})$. Therefore, by duality, it is enough to consider $p\in[2,4]$. For $p=4$, we have $w^{-1}\in A_2(\mathbb{T})\subset A_4(\mathbb{T})$ and the previous lemma applies. If $p=2$, the projection operator has norm 1. Thus, by Riesz-Thorin interpolation, we have an estimate for all $p\in[2,4]$. □

Define the projection operator onto Span$\{\varphi_n\}_{n\geqslant a+1}$ by

$$\mathcal{P}^w_{[a+1,\infty]}\overset{\text{def}}{=}\mathcal{P}^w_{[0,\infty]}-\mathcal{P}^w_{[0,a]}.$$

When $w\in A_2(\mathbb{T})$ and $p\in[4/3,4]$, $\{\mathcal{P}^w_{[0,n]}\}_{n\geqslant 0}$ is uniformly bounded on $L^p_w(\mathbb{T})$ if and only if $\{\mathcal{P}^w_{[n+1,\infty,]}\}_{n\geqslant 0}$ is uniformly bounded on $L^p_w(\mathbb{T})$. We will show the latter. To apply the same process as in section 3 for getting bounds for the polynomials $\{\varphi_n\}$, one needs the following identities.

Lemma 4.4. *If $\mathcal{P}^1_{[0,n]}$ corresponds to the unperturbed case $w=1$, then*

$$\begin{cases}\mathcal{P}^w_{[n+1,\infty]}=(I-\mathcal{P}^1_{[0,n]})\mathcal{P}^w_{[0,\infty]}+\mathcal{P}^1_{[0,n]}\mathcal{P}^w_{[n+1,\infty]}\\ \mathcal{P}^1_{[0,n]}w\mathcal{P}^w_{[n+1,\infty]}=0\end{cases}.$$

Proof. To prove the first identity, first note that applying both operators to a function f is the same as applying it to $\mathcal{P}^w_{[0,\infty]}f$, so it suffices to verify the identity for all functions in the range of $\mathcal{P}^w_{[0,\infty]}$ which is the closure of finite sums $\sum_{j=0}^N a_j\varphi_j(z)$. The formula then follows from $\mathcal{P}^1_{[0,n]}\varphi_k=\varphi_k$ for all $k\leqslant n$. To prove the second identity, it suffices to note that the range of $\mathcal{P}^w_{[n+1,\infty]}$ will be the closed span of $\{\varphi_{n+1},\varphi_{n+2},\ldots\}$; since $\varphi_{n+j}\perp_w\{1,z,\ldots,z^n\}$, it follows that $\mathcal{P}^1_{[0,n]}w\varphi_{n+j}=0$ for all $j\geqslant 1$, whence the identity. □

Proof of theorem 4.1. By duality, it is sufficient to consider $p>2$. Let $X_n\overset{\text{def}}{=}w^{1/p}\mathcal{P}^w_{[n+1,\infty]}w^{-1/p}$ and $X_\infty\overset{\text{def}}{=}w^{1/p}\mathcal{P}^w_{[0,\infty)}w^{-1/p}$. We need to estimate $\|X_n\|_{p,p}$. Rewriting the relations of the above lemma in terms of operators on $L^p(\mathbb{T})$, we get

$$\begin{cases}X_n=w^{1/p}(I-\mathcal{P}^1_{[0,n]})w^{-1/p}X_\infty+w^{1/p}\mathcal{P}^1_{[0,n]}w^{-1/p}X_n\\ w^{-1/p'}\mathcal{P}^1_{[0,n]}w^{1/p'}X_n=0\end{cases}.$$

Subtracting the bottom from the top and rearranging, we get back

$$(I - Q_{w,p})X_n = w^{1/p}(I - \mathcal{P}^1_{[0,n]})w^{-1/p}X_\infty.$$

Notice that $\sup_n \|w^{1/p}(I-\mathcal{P}^1_{[0,n]})w^{-1/p}X_\infty\|_{p,p} < \infty$ by Hunt-Muckenhoupt-Wheeden theorem and lemma 4.3. Furthermore, the proof of lemma 3.6 implies that $(I-Q_{w,p})$ on the left side of the equality has an inverse which is bounded in $L^p(\mathbb{T})$ uniformly in n for all $p \in [2, 2+\epsilon_\gamma] \subseteq [2,4]$ if ϵ_γ is small enough. Putting all of this together, we get

$$X_n = (I - Q_{w,p})^{-1}\Big(w^{1/p}(I - \mathcal{P}^1_{[0,n]})w^{-1/p}X_\infty\Big).$$

