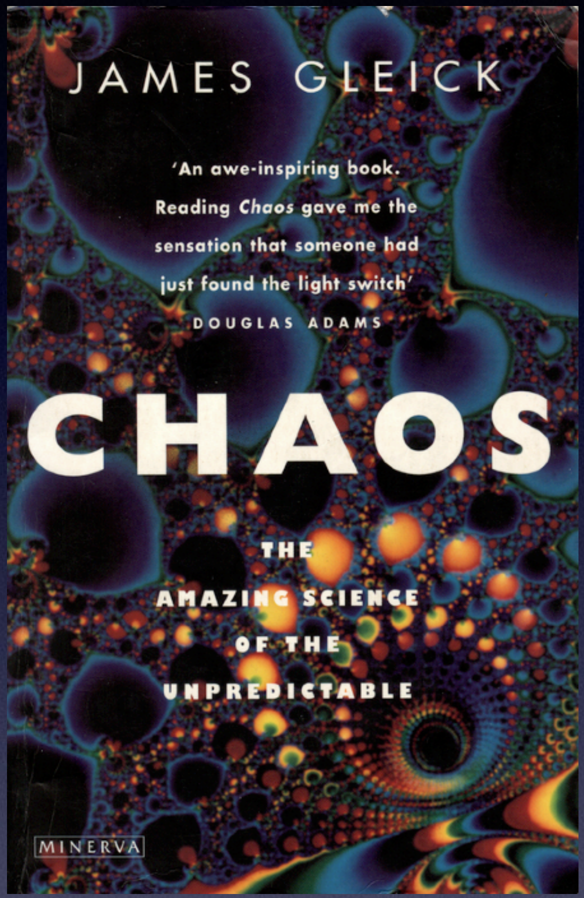
Enter the fantastic world of fractals

by Chris H. Rycroft
Harvard University

Thomas Crane Public Library
August 22nd, 2017

References

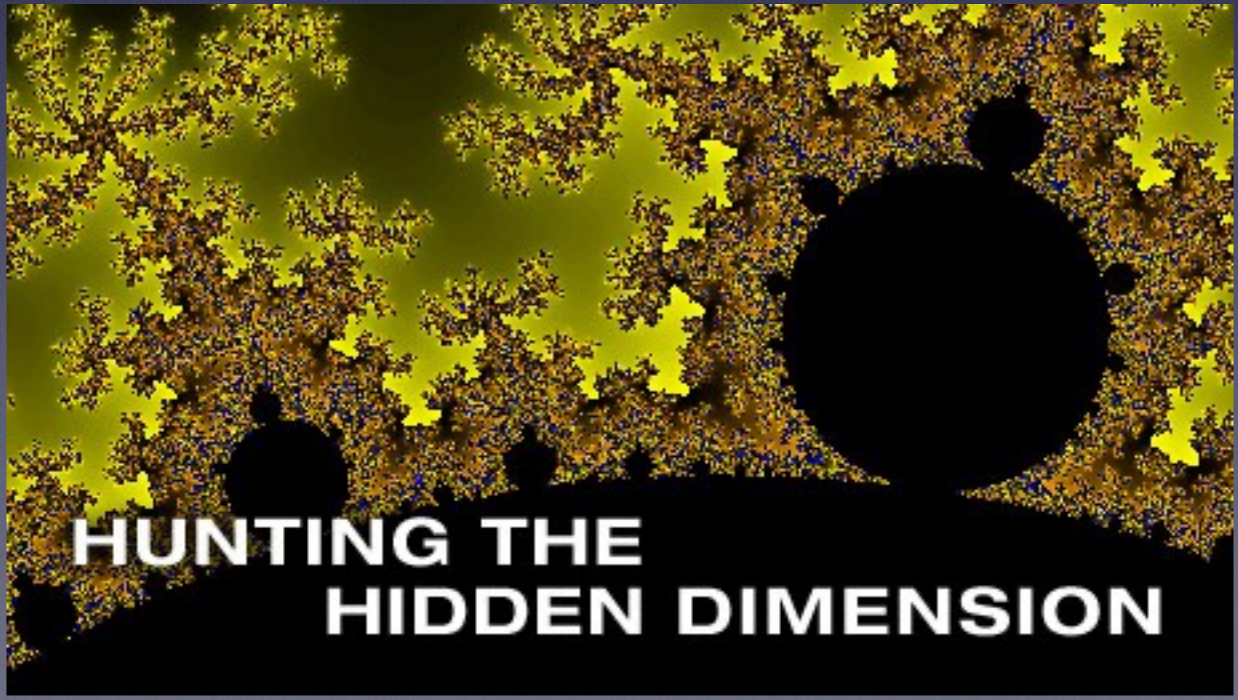


Chaos by James Gleick

Popular science overview

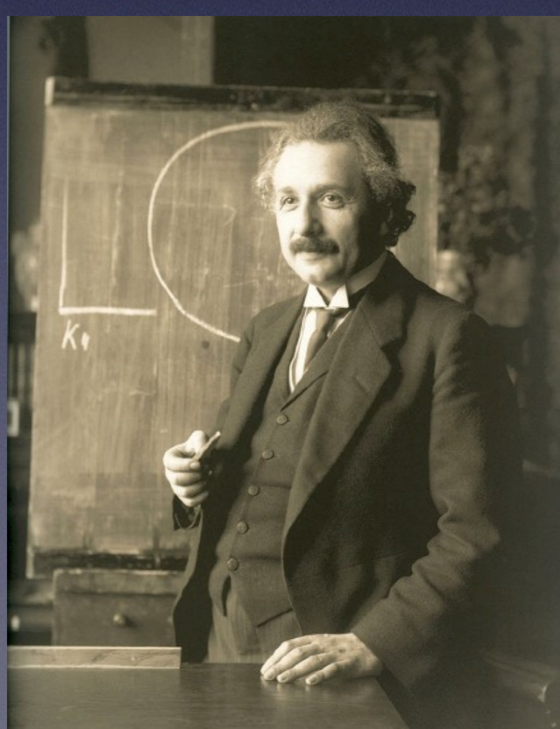
Hunting the Hidden Dimension

PBS Nova documentary

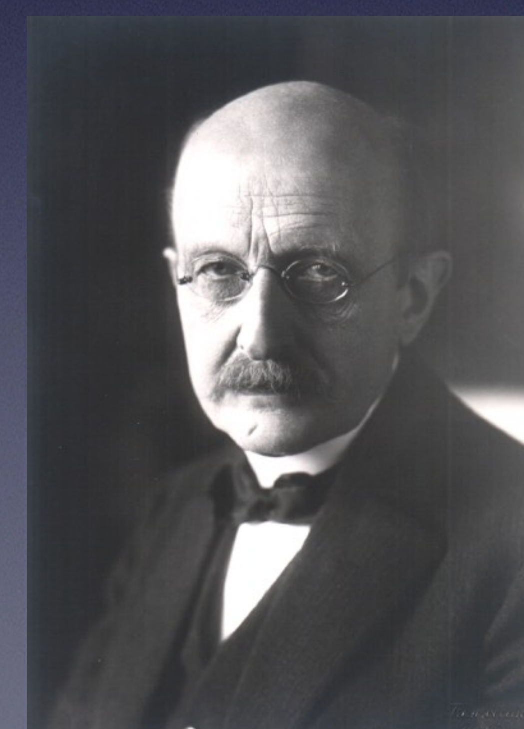


Quote from Gleick's *Chaos*

The most passionate advocates of the new science go so far as to say that twentieth century science will be remembered for just three things: relativity, quantum mechanics, and ***chaos****.*

