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Most homeomorphisms with a fixed point have a Cantor set of fixed points

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Abstract. We show that, for any $n \neq 2$, most orientation preserving homeomorphisms of the sphere S^{2n} have a Cantor set of fixed points. In other words, the set of such homeomorphisms that do *not* have a Cantor set of fixed points is of the first Baire category within the set of all homeomorphisms. Similarly, most orientation reversing homeomorphisms of the sphere S^{2n+1} have a Cantor set of fixed points for any $n \neq 0$. More generally, suppose that M is a compact manifold of dimension > 1 and $\neq 4$ and $\mathcal{H}$ is an open set of homeomorphisms $h : M \to M$ such that all elements of $\mathcal{H}$ have at least one fixed point. Then we show that most elements of $\mathcal{H}$ have a Cantor set of fixed points.

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Consider some topological space $\mathcal{T}$. We say that *most* elements of $\mathcal{T}$ have some property P if the set of elements of $\mathcal{T}$ that do not have the property P is a *first category set*, i.e., is a countable union of nowhere dense sets. Sets which are not first category sets are called second category sets. These notions are especially relevant if we know that all nonempty open subsets of $\mathcal{T}$ are second category sets, i.e., that $\mathcal{T}$ is a *Baire space*. Recall that each complete metric space is a Baire space, see [1,4].

If all elements of a Baire space $\mathcal{T}$ satisfy some property P, except for the elements in a first category set $\mathcal{S} \subset \mathcal{T}$, we say that P is a *generic property*, or that P is true for *most* elements of $\mathcal{T}$. Note that if in a Baire space $\mathcal{T}$ the properties P_i are generic for all $i \in \mathbb{N}$, then the set of elements of $\mathcal{T}$ that satisfy *all* properties P_i is generic as well. Previous analysis of generic properties related to the ones discussed here has appeared in [2,3,10].

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Consider some compact manifold M and a distance function $d : M \times M \to \mathbb{R}$ compatible with the topology of M. Also, consider the space $\mathcal{H}$ of homeomorphisms $h : M \to M$ with the topology of uniform convergence, i.e., the topology induced by the distance function $d : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, d(f,g) = \max_{x \in M} d(f(x), g(x))$. Then we have:

Propositon 1. *The space $\mathcal{H}$ is a Baire space.*

Proof. This fact has been proved in [2] for the case where the manifold M is a sphere S^n. The same proof carries over to any compact manifold, with or without boundary. The idea of the proof is that the sets

$$\mathcal{F}_n = \{f \in \overline{\mathcal{H}} : \text{diameter } (f^{-1}(y)) \geq 1/n \quad \text{for some } y \in M\}$$

are nowhere dense in $\overline{\mathcal{H}}$ for each n, where $\overline{\mathcal{H}}$ is the closure of $\mathcal{H}$ in the space of all continuous functions from M to itself. Since $\overline{\mathcal{H}}$ is a Baire space, it follows that $\overline{\mathcal{H}} \setminus \bigcup_n \mathcal{F}_n$ is a Baire space. But $\overline{\mathcal{H}} \setminus \bigcup_n \mathcal{F}_n = \mathcal{H}$, so $\mathcal{H}$ is a Baire spece. See [2] for more details. □

Lemma 2. *Consider some compact manifold M, an orientation preserving homeomorphism $g : M \to M$ that has a fixed point $x \in M$, and $\epsilon, \delta \in (0,1)$. Then there exists some homeomorphism $f : M \to M$ such that for some $\delta' \in (0, \delta)$ we have:*

(i) *$f(y) = y$, for all y such that $d(x,y) < \delta'$,*
(ii) *$d(f,g) < \epsilon$,*
(iii) *$f(y) = g(y)$ for any y such that $d(x,y) > \delta$.*

Proof. In other words, we are saying that given an orientation preserving homeomorphism $g : M \to M$ and $x \in M$ such that $g(x) = x$, we can modify g on a small neighborhood of x such that the modified g fixes not only x, but an entire (even smaller) neighborhood of x. Again, a proof of this lemma appears in [2] for $M = S^n$, and the same proof carries over to the general case described above. (Note though that there is a typographical error in the proof in [2]: the correct definition of the set $C(a,b)$ is $\{x \in \mathbb{R}^n : a \leq ||x|| \leq b\}$). □

Lemma 3. *Let M be a compact manifold of dimension different from 4. Then the set of homeomorphisms from M to M with a finite number of fixed points is dense within the set of all homeomorphisms from M to M.*

If the dimension of M is three or less, then a proof of this lemma follows from classical results in [7,8]. If the dimension of M is five or more, a proof of this lemma can be found in [5]. Also, a *different proof* for all dimensions different from four appears in [6]. Reference [6] is available via `google groups`. We thank Maria Alejandra Rodriguez Hertz, Michael R. Kelly, and Laurent Siebenmann for pointing out these references to us.