Therefore, $\{X_n\}_{n\geqslant 0}$ is uniformly bounded, completing the proof. □

5. Weights in $A_2(\mathbb{T})$ and their Aleksandrov-Clark measures

Several generalizations of $A_2(\mathbb{T})$ and $A_\infty(\mathbb{T})$ classes were studied in the literature (see, e.g., [29]). We will need two definitions here.

Definition. We say that $w \in A_2^{\mathcal{P}}(\mathbb{T})$ if

$$[w]_{A_2^{\mathcal{P}}(\mathbb{T})} \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}}\Big(\mathcal{P}(w,z)\mathcal{P}(w^{-1},z)\Big) < \infty \tag{5.1}$$

and $w \in A_\infty^{\mathcal{P}}(\mathbb{T})$ if

$$[w]_{A_\infty^{\mathcal{P}}(\mathbb{T})} \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}}\Big(\mathcal{P}(w,z)\exp(-\mathcal{P}(\log w, z))\Big) < \infty\,. \tag{5.2}$$

By Jensen's inequality, we have

$$[w]_{A_\infty^{\mathcal{P}}(\mathbb{T})} \leqslant [w]_{A_2^{\mathcal{P}}(\mathbb{T})}\,. \tag{5.3}$$

The following lemma is part of the folklore of modern Harmonic Analysis, we include its proof for completeness.

Lemma 5.1. *We have $A_2(\mathbb{T}) = A_2^{\mathcal{P}}(\mathbb{T}) \subseteq A_\infty^{\mathcal{P}}(\mathbb{T})$.*

Proof. By (5.3), we get the second inclusion. The inclusion $A_2^{\mathcal{P}}(\mathbb{T}) \subseteq A_2(\mathbb{T})$ follows from a bound

$$\frac{1}{|I|^2}\left(\int_I w\,d\theta\right)\left(\int_I w^{-1}d\theta\right) \lesssim \mathcal{P}(w,z_I)\mathcal{P}(w^{-1},z_I)\,,$$

where $z_I \stackrel{\text{def}}{=} c_I(1-0.1|I|)$ and c_I denotes the center of I. Thus, we only need to show $A_2(\mathbb{T}) \subseteq A_2^{\mathcal{P}}(\mathbb{T})$. Due to the rotational symmetry of $\mathbb{D}$, it is enough to take a point $z = 1-\epsilon, \epsilon \in [0,1)$ and prove that

$$\left(\int_{-\pi}^{\pi}\frac{\epsilon}{\epsilon^2+\theta^2}w(\theta)d\theta\right)\left(\int_{-\pi}^{\pi}\frac{\epsilon}{\epsilon^2+\theta^2}w^{-1}(\theta)d\theta\right) < C([w]_{A_2(\mathbb{T})})\,. \tag{5.4}$$

We can assume without loss of generality that

$$\langle w\rangle_{[0,\epsilon]} = 1, \quad \langle w^{-1}\rangle_{[0,\epsilon]} \leqslant [w]_{A_2(\mathbb{T})}\,.$$

In [20], Lerner and Perez proved, in particular, that:

Given $p \in (1,\infty)$, we have $w \in A_p(\mathbb{R})$ if and only if for every $\gamma > 0$ there is $C(\gamma, [w]_{A_p})$ such that

$$\frac{|E|}{|I|}\log^\gamma\left(\frac{|I|}{|E|}\right) \leqslant C(\gamma,[w]_{A_p})\left(\frac{w(E)}{w(I)}\right)^{1/p},$$

where I is any interval in $\mathbb{R}$ and $E \subset I$.