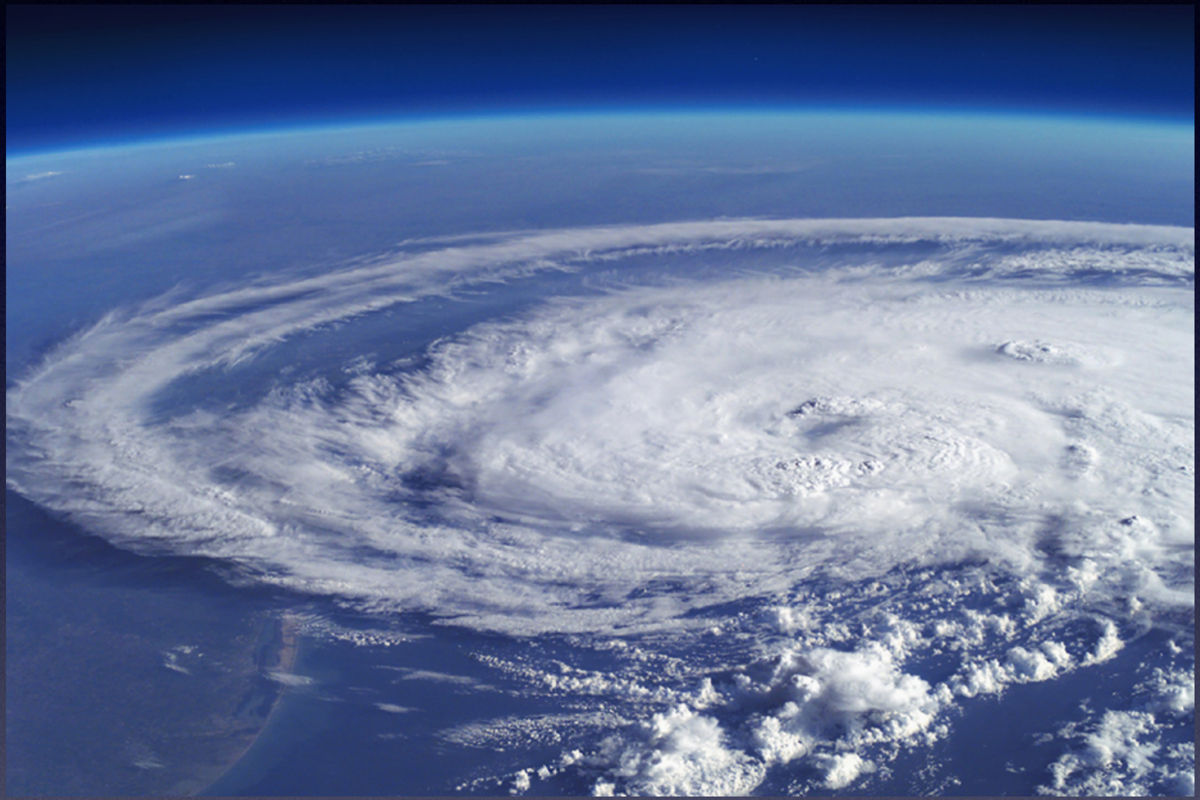


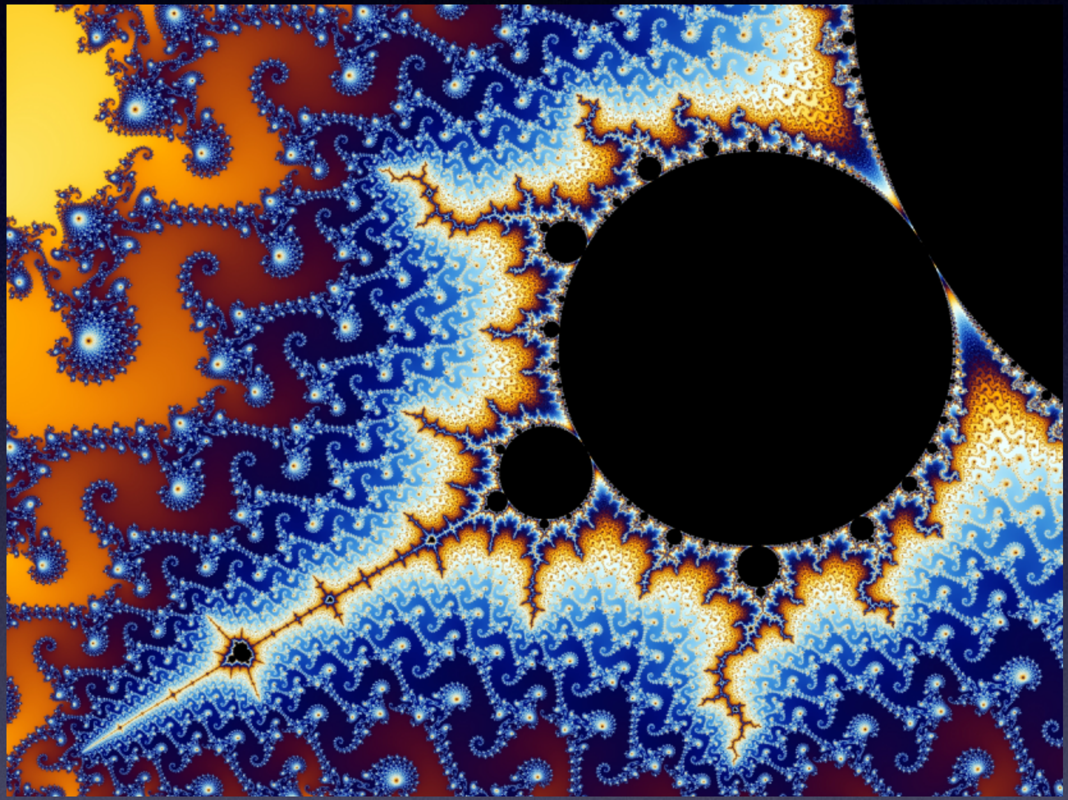
Albert Einstein



Max Planck

Talk outline





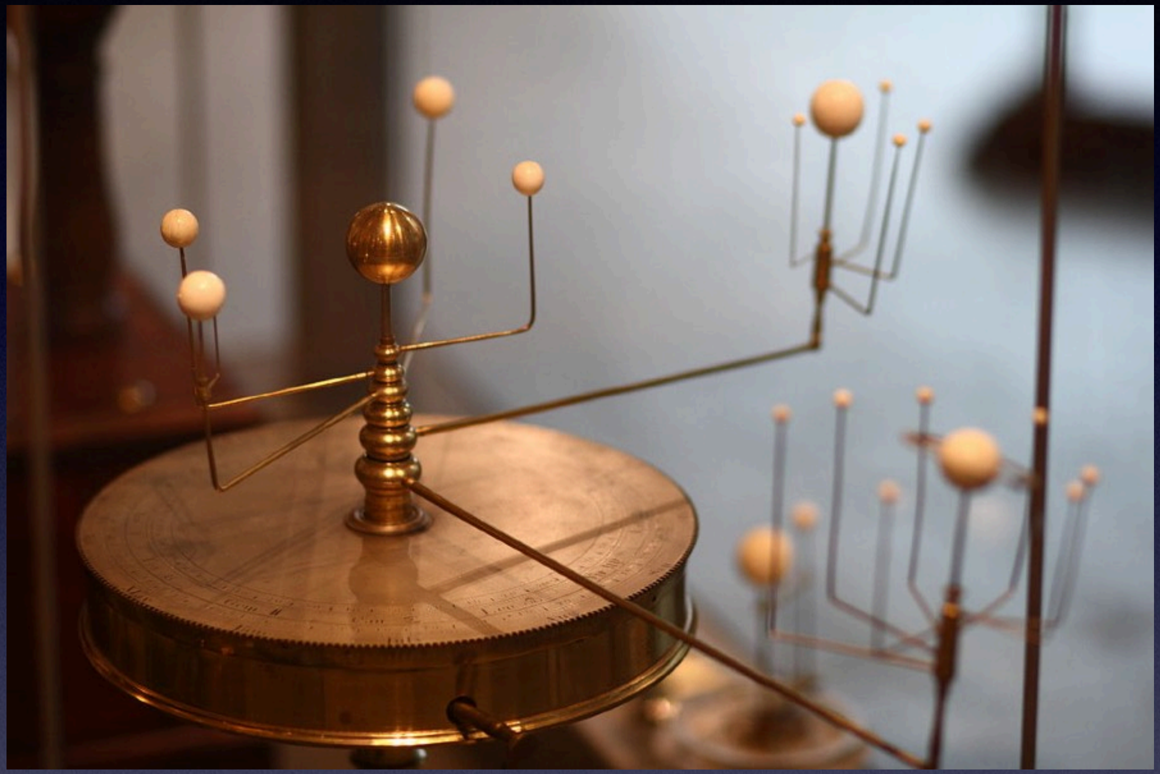
Chaos theory

The science of the unpredictable

Fractal geometry

A new way to look at natural forms and shapes

The clockwork universe

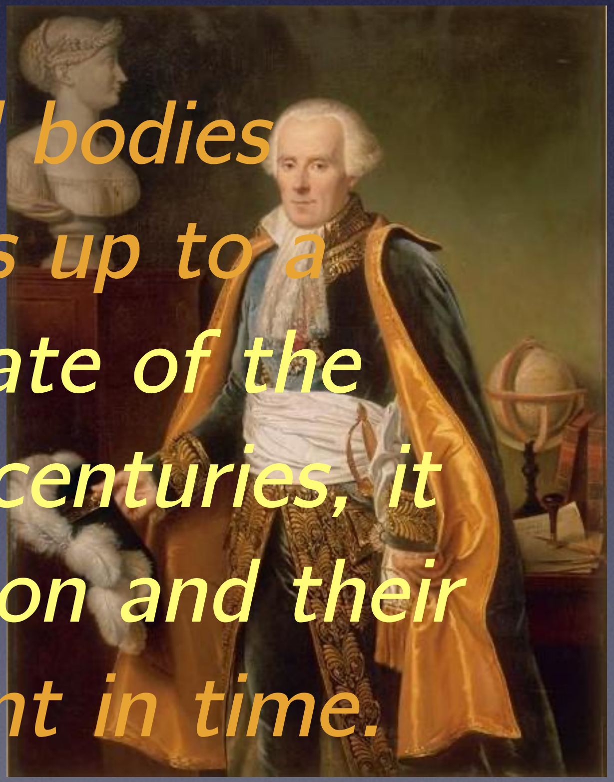


Astronomical observations of the Sun, Moon, and planets follow regular, predictable patterns

A method of thinking

The present state of the system of nature is evidently a consequence of what it was in the preceding moment, and if we conceive of an intelligence that at a given instant comprehends all the relations of the entities of this universe, it could state the respective position, motions ... of all these entities at any time in the past or future.

The simplicity of the law by which the celestial bodies move ... permit analysis to follow their motions up to a certain point; and in order to determine the state of the system of these great bodies in past or future centuries, it suffices for the mathematician that their position and their velocity be given by observation for any moment in time.



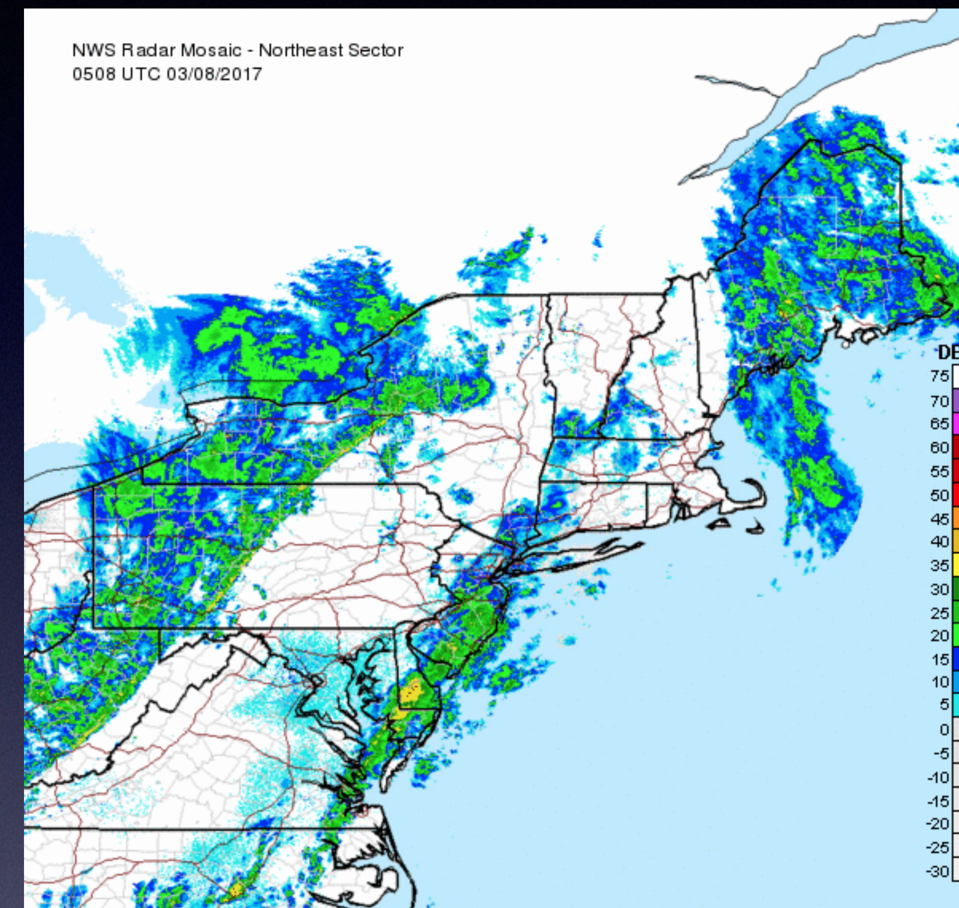
– Pierre-Simon Laplace (1776)

A method of thinking

Physicists like to think that all you have to do is say these are the conditions, now what happens next?