By a *Cantor set* we mean a topological space homeomorphic to the usual Cantor set. Our main result is the following:

Theorem 4. *Let M be a compact manifold of dimension larger than one and different from four, and let $\mathcal{H}$ be an open set of homeomorphisms from M to*

M such that all the elements of $\mathcal{H}$ have at least one fixed point. Then most elements of $\mathcal{H}$ have a Cantor set of fixed points.

Proof. Consider first the case when M is connected. Also, consider the case when all the homeomorphisms in $\mathcal{H}$ are orientation preserving. According to [2], most elements in $\mathcal{H}$ have a perfect set of fixed points. The proof in [2] mentions just the case when M is an even-dimensional sphere and $\mathcal{H}$ contains all orientation preserving homeomorphisms from M to M, but exactly the same proof works for any compact manifold M and any open set $\mathcal{H}$ of orientation preserving homeomorphisms. Note that we *do not* need the hypothesis that the dimension of M is different from 4 to conclude that most elements of $\mathcal{H}$ have a perfect set of fixed points (but we do need $\dim(M) > 1$).

Recall that any compact metric space that is perfect and totally disconnected is homeomorphic to the Cantor set (see [1] or [4]). Therefore, we would like to prove that most elements of $\mathcal{H}$ have a totally disconnected set of fixed points.

For any $h : M \to M$ we denote by F_h the set of fixed points of h. For any fixed $n \in \mathbb{N}$ we define:

$$\mathcal{C}_n = \left\{ h \in \mathcal{H} : F_h \text{ has at least one connected component with diameter } > \frac{1}{n} \right\}.$$

We will show that $\mathcal{C}_n$ is a nowhere dense set in $\mathcal{H}$. Denote by $\mathcal{H}_0$ the set of elements of $\mathcal{H}$ with a finite set of fixed points. According to the previous Lemma, $\mathcal{H}_0$ is dense in $\mathcal{H}$.

Consider some $f \in \mathcal{H}_0$. We will prove that a neighborhood of f in $\mathcal{H}$ is disjoint from $\mathcal{C}_n$. The set F_f is finite, so there is some $\delta > 0$ such that the minimum distance between any two points of F_f is larger than 2δ. Consider now some $\delta' > 0$ such that $\delta' < \min(\frac{1}{4n}, \delta)$. Note that the sets $\{y \in M : d(x_1, y) < \delta'\}$ and $\{y \in M : d(x_2, y) < \delta'\}$ are disjoint for any $x_1, x_2 \in F_f, x_1 \neq x_2$. Define

$$M_f = M \setminus \bigcup_{x \in F_f} \{y \in M : d(x, y) < \delta'\}.$$

Then M_f is compact. Let us notice that $d(x, f(x)) > 0$ on M_f, since f has no fixed points in M_f. Denote $\epsilon = \min\{d(x, f(x)) : x \in M_f\}$. The minimum exists and is strictly positive since M_f is a compact set. Consider now some $g \in \mathcal{H}$ such that $d(f, g) < \min(\epsilon, \delta')$. We will show that g does not belong to $\mathcal{C}_n$. Indeed, g has no fixed points on M_f, since for any $x \in M_f$ we have that $d(x, g(x)) \geq |d(x, f(x)) - d(f(x), g(x))| \geq d(x, f(x)) - d(f(x), g(x)) \geq \epsilon - d(f(x), g(x)) > 0$. Therefore g has only fixed points in $\bigcup_{x \in F_f} \{y \in M : d(x, y) < \delta'\}$, and this is a union of disjoint open sets. Therefore any connected component of F_g is contained in a set $\{y \in M : d(x, y) < \delta'\}$ for some $x \in F_f$, so the diameter of the connected component is at most $2\delta'$. But $2\delta' < \frac{1}{n}$, so $g \notin \mathcal{C}_n$.