Since each $w \in A_2(\mathbb{T})$ can be considered as a 2π-periodic weight on $\mathbb{R}$ with $[w]_{A_2(\mathbb{R})} \lesssim [w]_{A_2(\mathbb{T})}$, the result of Lerner and Perez holds for $\mathbb{T}$ as well. We take $p = 2$, $E = [0,\epsilon], I = [0,x], 2\epsilon < x < \pi$ to get

$$\frac{1}{x}\int_0^x w(s)ds \leqslant C(\gamma,[w]_{A_2(\mathbb{T})})\frac{x}{\epsilon}\log^{-2\gamma}\left(\frac{x}{\epsilon}\right).$$

Therefore when $\gamma > 1/2$ is fixed,

$$\int_0^\pi \frac{\epsilon w(x)}{\epsilon^2+x^2}dx \lesssim \epsilon^{-1}\int_0^{2\epsilon} w(x)dx + \epsilon\int_{2\epsilon}^\pi \frac{w(x)}{x^2}dx \leqslant C([w]_{A_2}) + \epsilon\int_{2\epsilon}^\pi \frac{1}{x^2}\left(\int_{2\epsilon}^x w(\tau)d\tau\right)' dx \lesssim$$

$$C([w]_{A_2}) + \epsilon\int_{2\epsilon}^\pi w(x)dx + C(\gamma,[w]_{A_2(\mathbb{T})})\int_{2\epsilon}^\pi \frac{\log^{-2\gamma}(x/\epsilon)}{x}dx < C([w]_{A_2(\mathbb{T})})\,,$$

where in the second inequality we used that A_2 weights are doubling, along with our normalization. The integral over $[-\pi, 0]$ can be estimated in the same way. Thus,

$$\int_{\mathbb{T}} \frac{\epsilon w(x)}{\epsilon^2 + x^2} dx < C([w]_{A_2(\mathbb{T})}) \tag{5.5}$$

and we get a similar estimate for w^{-1} because $w^{-1} \in A_2(\mathbb{T})$. We obtained (5.4) and the lemma is proved. □

The following lemma was proved in [7] (see lemma 2 in this reference). We provide the sketch of the proof here.

Lemma 5.2. *If $w \in A_\infty^P(\mathbb{T})$ and $d\mu = \frac{w}{2\pi} d\theta$, then μ_α is absolutely continuous and $d\mu_\alpha = \frac{w_\alpha}{2\pi} d\theta$ for every $\alpha \in \mathbb{T}$. Moreover, $w_\alpha \in A_\infty^P(\mathbb{T})$.*

Proof. Given probability measure $\mu : d\mu = \frac{w}{2\pi} d\theta + d\mu_{\mathbf{s}}$, consider a generalized entropy

$$\mathcal{K}(\mu, z) = \log \mathcal{P}(\mu, z) - \mathcal{P}(\log w, z), \qquad z \in \mathbb{D}.$$

If we introduce f, the Schur function of measure μ, through the formula

$$\frac{1 + zf(z)}{1 - zf(z)} = F(z) = \int_{\mathbb{T}} \frac{1 + \bar{\xi} z}{1 - \bar{\xi} z} d\mu(\xi), \quad z \in \mathbb{D}, \quad \xi = e^{i\theta}, \tag{5.6}$$

then the straightforward but lengthy calculation shows that

$$\mathcal{K}(\mu, z) = \frac{1}{2\pi} \int_{\mathbb{T}} \log \left(\frac{1 - |zf(z)|^2}{1 - |f(\xi)|^2} \right) \frac{1 - |z|^2}{|1 - \bar{\xi} z|^2} d\theta \,. \tag{5.7}$$

On the other hand, it is known that the Schur function of each measure μ_α is given by $f_\alpha = \alpha f$. Therefore, $\mathcal{K}(\mu_\alpha, z) = \mathcal{K}(\mu, z)$. Notice that $w \in A_\infty^P(\mathbb{T})$ is equivalent to $\mathcal{K}(w, z) \in L^\infty(\mathbb{D})$. Thus, if $w \in A_\infty^P(\mathbb{T})$, then $\mathcal{K}(\mu_\alpha, z) \in L^\infty(\mathbb{D})$. On the other hand, this condition implies that μ_α has no singular part. Indeed, if $d\mu_\alpha = \frac{w_\alpha}{2\pi} d\theta + d\mu_{\mathbf{s}}^{(\alpha)}$ where $\mu_{\mathbf{s}}^{(\alpha)}$ is a singular measure, then

$$\log \left(\mathcal{P}(\mu_{\mathbf{s}}^{(\alpha)}, z) + \mathcal{P}(w_\alpha, z) \right) - \mathcal{P}(\log w_\alpha, z) \leqslant C, \quad z \in \mathbb{D}\,.$$