– Richard Feynman

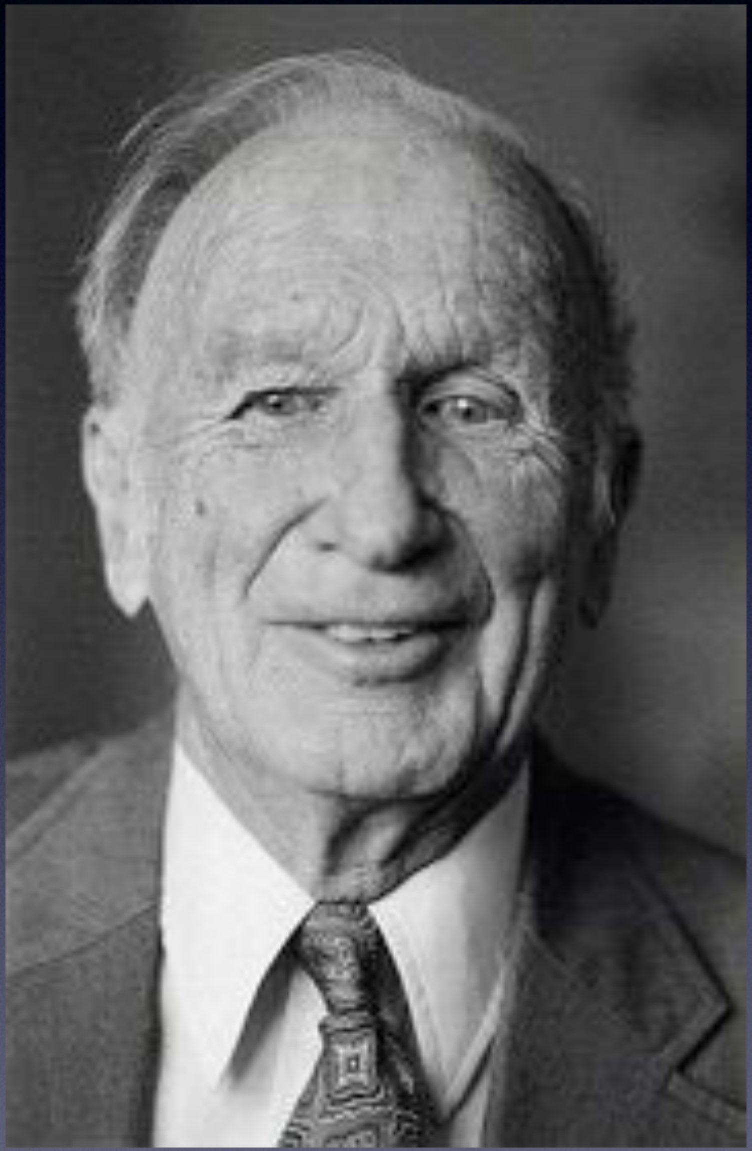
Weather prediction



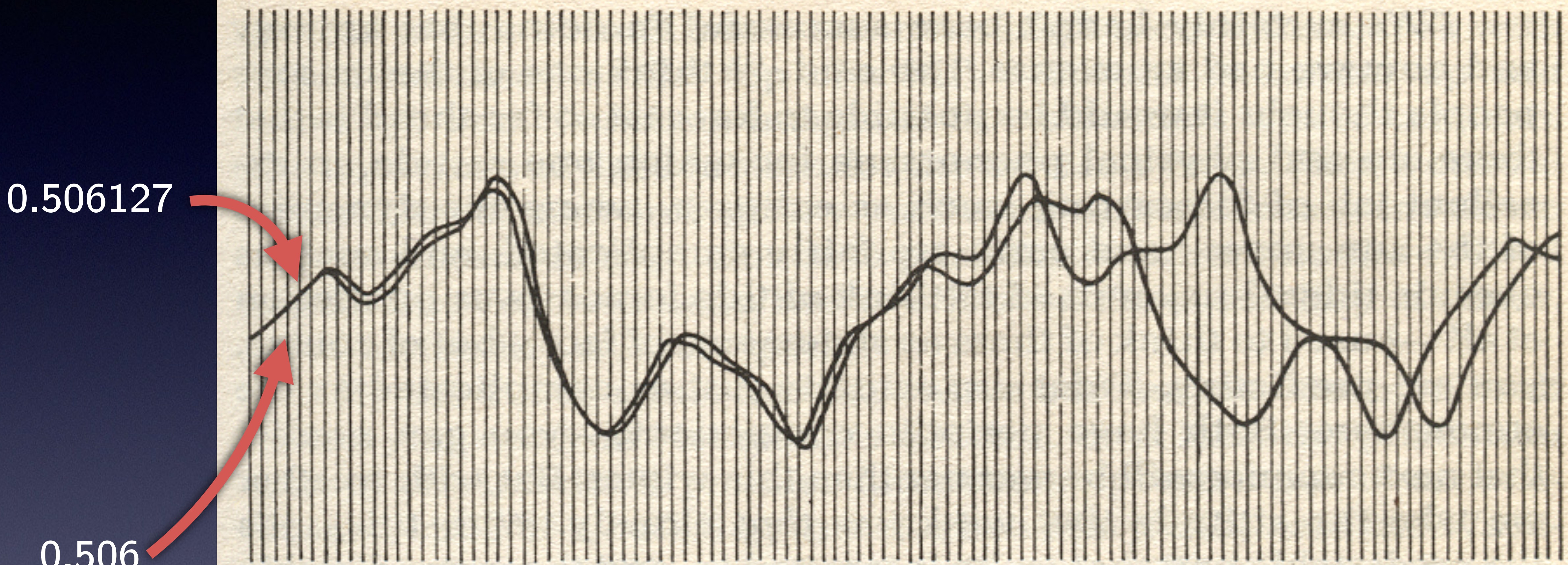
1950’s thinking: if we could build an increasingly sophisticated network of weather sensors, then the weather could be predicted for weeks and months in advance

Edward Lorenz

- Born in West Hartford, CO in 1917
- Studied mathematics at Dartmouth and Harvard
- Worked as a meteorologist during World War II and continued studying the field afterward
- In 1963, worked on a simplified model of atmospheric convection at MIT



Lorenz's observation



How two weather patterns diverge. From nearly the same starting point, Edward Lorenz saw his computer weather produce patterns that grew farther and farther apart until all resemblance disappeared. (From Lorenz's 1961 printouts.)

"Sensitive dependence on initial conditions"

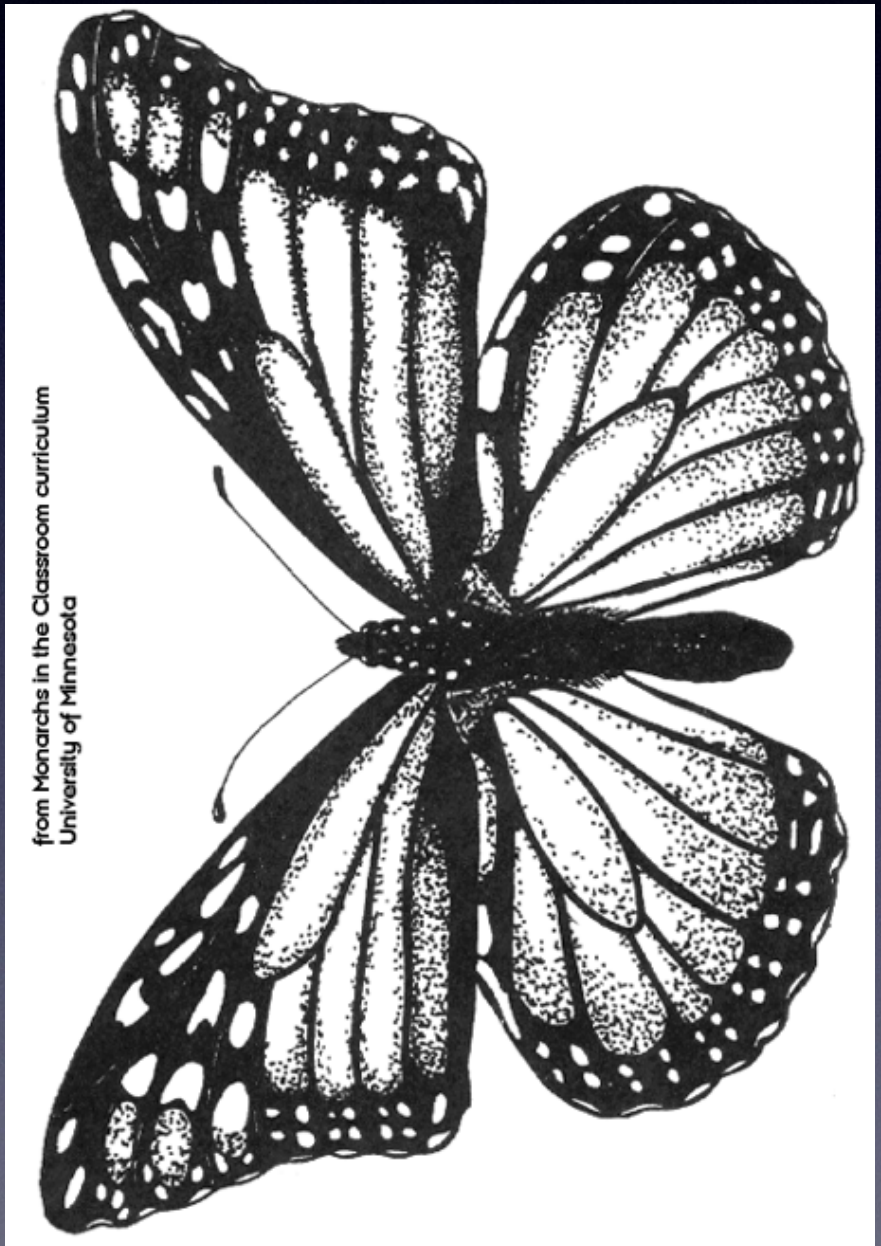
Lorenz's conclusion

Published in the *Journal of Atmospheric Sciences*, 1963

When our results concerning the instability of non-periodic flow are applied to the atmosphere, which is ostensibly nonperiodic, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be non-existent.

The butterfly effect (Lorenz, 1969)

- A butterfly flaps its wings in Tahiti
- At a later time, the effect of the butterfly's wings changes the weather in New York



A second opinion from Dr. Ian Malcolm (Jurassic Park, 1993)

Tiny variations ... never repeat and vastly affect the outcome—that's chaos theory.

Some further questions

1. Weather is a very complicated system. Is that necessary to see this type of behavior?

2. Why are some systems predictable and some aren’t?

3. Is there really any connection between the behavior of different unpredictable systems? Aren’t droplets and weather just different?

