Exact Sequences

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1 Basic definitions

Definition. Let

$$\cdots \to G_{k-1} \xrightarrow{f_{k-1}} G_k \xrightarrow{f_k} G_{k+1} \to \cdots$$

be a sequence of homomorphisms. This sequence is said to be exact if for each k, $\operatorname{im}(f_{k-1}) = \ker(f_k)$.

For the remainder of the document, let 1 be the trivial group. The group 1 has the special property that for any group G, there is exactly one homomorphism $1 \to G$ and exactly one homomorphism $G \to 1$. This is because 1 has exactly one element, namely the identity, and homomorphisms must preserve the identity. In the language of category theory, we say that 1 is the zero object in the category of groups. Because of this property, we will write $1 \to G$ and $G \to 1$ without specifying which homomorphism we are talking about, since there is only one.

Exercise 1. Let I be a group with the property that for every group G, there is exactly one homomorphism $I \to G$. Let F be a group with the property that for every group G, there is exactly one homomorphism $G \to F$. Prove that both I and F are isomorphic to 1.

Exercise 2. Let G, H, and K be groups. Prove that

1. The sequence $1 \to H \xrightarrow{f} G$ is exact if and only if f is injective.
2. The sequence $G \xrightarrow{h} K \to 1$ is exact if and only if h is surjective.
3. The sequence $1 \to H \xrightarrow{f} G \xrightarrow{h} K \to 1$ is exact if and only if f is injective, h is surjective, and h induces an isomorphism $K \simeq G/\operatorname{im}(f)$.

Definition. An exact sequence of the form $1 \to H \to G \to K \to 1$ is called a short exact sequence.

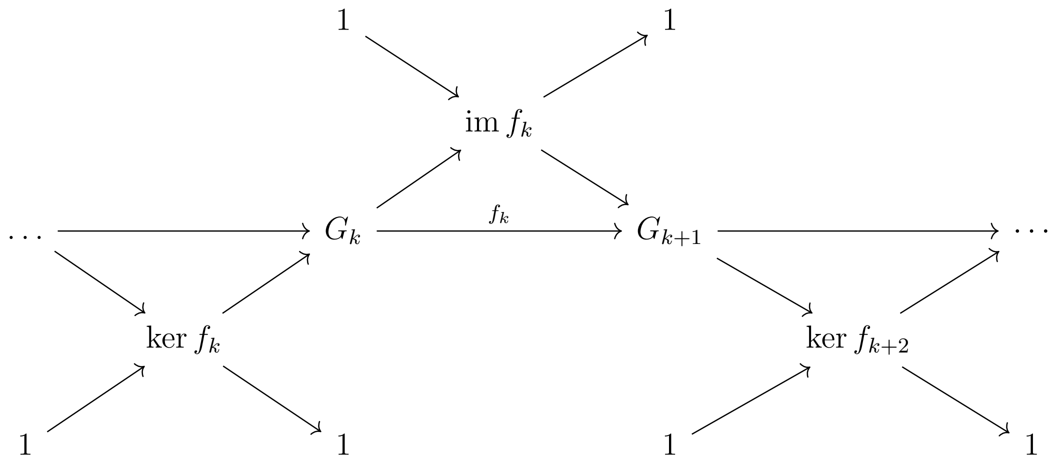
Here are two reasons why short exact sequences are important. First, a short exact sequence $1 \to H \xrightarrow{f} G \to K \to 1$ encodes the data that $K \simeq G/f(H)$ in such a way that makes clear what the homomorphisms involved are. Second, given an exact sequence

$$\cdots \to G_{k-1} \xrightarrow{f_{k-1}} G_k \xrightarrow{f_k} G_{k+1} \to \cdots ,$$

for each k we get a short exact sequence

$$1 \to \ker(f_k) \to G_k \to \operatorname{im}(f_k) \to 1,$$

and these short exact sequences fit together to fill the diagram



where we have used the fact that $\ker f_n = \operatorname{im} f_{n-1}$ for all n. Thus any long exact sequence may be broken up into a bunch of short exact sequences and analyzed as such.