This implies

$$\mathcal{P}(\mu_{\mathbf{s}}^{(\alpha)}, z) \leqslant \mathcal{P}(\mu_{\mathbf{s}}^{(\alpha)}, z) + \mathcal{P}(w_\alpha, z) \leqslant C \exp\left(\mathcal{P}(\log w_\alpha, z)\right) \leqslant C\mathcal{P}(w_\alpha, z)$$

by Jensen inequality, hence, $\mu_{\mathbf{s}}^{(\alpha)} = 0$.

□

Proof of theorem 1.5. The first claim is immediate from lemma 5.1 and lemma 5.2. Now, let us show that $w_\alpha \in A_2(\mathbb{T})$. We will consider $w_{-1} = w_{\text{dual}}$ only, the cases of other α can be handled similarly. We can write $F(e^{i\theta}) = w + i\widetilde{w}$, where $\widetilde{w}$ is a harmonic conjugate function. Then, since $\operatorname{Re} F_{-1} = \operatorname{Re} F^{-1} = \operatorname{Re} F / |F|^2$, we get

$$w_{\text{dual}} = \frac{w}{w^2 + \widetilde{w}^2}\,.$$

Without loss of generality, we can consider an interval $I_\epsilon \stackrel{\text{def}}{=} [-\epsilon, \epsilon]$ when checking $A_2(\mathbb{T})$ condition for w_{dual}. We need to control

$$K \stackrel{\text{def}}{=} \epsilon^{-2} \left(\int_{-\epsilon}^{\epsilon} \frac{w}{w^2 + \widetilde{w}^2} d\theta \right) \left(\int_{-\epsilon}^{\epsilon} \frac{\widetilde{w}^2 + w^2}{w} d\theta \right) \tag{5.8}$$

under assumptions

$$\langle w \rangle_{I_\epsilon} = 1, \quad \langle w^{-1} \rangle_{I_\epsilon} \leqslant [w]_{A_2(\mathbb{T})}\,. \tag{5.9}$$

Clearly,

$$\epsilon^{-2} \left(\int_{-\epsilon}^{\epsilon} \frac{w}{w^2 + \widetilde{w}^2} d\theta \right) \left(\int_{-\epsilon}^{\epsilon} w d\theta \right) \lesssim [w]_{A_2(\mathbb{T})} \tag{5.10}$$

by definition and we are left with estimating

$$\epsilon^{-2} \left(\int_{-\epsilon}^{\epsilon} \frac{w}{w^2 + \widetilde{w}^2} d\theta \right) \left(\int_{-\epsilon}^{\epsilon} \frac{\widetilde{w}^2}{w} d\theta \right)\,. \tag{5.11}$$

We can write

$$\widetilde{w} = h_1 + h_2, \quad h_1 \stackrel{\text{def}}{=} \mathfrak{h}(w\chi_{[-2\epsilon, 2\epsilon]}), \quad h_2 \stackrel{\text{def}}{=} \mathfrak{h}(w\chi_{[-2\epsilon, 2\epsilon]^c})\,,$$

where $\mathfrak{h}$ is harmonic conjugation, a standard singular integral operator. Hence,

$$\int_{-\epsilon}^{\epsilon} w^{-1}|h_1|^2 d\theta \leqslant \int_{\mathbb{T}} w^{-1}|h_1|^2 d\theta = \int_{\mathbb{T}} w^{-1}|\mathfrak{h}(w^{1/2}\cdot w^{1/2}\chi_{[-2\epsilon,2\epsilon]}|^2 d\theta \leqslant C([w]_{A_2(\mathbb{T})})\int_{-2\epsilon}^{2\epsilon} w d\theta$$