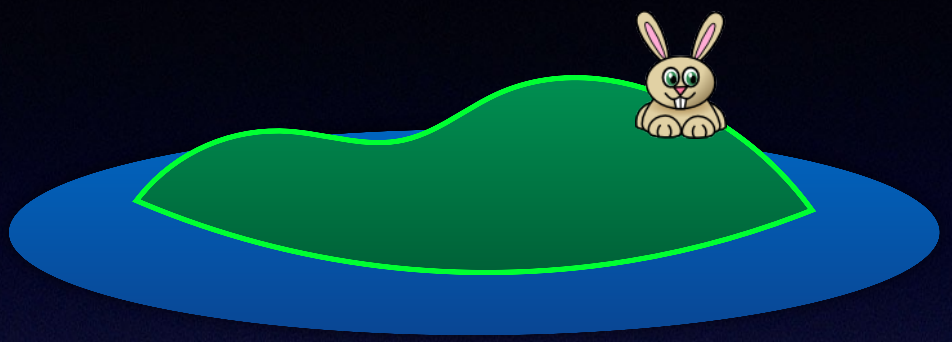
Population growth

- Let x_n be the number of rabbits on an island after n years
- Rabbits multiply by two at each year

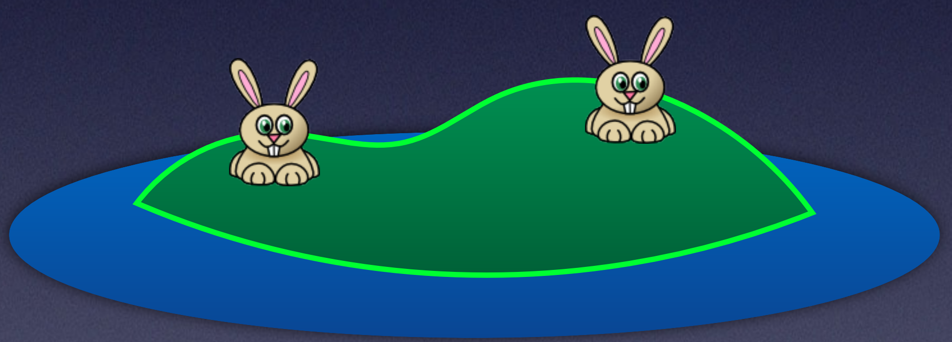
$$x_{n+1} = 2x_n$$

- More generally for some growth rate r

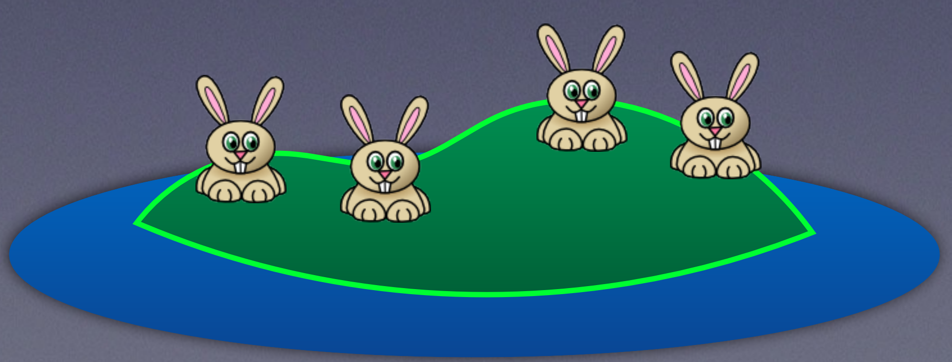
$$x_{n+1} = rx_n$$



$x_0 = 1$ Year 0



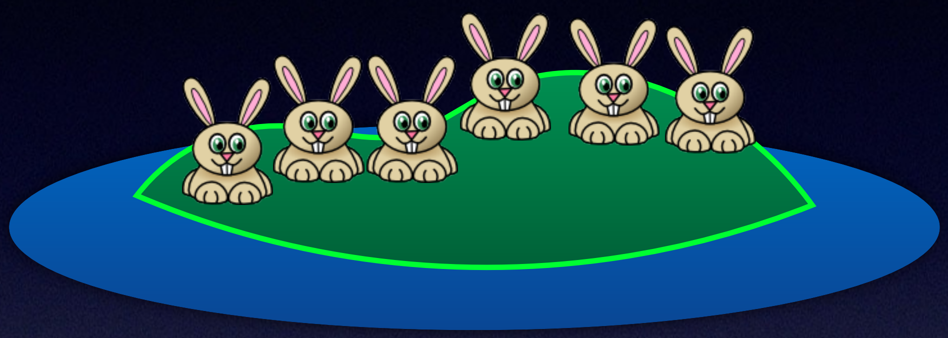
$x_1 = 2$ Year 1



$x_2 = 4$ Year 2

A better model

- Simple model assumes that the rabbit population will grow indefinitely—not realistic for an island with limited food

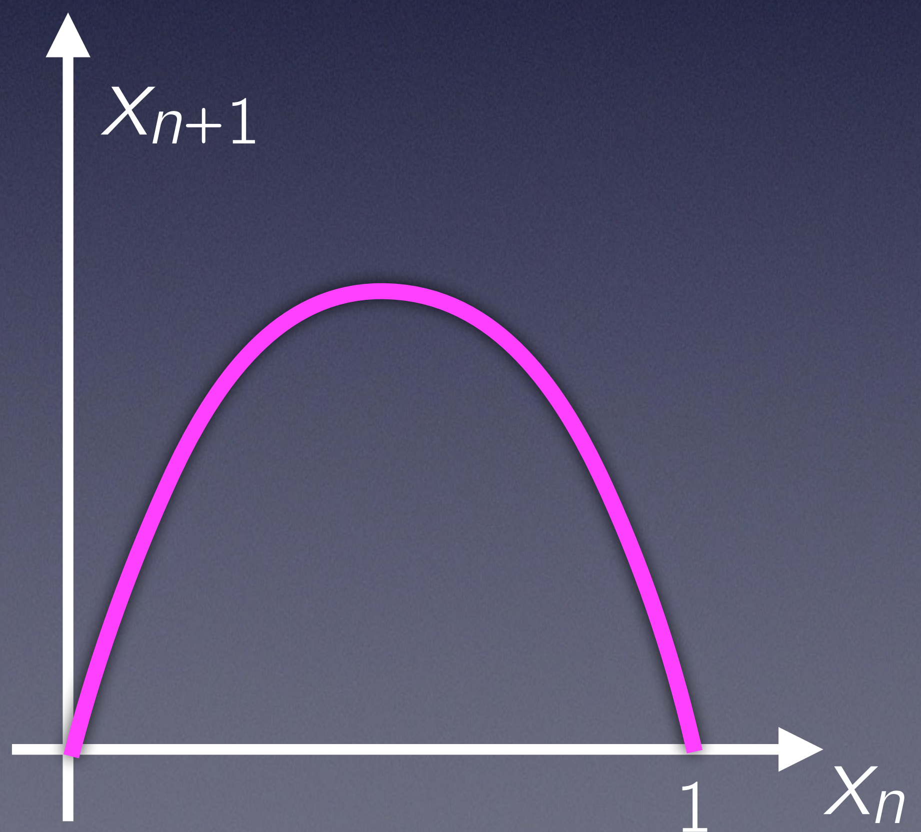


- Growth rate will actually decrease as more rabbits are present. Can say rate is

$$\text{Rate} = r(1 - x_n)$$

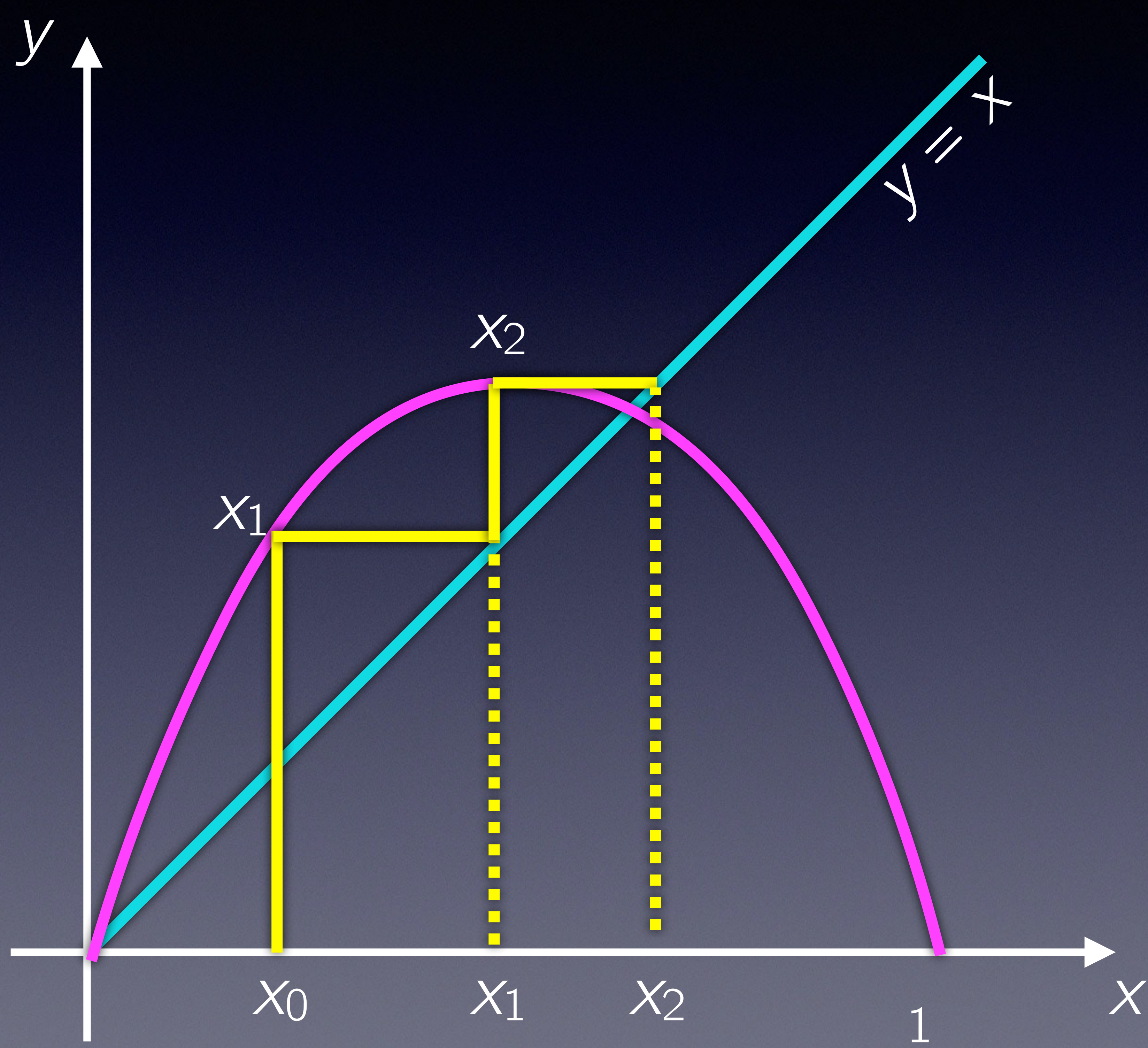
- Hence

$$\begin{aligned} x_{n+1} &= \text{Rate} \times x_n \\ &= r(1 - x_n)x_n \end{aligned}$$