Let $1 \to N \to G \to Q \to 1$ be a short exact sequence. Because $N \to G$ is injective, N is isomorphic to its image in G. Thus we may view N as a subgroup of G. Moreover, since the image of N in G is the kernel of the map $G \to Q$, the image of N in G is a normal subgroup of G. Similarly, we may view Q as being a quotient of G via the identification in Exercise 2. By abuse of notation, we may write $Q = G/N$.

In this situation, we call G an extension of Q by N. A fundamental problem in group theory is to classify the ways in which we may combine two small groups together to get a larger group. That is, given groups Q and N, what are all the extensions of Q by N? As we will show in the next section, one such extension is $N \times Q$, but in general there are many more. This problem, known as the extension problem, is very difficult.

2 Splitting of abelian groups

Exercise 3. Let H and K be groups. Prove that the sequence

$$1 \to H \xrightarrow{\iota_H} H \times K \xrightarrow{\pi_K} K \to 1$$

is exact, where ι_H and π_K are the natural inclusion and projection respectively to and from the direct product $H \times K$.

The exact sequence from this exercise has a quite a bit more structure than the average exact sequence. Specifically, the injection ι_H has a left inverse (namely the projection π_H) and the surjection π_K has a right inverse (namely the inclusion ι_K). Set theoretically, every injection has a left inverse and every surjection has a right inverse, but rarely will these set theoretic maps be group homomorphisms.

In fact, the existence of left or right inverses for maps in an exact sequence tells us quite a bit of information about the groups in the sequence. To see why, we first restrict our attention to the special case of abelian groups. For the remainder of this section, we will be working with abelian groups. To emphasize this, we will use "additive notation." Specifically, we write 0 for the trivial group and $\oplus$ for the direct *sum* of groups, which is formally identical to the direct product (at least for finite collections of groups).

Exercise 4 (Splitting lemma)**.** Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence of abelian groups. Then the following are equivalent:

1. There exists a homomorphism $r : B \to A$ such that $rf = \mathrm{id}_A$.
2. There exists a homomorphism $s : C \to B$ such that $gs = \mathrm{id}_C$.
3. There is an isomorphism $h : B \to A \oplus C$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & & \searrow^{\iota_A} & \downarrow h & \nearrow_{\pi_C} & & & \\ & & & & A \oplus C & & & & \end{array}$$

commutes. (That is, $\iota_A = hf$ and $\pi_C = gh^{-1}$.)

A short exact sequence satisfying any of these equivalent conditions is called split. In practice, we usually check that an exact sequence is split by showing that the surjection g is right invertible. The importance of an exact sequence splitting is evident; when an exact sequence splits, the group in the middle splits as a direct sum of the groups on either side.

Most exact sequences, however, do not split. This reflects the difficulty of the extension problem. In general, there are many extensions of a given abelian group C by another abelian group A other than $A \oplus C$.

Exercise 5. Show that the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split using all three equivalent characterizations of a split exact sequence.

Exercise 6. Where did you use the condition that A, B, and C are abelian in your proof of the splitting lemma?

3 Splitting in general

Exercise 7. Show that the splitting lemma fails for the exact sequence

$$1 \to A_3 \to S_3 \to \{\pm 1\} \to 1,$$

where the map $A_3 \to S_3$ is the inclusion. Specifically, show that condition (2) holds, but the other conditions do not.

Since the splitting lemma fails for non-abelian groups, we cannot define a split exact sequence using all of the non-equivalent conditions. Instead, we adopt (2) as our definition. Some authors call a sequence satisfying (2) right split, while a sequence satisfying (1) is called left split.

Exercise 8. A short exact sequence

$$1 \to H \xrightarrow{f} G \xrightarrow{g} K \to 1$$

splits (that is, there exists a homomorphism $K \to G$ with $gs = \mathrm{id}_K$) if and only if $G \cong H \rtimes K$ via an isomorphism which commutes with f and g (as in the splitting lemma.)

So for exact sequences of non-abelian groups, splitting means that G can be split as a *semi*-direct product of H and K.

Exercise 9. In the situation of the previous exercise, what is the automorphism that the semi-direct product is taken with respect to?

Exercise 10. Show that the exact sequence

$$1 \to \{\pm 1\} \to Q \to V \to 1$$

is not split, where Q is the quaternion group, V is the Klein four-group, and the homomorphism $\{\pm 1\} \to Q$ is the inclusion.