if we use the Hunt-Muckenhoupt-Wheeden theorem with weight $w^{-1}\in A^2(\mathbb{T})$ and $w^{-1/2}\mathfrak{h}w^{1/2}$ applied to function $w^{1/2}\chi_{[-2\epsilon,2\epsilon]}$. In (5.11), this gives the contribution

$$\epsilon^{-2}\left(\int_{-\epsilon}^{\epsilon}\frac{w}{w^2+\widetilde{w}^2}d\theta\right)\left(\int_{-\epsilon}^{\epsilon}\frac{h_1^2}{w}d\theta\right)\leqslant C([w]_{A_2(\mathbb{T})})\epsilon^{-2}\left(\int_{-2\epsilon}^{2\epsilon}wd\theta\right)\left(\int_{-2\epsilon}^{2\epsilon}w^{-1}d\theta\right)\leqslant C([w]_{A_2(\mathbb{T})})\,. \tag{5.12}$$

We are left with controlling

$$\epsilon^{-2}\left(\int_{-\epsilon}^{\epsilon}\frac{w}{w^2+\widetilde{w}^2}d\theta\right)\left(\int_{-\epsilon}^{\epsilon}\frac{h_2^2}{w}d\theta\right)\,. \tag{5.13}$$

Notice that

$$h_2(\varphi)=\operatorname{Im}U(e^{i\varphi}),\qquad |\varphi|<\epsilon\,,$$

where

$$U(\zeta)\stackrel{\text{def}}{=}\frac{1}{2\pi}\int_{|\theta|>2\epsilon}\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}wd\theta,\quad \zeta\in\mathbb{D}\,.$$

When $|\zeta-1|<\epsilon$, we have

$$|U'(\zeta)|\lesssim\int_{|\theta|>2\epsilon}\frac{1}{|e^{i\theta}-1|^2}wd\theta\lesssim\epsilon^{-1}\int_{\mathbb{T}}\frac{\epsilon}{\theta^2+\epsilon^2}wd\theta\leqslant\epsilon^{-1}C([w]_{A_2(\mathbb{T})}),$$

where we used the bound (5.5). Therefore,

$$|\operatorname{Im}U(e^{i\varphi})-\operatorname{Im}U(1-\epsilon)|\leqslant C([w]_{A_2(\mathbb{T})}),\quad |\varphi|<\epsilon$$

as follows from the Fundamental Theorem of Calculus. Therefore,

$$\int_{-\epsilon}^{\epsilon}\frac{h_2^2}{w}d\theta\lesssim(\operatorname{Im}U(1-\epsilon))^2\int_{-\epsilon}^{\epsilon}w^{-1}d\theta+C([w]_{A_2(\mathbb{T})})\int_{-\epsilon}^{\epsilon}w^{-1}d\theta\,. \tag{5.14}$$

The second term gives the following contribution in (5.13):

$$\epsilon^{-2}\left(\int_{-\epsilon}^{\epsilon}\frac{w}{w^2+\widetilde{w}^2}d\theta\right)C([w]_{A_2(\mathbb{T})})\int_{-\epsilon}^{\epsilon}w^{-1}d\theta\leqslant C([w]_{A_2(\mathbb{T})})\left(\langle w^{-1}\rangle_{I_\epsilon}\right)^2\leqslant C([w]_{A_2(\mathbb{T})})\,, \tag{5.15}$$

where we used (5.9). For the first term in (5.14), recall that $\operatorname{Re}(F^{-1})=w/(w^2+\widetilde{w}^2)$ a.e. on $\mathbb{T}$ and estimate

$$\epsilon^{-2}\left(\int_{-\epsilon}^{\epsilon}\frac{w}{w^2+\widetilde{w}^2}d\theta\right)(\operatorname{Im}U(1-\epsilon))^2\int_{-\epsilon}^{\epsilon}w^{-1}d\theta\lesssim\left(\mathcal{P}(\operatorname{Re}(F^{-1}),1-\epsilon)\cdot(\operatorname{Im}U(1-\epsilon))^2\right)\cdot\left(\epsilon^{-1}\int_{-\epsilon}^{\epsilon}w^{-1}d\theta\right).$$