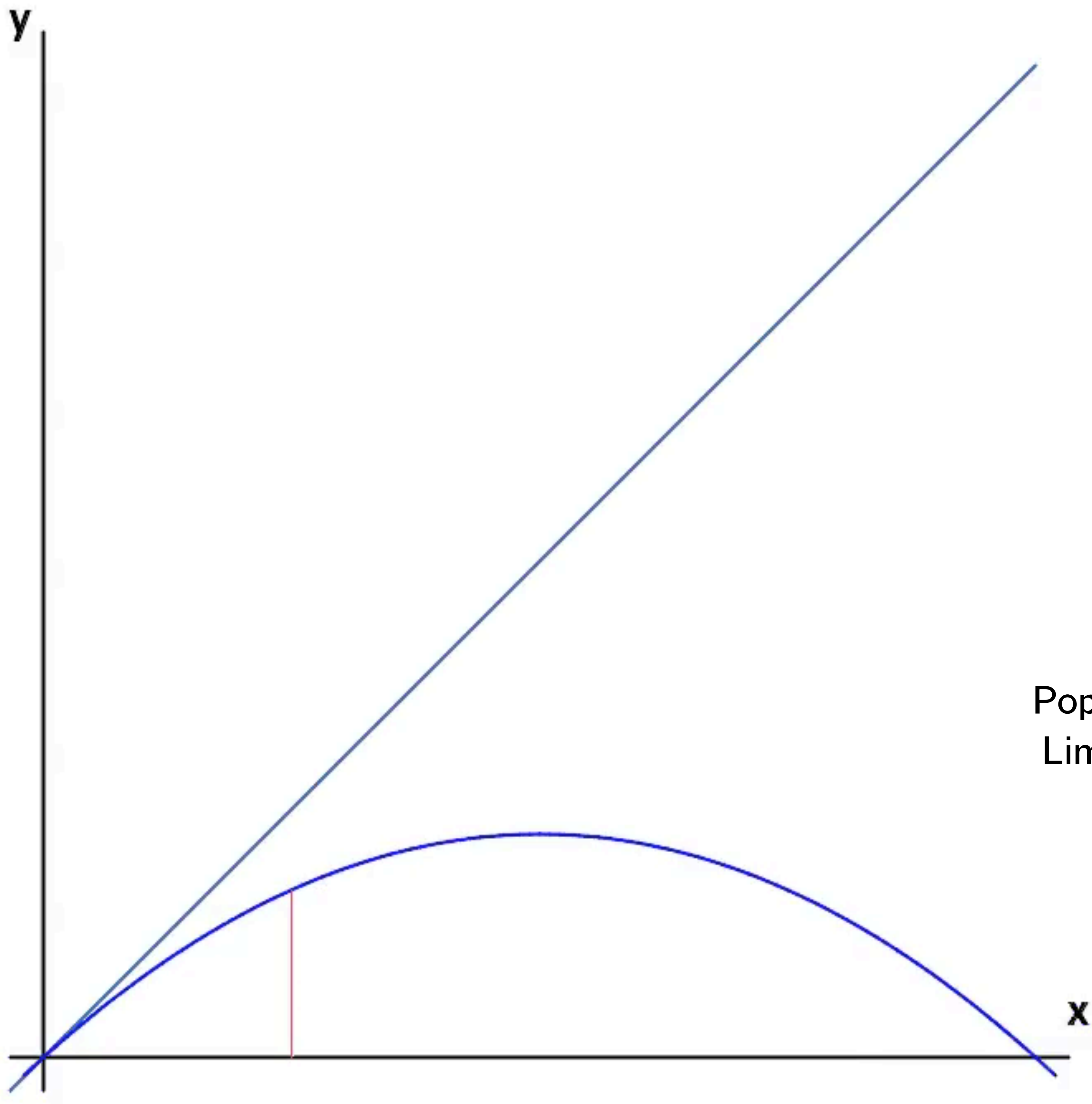


Cobweb plotting

For graphically determining iterative equations

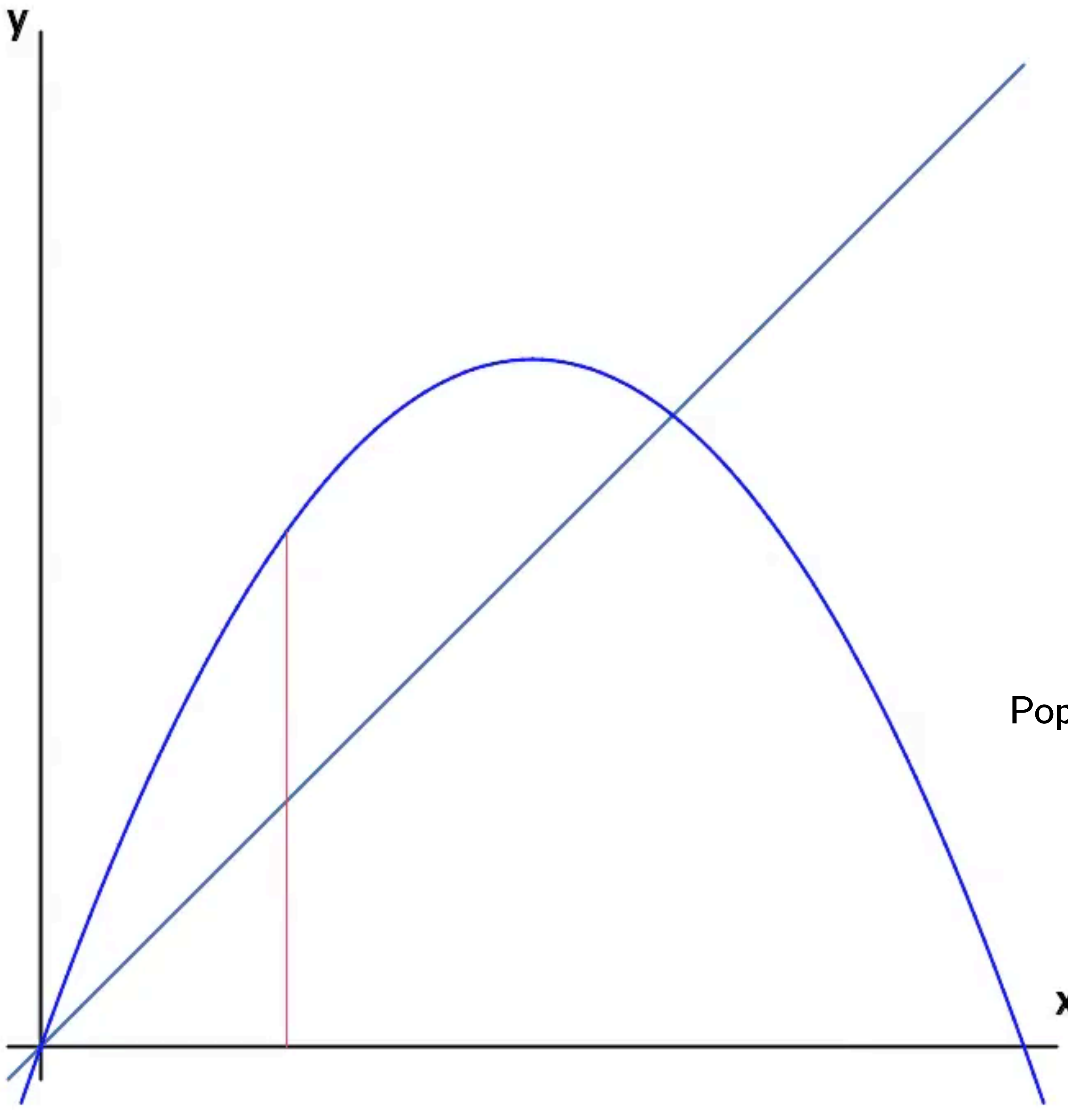


Cobweb plot ($r = 1$)



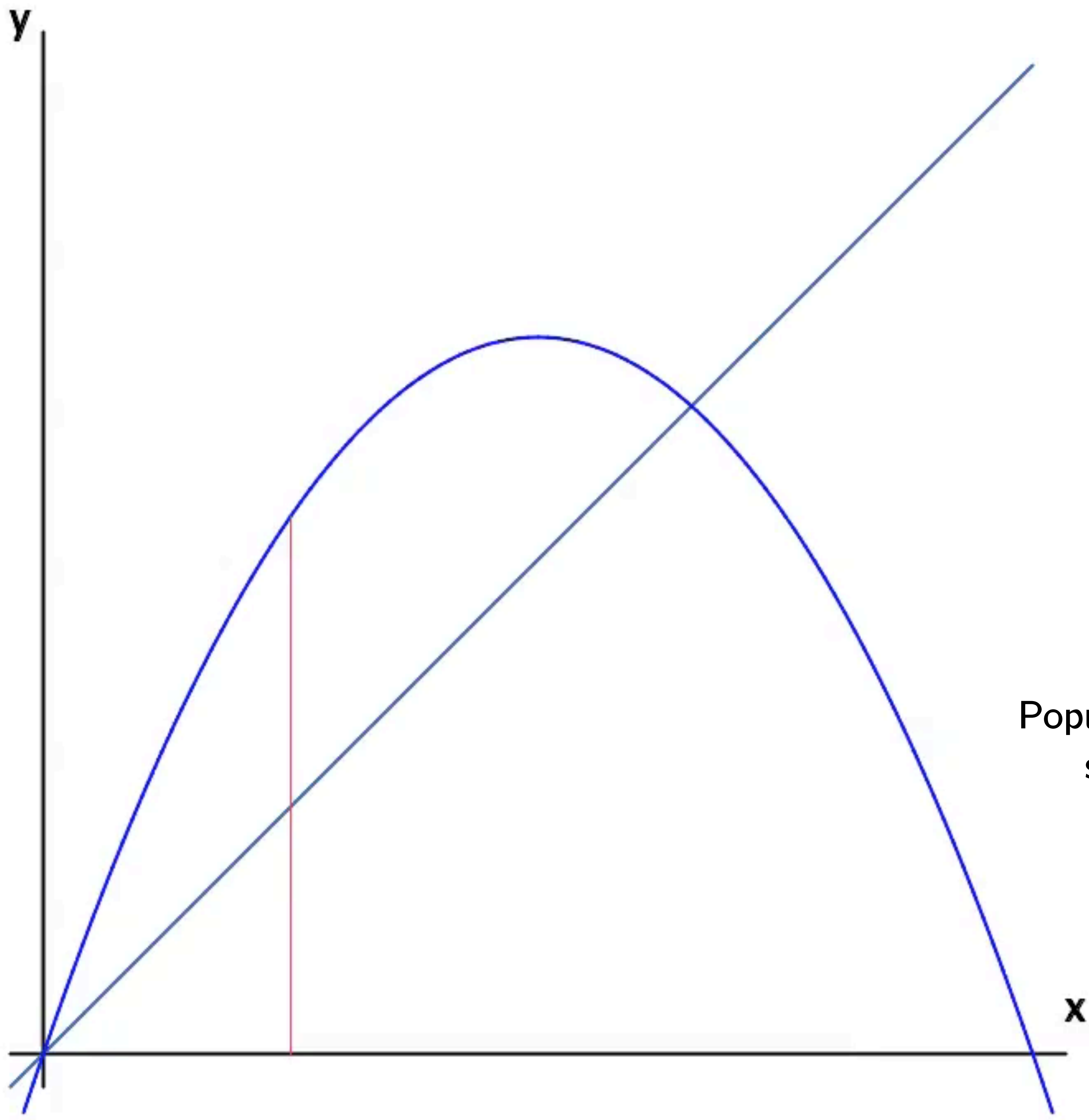
Population dies out.
Limiting value is 0.