Therefore we have proved that a small open neighborhood of f in $\mathcal{H}$ is disjoint from $\mathcal{C}_n$. Consider now the set $\mathcal{F}_n \subset \mathcal{H}$ which is the union of such open neighborhoods for each element f of $\mathcal{H}_0$. Then $\mathcal{F}_n \cap \mathcal{C}_n = \emptyset$. On the other hand $\mathcal{F}_n$ is open, and since $\mathcal{H}_0 \subset \mathcal{F}_n$, it follows that $\mathcal{F}_n$ is dense in $\mathcal{H}$. Then $\mathcal{C}_n$ is a nowhere dense set. Let us now notice that the set of elements of $\mathcal{H}$ that have a totally disconnected set of fixed points is exactly $\mathcal{H} \setminus \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$. Since each $\mathcal{C}_n$ is a nowhere dense set, it follows that most elements of $\mathcal{H}$ have a totally disconnected set of fixed points. This concludes the proof for the case that all homeomorphisms in $\mathcal{H}$ are orientation preserving.

Consider now the case where all the elements of $\mathcal{H}$ are orientation reversing homeomorphisms. The condition (i) in Lemma 2 cannot hold for the orientation reversing homeomorphisms. Still, the same idea applies to this case, once we notice that Lemma 2 works in the case of orientation reversing homeomorphisms f, g, if we just replace the condition (i) by:

$$(i') \quad f(y) = \tilde{y}, \text{ for all } y \text{ such that } d(x,y) < \delta',$$

where $\tilde{y} = \phi^{-1}(R_{a_1}(\phi(y))), \phi : \{y \in M : d(x,y) < \delta'\} \to \{a \in \mathbb{R}^n : ||a|| < \delta'\}$ is a homeomorphism such that $\phi(x) = 0$, and R_{a_1} is the reflection of $\mathbb{R}^n$ with respect to the first coordinate $a_1, R_{a_1}(a_1, a_2, \ldots, a_n) = (-a_1, a_2, \ldots, a_n)$. In other words, instead of the property that f fixes each point in a neighborhood of x, we have that f fixes each point in a neighborhood of x up to reflection. With this change the proof is analogous to the orientation preserving case.

If $\mathcal{H}$ is made up of both orientation-preserving and orientation-reversing homeomorphisms, we can just prove the desired result for the two sets of homeomorphisms separately, since they are disjoint open subsets of $\mathcal{H}$.

Finally, if M is not connected, we can partition $\mathcal{H}$ into finitely many subsets corresponding to the way the connected components of M are mapped to each other and prove the desired result for these disjoint open subsets of homeomorphisms separately. □

For the case $M = S^{2n}, n \neq 2$ and $\mathcal{H}$ containing all orientation preserving homeomorphisms of S^{2n}, we obtain a proof of the conjecture in [2] (except for the case of the sphere S^4, which is still an open problem):

Corollary 5. *Generically, orientation preserving homeomorphisms of* S^{2n}, $n \neq 2$, *have a Cantor set of fixed points.*

For the case $M = S^{2n+1}$ and $\mathcal{H}$ containing all orientation reversing homeomorphisms of S^{2n+1}, we obtain:

Corollary 6. *Generically, orientation reversing homeomorphisms of* S^{2n+1}, $n > 0$, *have a Cantor set of fixed points.*

Remark 1. Let us also notice that for each compact manifold M with $\dim(M) > 0$ and $\neq 4$ there exists some orientation-preserving homeomorphism $h : M \to M$ such that $h(\overline{U}) \subset U$ for some open set $U \subset M$ which is homeomorphic to a ball of dimension $\dim(M)$. Then, there exists a neighborhood $\mathcal{N}$ of h such that $g(\overline{U}) \subset U$ for all homeomorphisms g in $\mathcal{N}$. Then, by using Theorem 4 (or, if $\dim(M) = 1$, by using results from [3]) we conclude that most elements of $\mathcal{N}$ have a Cantor set of fixed points. In particular, it follows that *the set of homeomorphisms of M that have a Cantor set of fixed points is a second category set in the space of all homeomorphisms of M.*

Remark 2. One can also formulate our main theorem for a manifold with boundary M, either for homeomorphisms of M, or for embeddings of M in M. The same proof works in this case (the proof of Lemma 3 for embeddings appears in [5]). In particular, it follows that *generically, the homeomorphisms of an n-dimensional disk D^n ($n > 1$ and $\neq 4$), have a Cantor set of fixed points.*

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