For the last factor, one can write

$$\epsilon^{-1}\int_{-\epsilon}^{\epsilon}w^{-1}d\theta\lesssim[w]_{A_2(\mathbb{T})}\,.$$

Since $\operatorname{Re}(F^{-1})$ is harmonic, μ_{dual} is absolutely continuous, and $\operatorname{Re}(F^{-1})=\operatorname{Re}F/|F|^2$, we get

$$\mathcal{P}(\operatorname{Re}(F^{-1}),1-\epsilon)\cdot(\operatorname{Im}U(1-\epsilon))^2=\frac{\operatorname{Re}F(1-\epsilon)}{|F(1-\epsilon)|^2}(\operatorname{Im}U(1-\epsilon))^2.$$

Notice that our normalization gives

$$1=(2\epsilon)^{-1}\int_{-\epsilon}^{\epsilon}wd\theta\lesssim\operatorname{Re}F(1-\epsilon)\sim\int_{-\pi}^{\pi}\frac{\epsilon}{\theta^2+\epsilon^2}wd\theta\leqslant C([w]_{A_2(\mathbb{T})})\,, \tag{5.16}$$

where the last bound is (5.5). Let us compare $\operatorname{Im}U(1-\epsilon)$ and $\operatorname{Im}F(1-\epsilon)$. By definition of F and U,

$$|U(1-\epsilon)-F(1-\epsilon)|\lesssim\frac{1}{\epsilon}\int_{-2\epsilon}^{2\epsilon}wd\theta\leqslant C([w]_{A_2(\mathbb{T})})\,.$$

Thus,

$$\begin{aligned}\frac{\operatorname{Re}F(1-\epsilon)}{|F(1-\epsilon)|^2}(\operatorname{Im}U(1-\epsilon))^2&\lesssim\frac{\operatorname{Re}F(1-\epsilon)}{|F(1-\epsilon)|^2}(|F(1-\epsilon)|^2+C([w]_{A_2(\mathbb{T})}))\\&<C([w]_{A_2(\mathbb{T})})\left(\operatorname{Re}F(1-\epsilon)+\frac{1}{\operatorname{Re}F(1-\epsilon)}\right),\end{aligned}$$

which, thanks to (5.16), is bounded by $C([w]_{A_2(\mathbb{T})})$. Summing up, we estimate K in (5.8) by $K \leqslant C([w]_{A_2(\mathbb{T})})$ and the lemma is proved. □

6. Appendix: Fisher-Hartwig weights

The Fisher-Hartwig weights are a large class of weights on the circle, which generalizes the class of Jacobi weights. It was at the focus of recent research (see, e.g., [8]) mainly due to some connections with probability and mathematical physics. For these weights, the asymptotics of polynomials is now well-understood [8]. In this section, we provide an upper bound for the function $p_{\mathrm{cr}}(t)$ using some results obtained in [21]. In particular, the analysis developed for Fisher-Hartwig weights will give us the proof of the following lemma.

Lemma 6.1. *If $t \in (1,2)$, we have $p_{\mathrm{cr}}(t) < C(t-1)^{-1/2}$.*

We provide its proof in the end of this section. For $\beta \geqslant 0$, consider the weight $w_\beta = |z-1|^{2\beta}$ on the unit circle for and the associated orthogonal polynomials $\{\Phi_n(z, w_\beta)\}$. This is a particular choice for the Fisher-Hartwig weight with the single point of singularity located at $z = 1$. Note that in order for $w_\beta \in A_2(\mathbb{T})$, one needs $2\beta < 1$, i.e. $\beta \in [0, \frac{1}{2})$. We start with the the following proposition:

Proposition 6.2. *Suppose $\beta \in [0, \frac{1}{2})$. Then*

$$[w_\beta]_{A_2(\mathbb{T})} \sim \frac{1}{1-4\beta^2} \sim \frac{1}{1-2\beta}\,.$$

Furthermore, if $\beta \in [0, 1/4]$, then

$$[w_\beta]_{A_2(\mathbb{T})} - 1 \sim \beta^2\,.$$

Remark. The first asymptotics is useful in particular when $[w_\beta]_{A_2(\mathbb{T})} > 2$, i.e. when our weight varies quite a bit, whereas when $[w_\beta]_{A_2(\mathbb{T})} - 1 < 1$, the second formula is more helpful.