Cobweb plot ($r = 2$)



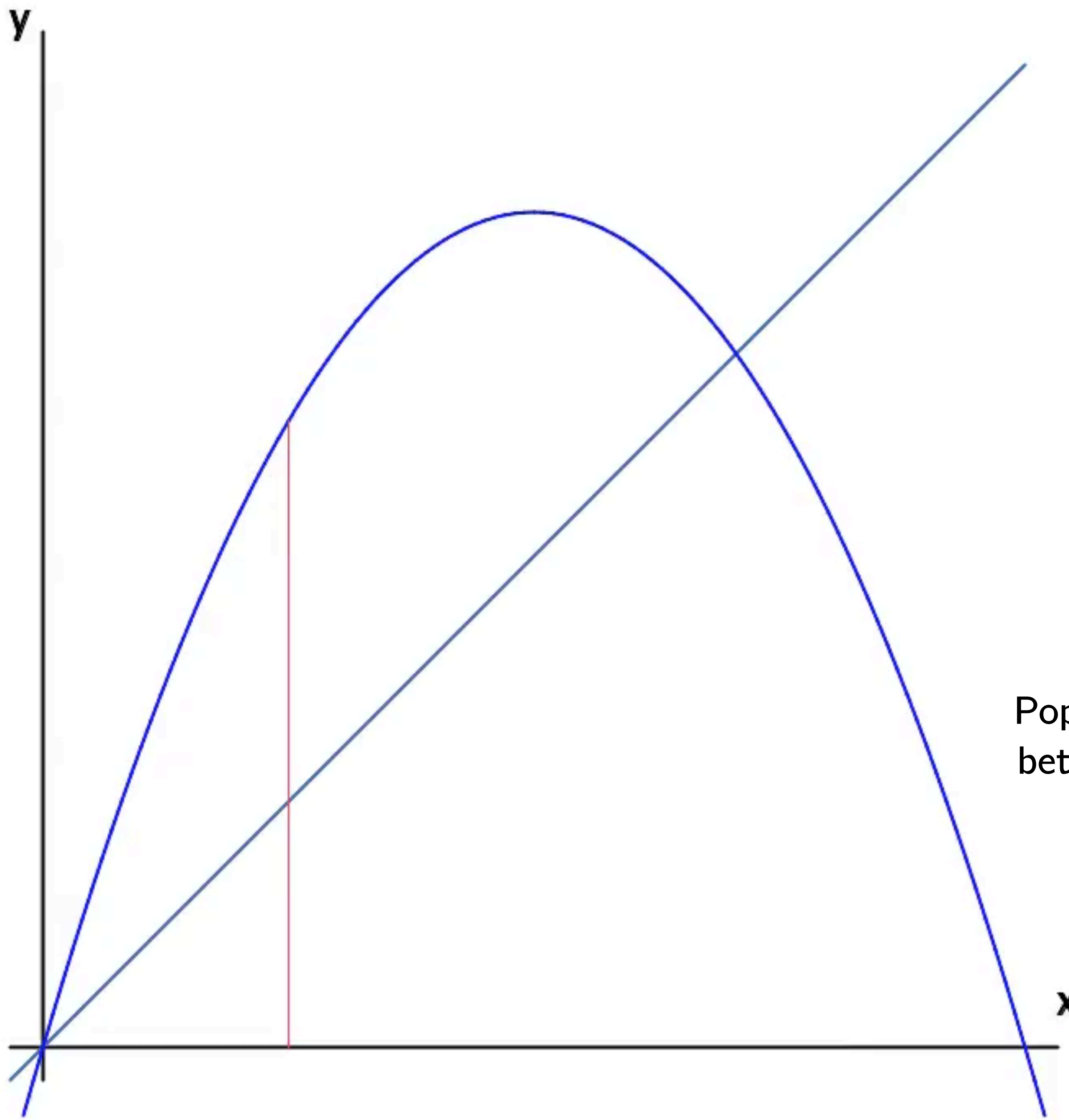
Population reaches a steady value.

Cobweb plot ($r = 2.5$)



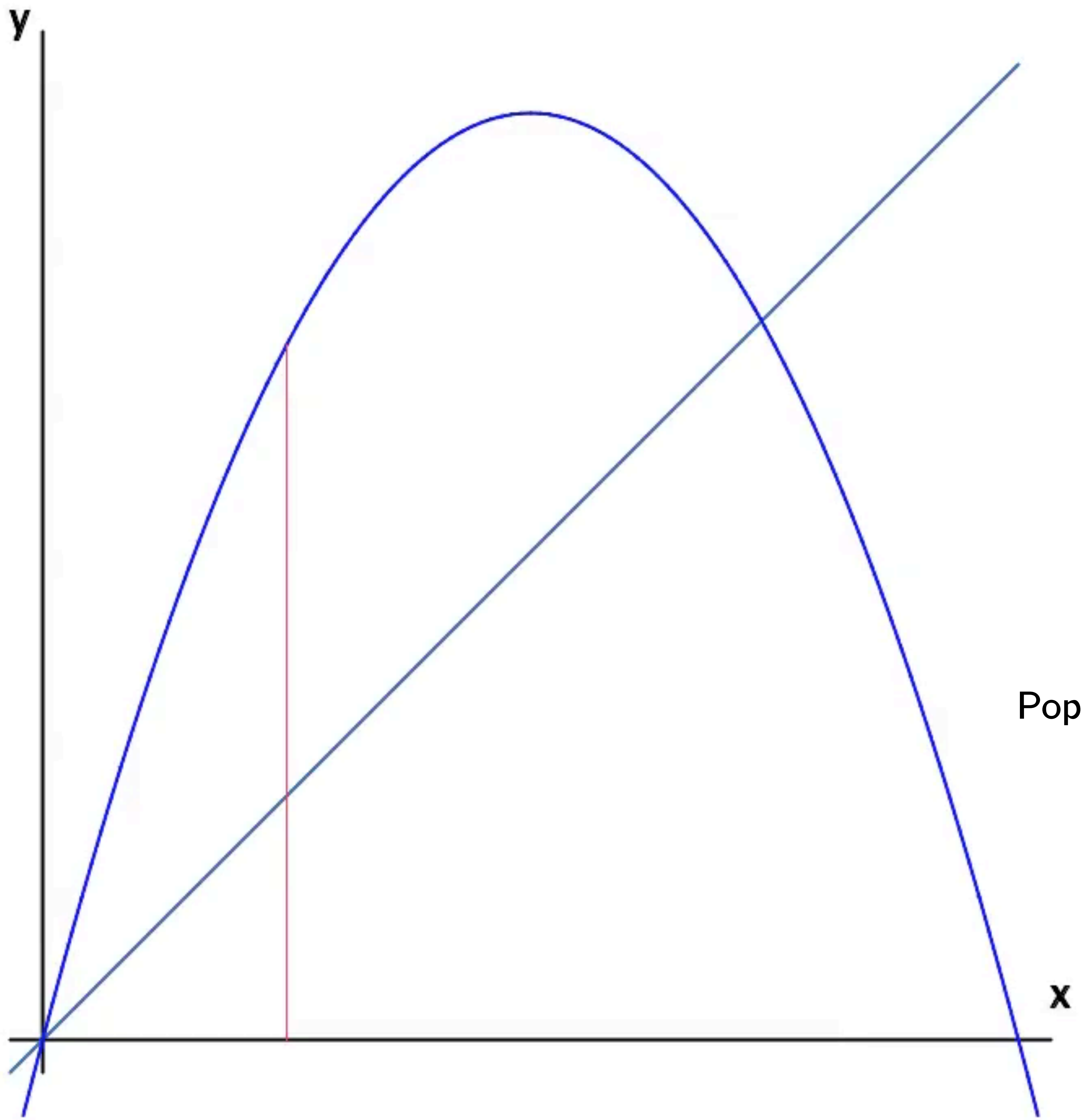
Population reaches a steady value.

Cobweb plot ($r = 3$)