Proof. It is the straightforward calculation in which the integrals over intervals I involved in the definition of $A_2(\mathbb{T})$ can be explicitly computed and estimated. We omit considering all cases here. The formula which best explains the resulting bound is

$$\langle \widetilde{w} \rangle_I \langle \widetilde{w}^{-1} \rangle_I = \frac{1}{1-4\beta^2}, \quad \widetilde{w} = |\theta|^{2\beta}$$

for $I = [0, a]$ and any $0 \leqslant a \leqslant \pi$.

□

The next proposition makes use of some statements from [21]. Similar results for Jacobi weights were obtained in [3].

Proposition 6.3. *Let $w_\beta = |z-1|^{2\beta}$, $\beta \in [0, 1/2)$. Then,*

$$\|\Phi_n(\cdot, w_\beta)\|_{L^p_{w_\beta}(\mathbb{T})} \sim_{\beta,p} \begin{cases} 1, & 2\beta - p\beta + 1 > 0 \\ \log n, & 2\beta - p\beta + 1 = 0 \\ n^{-(2\beta - p\beta + 1)}, & 2\beta - p\beta + 1 < 0 \end{cases}.$$

In particular, $\sup_n \|\Phi_n(\cdot, w_\beta)\|_{L^p_{w_\beta}(\mathbb{T})} < \infty$ if and only if $p < 2 + \frac{1}{\beta}$.

Proof. First, write

$$\|\Phi_n(\cdot, w_\beta)\|^p_{L^p(w_\beta)} = \int_{|\theta|>\delta} |\Phi_n(z, w_\beta)|^p w_\beta d\theta + \int_{|\theta|<\delta} |\Phi_n(z, w_\beta)|^p w_\beta d\theta\,,$$

where δ is a parameter independent of n. To control the first term, we use formula (1.13) of [21] to get

$$\int_{|\theta|>\delta} |\Phi_n(z, w_\beta)|^p w_\beta d\theta \leqslant C(\beta, p, \delta) \int_{|\theta|>\delta} w_\beta^{1-p/2} d\theta \leqslant C(\beta, p, \delta)\,.$$

As for the second term, using the asymptotics provided in (1.17) of [21] and applying a change of variables $x = n\theta/2$, we get

$$\int_{|\theta|<\delta} |\Phi_n(z, w_\beta)|^p w_\beta d\theta \sim_\beta n^{p\beta - 2\beta - 1} \int_0^{\delta n/2} x^{2\beta - p(\beta - 1/2)} |iJ_{\beta+1/2}(x) + J_{\beta-1/2}(x)|^p dx\,,$$

where $J_\nu(x)$ is the Bessel function of the first kind. One can then split this new integral in x up into two: when $x \in (0,1)$ and when $x \geqslant 1$. We then use the known asymptotics for Bessel functions (see, e.g., [1]) to get

$$\int_{|\theta|<\delta} |\Phi_n(z,w_\beta)|^p w_\beta d\theta \sim_\beta n^{-(2\beta-p\beta+1)}\Big(1+\int_1^{n\delta/2} x^{2\beta-p\beta}dx\Big) \sim_{\beta,p} \begin{cases} 1, & 2\beta-p\beta+1>0 \\ \log n, & 2\beta-p\beta+1=0 \\ n^{-(2\beta-p\beta+1)}, & 2\beta-p\beta+1<0 \end{cases}.$$

In particular, this quantity is bounded precisely when $2\beta-p\beta+1>0$, i.e. when $\beta < \frac{1}{p-2}$. The proposition now follows from combining the given estimates. □

Now, we are ready to prove the main lemma of this section.

Proof of lemma 6.1. From the first proposition in appendix, we get $[w_\beta]_{A_2(\mathbb{T})} - 1 \sim \beta^2$ if β is small. The second proposition shows that $\sup_n \|\Phi_n(\xi,w_\beta)\|_{L^p_{w_\beta}(\mathbb{T})} < \infty$ if and only if $p < 2+\beta^{-1}$. Combining these results we get the statement of the lemma. □

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