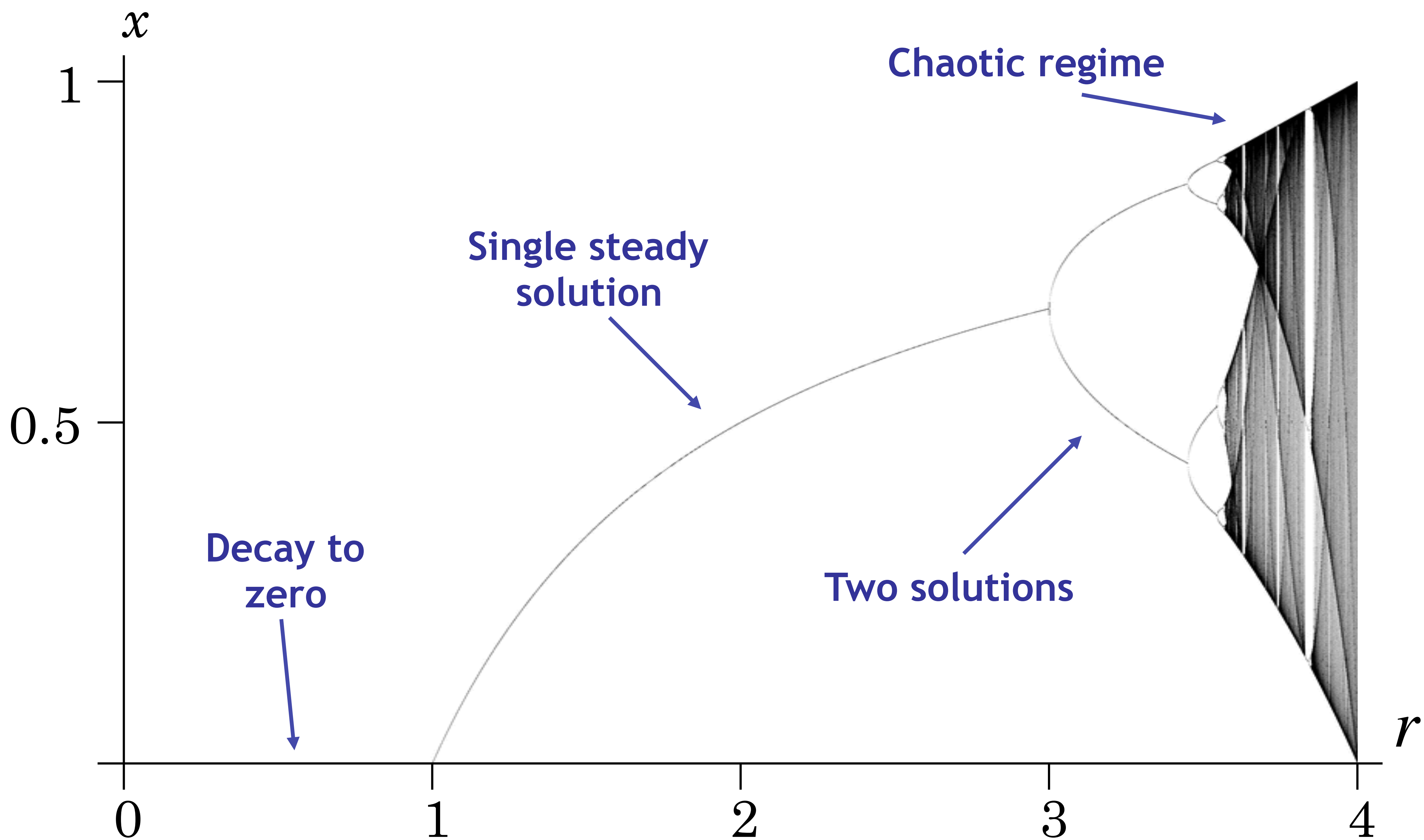
Population oscillates between two values.

Cobweb plot ($r = 4$)

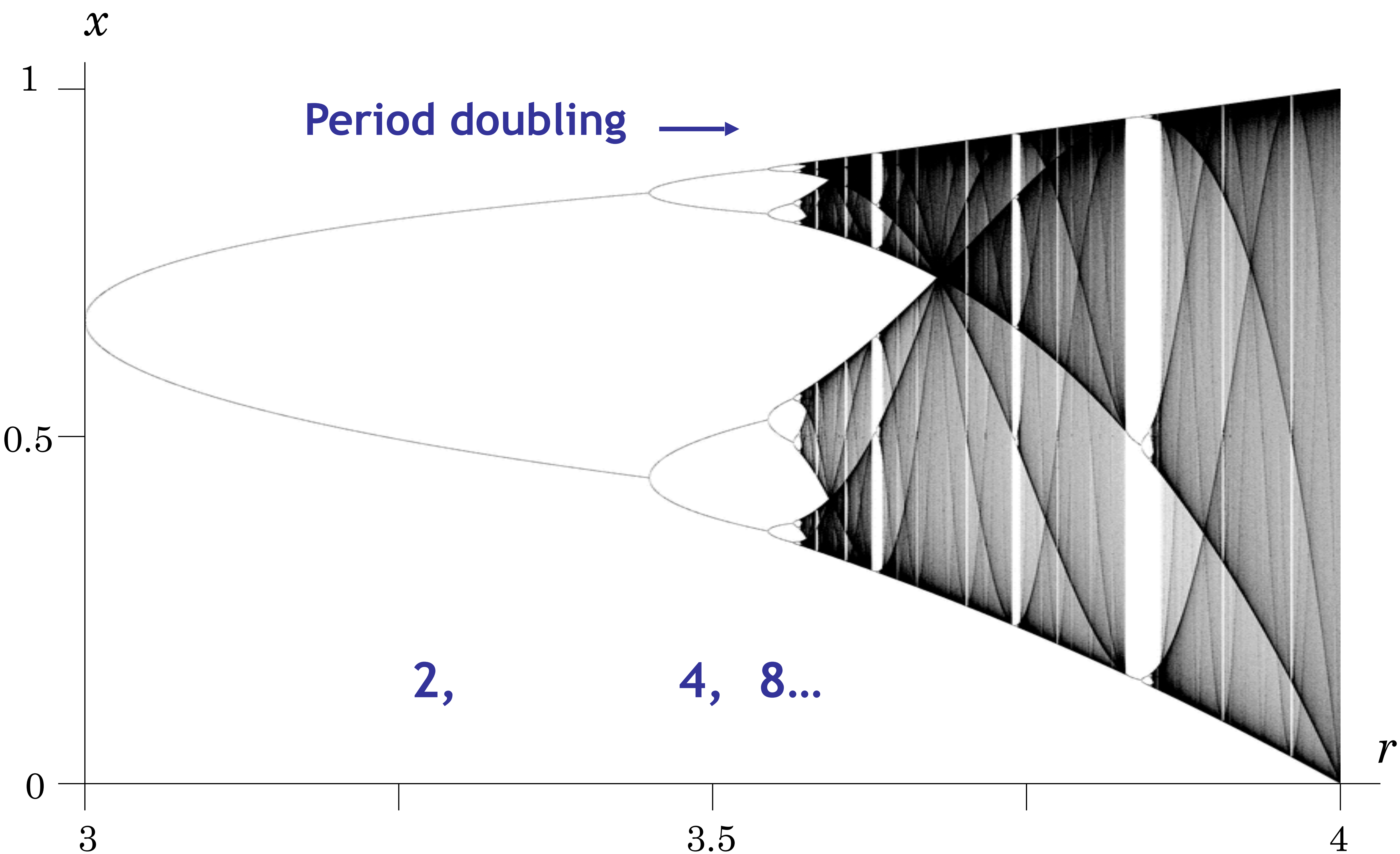


Population behaves chaotically

Bifurcation in steady state solutions



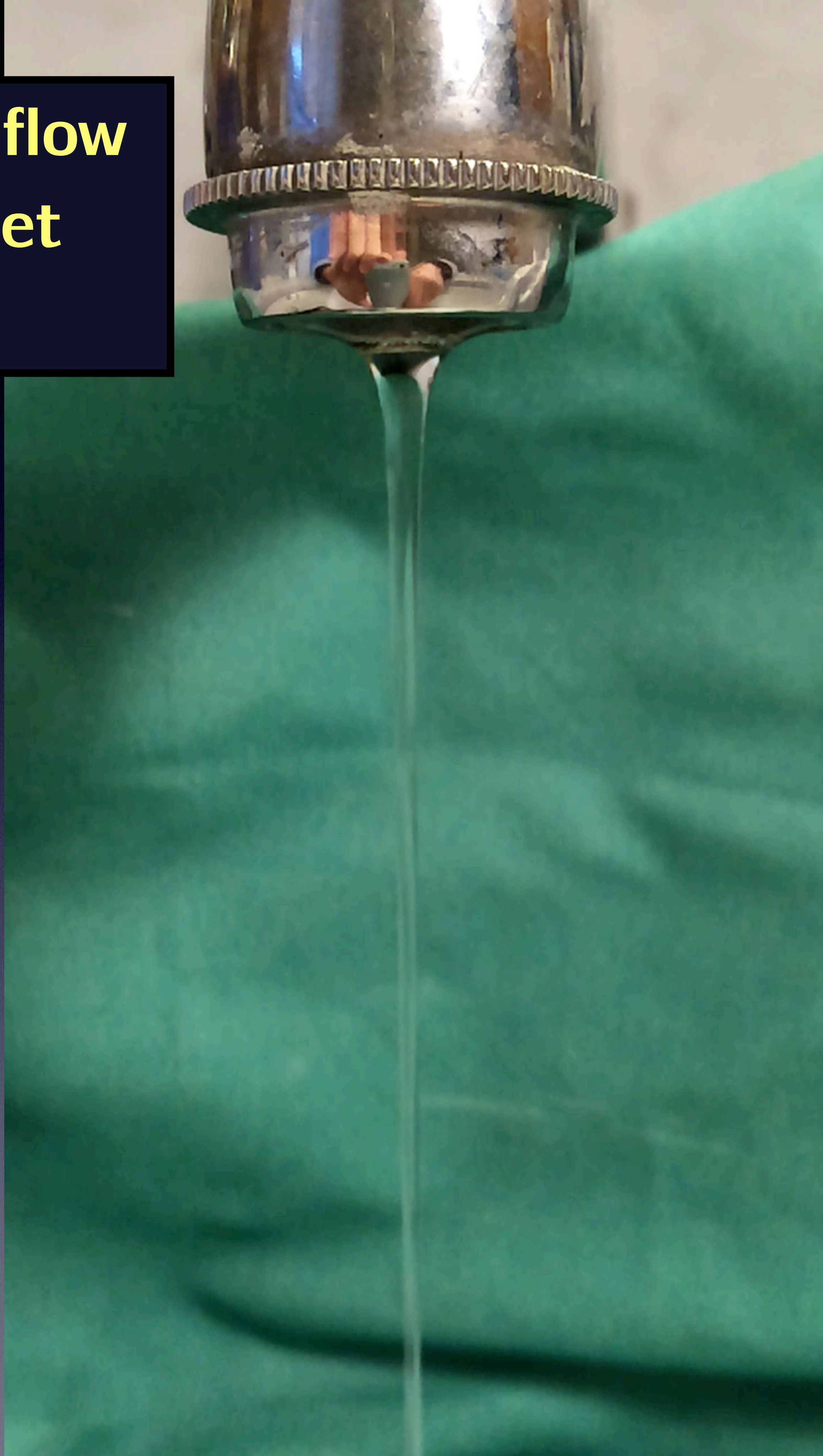
The route to chaos



Four rates of flow from a faucet (1,2,3,4)

Rate 1
(lowest)

Rate 4
(highest)



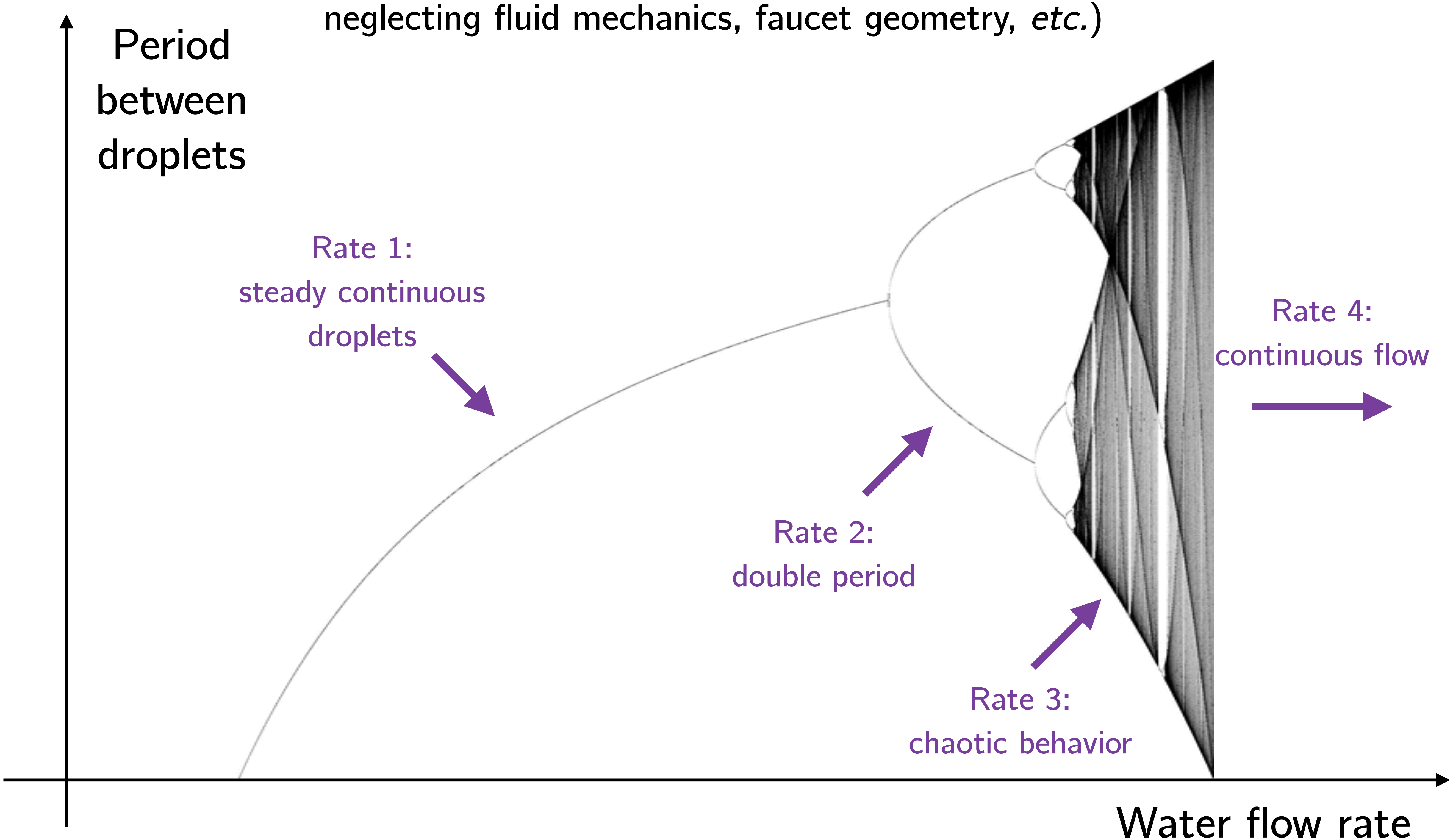
Four rates of flow from a faucet (1,2,3,4)

Rate 2

Rate 3

A period-doubling transition

(An approximate explanation of the behavior, completely neglecting fluid mechanics, faucet geometry, *etc.*)



Some further questions

1. Weather is a very complicated system. Is that necessary to see this type of behavior?

   No. Sometimes very simple systems show this behavior.

2. Why are some systems predictable and some aren't?

   Key idea is that system states that start close together will diverge over time (like kneading bread).

3. Is there really any connection between the behavior of different unpredictable systems? Aren't droplets and weather just different?

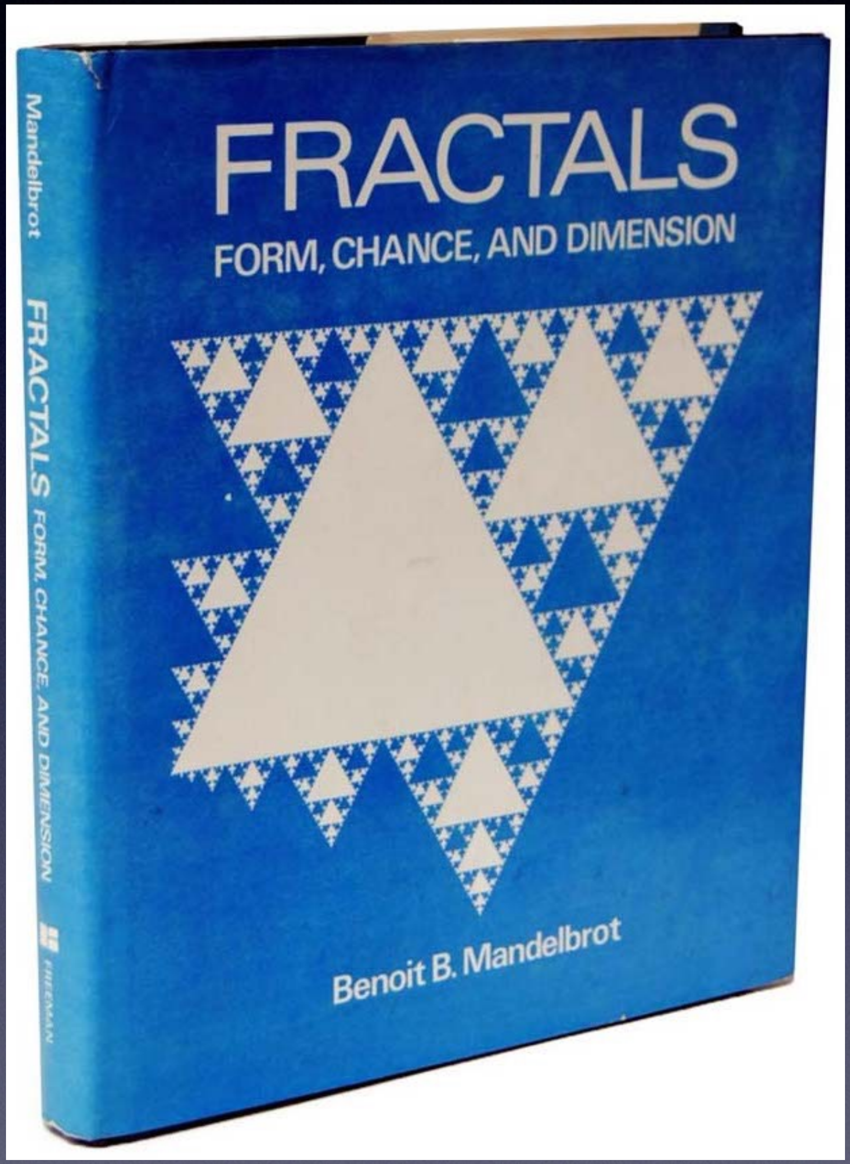
   Many systems show a universal transition to chaos, via "period doubling" as the amplitude/power is ramped up.

Benoit Mandelbrot (1924–2008)

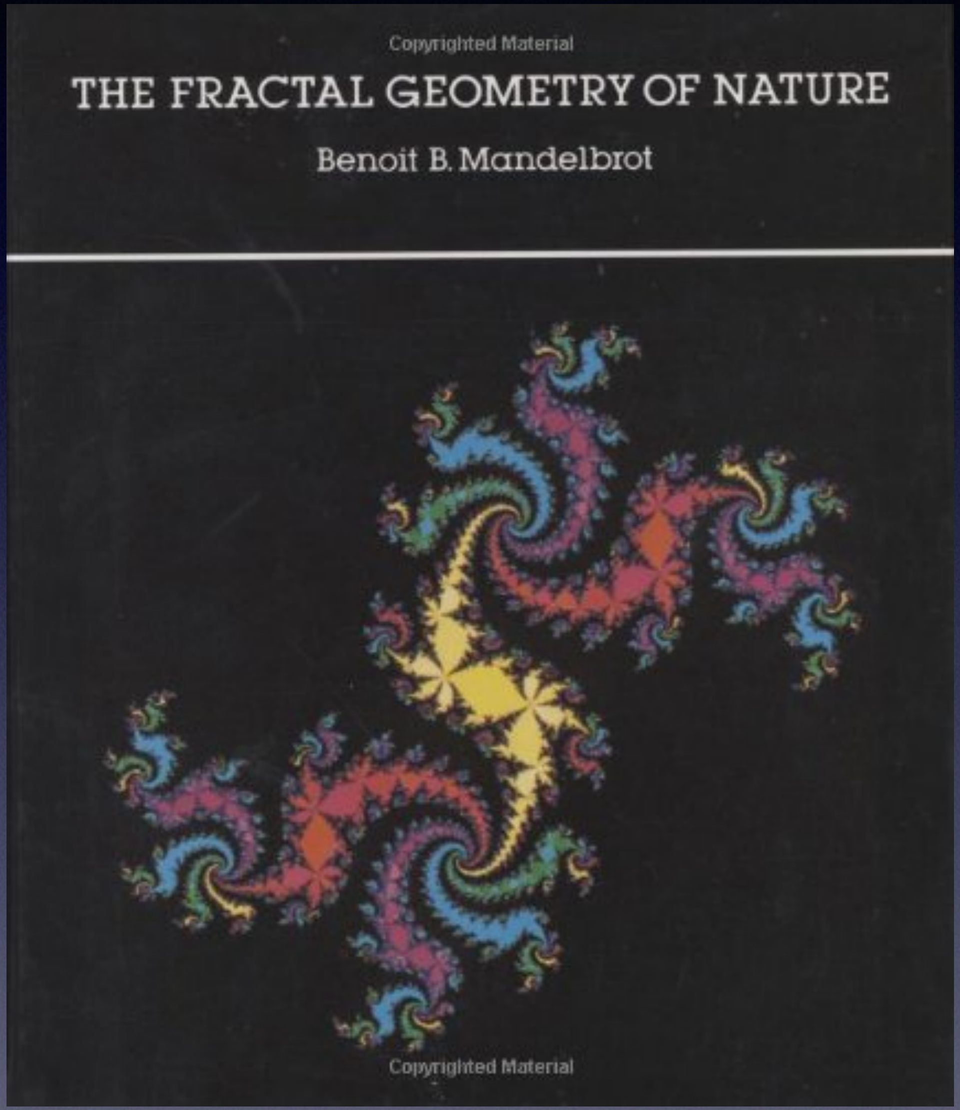
- Born in Poland, although moved to France in 1936
- Studied and taught in France
- Moved to IBM in 1958, initially worked on telephone communications, although became part of their pure research division
- Later worked at Harvard and Yale
- Highly familiar with computation

Mandelbrot at IBM

Mandelbrot's books



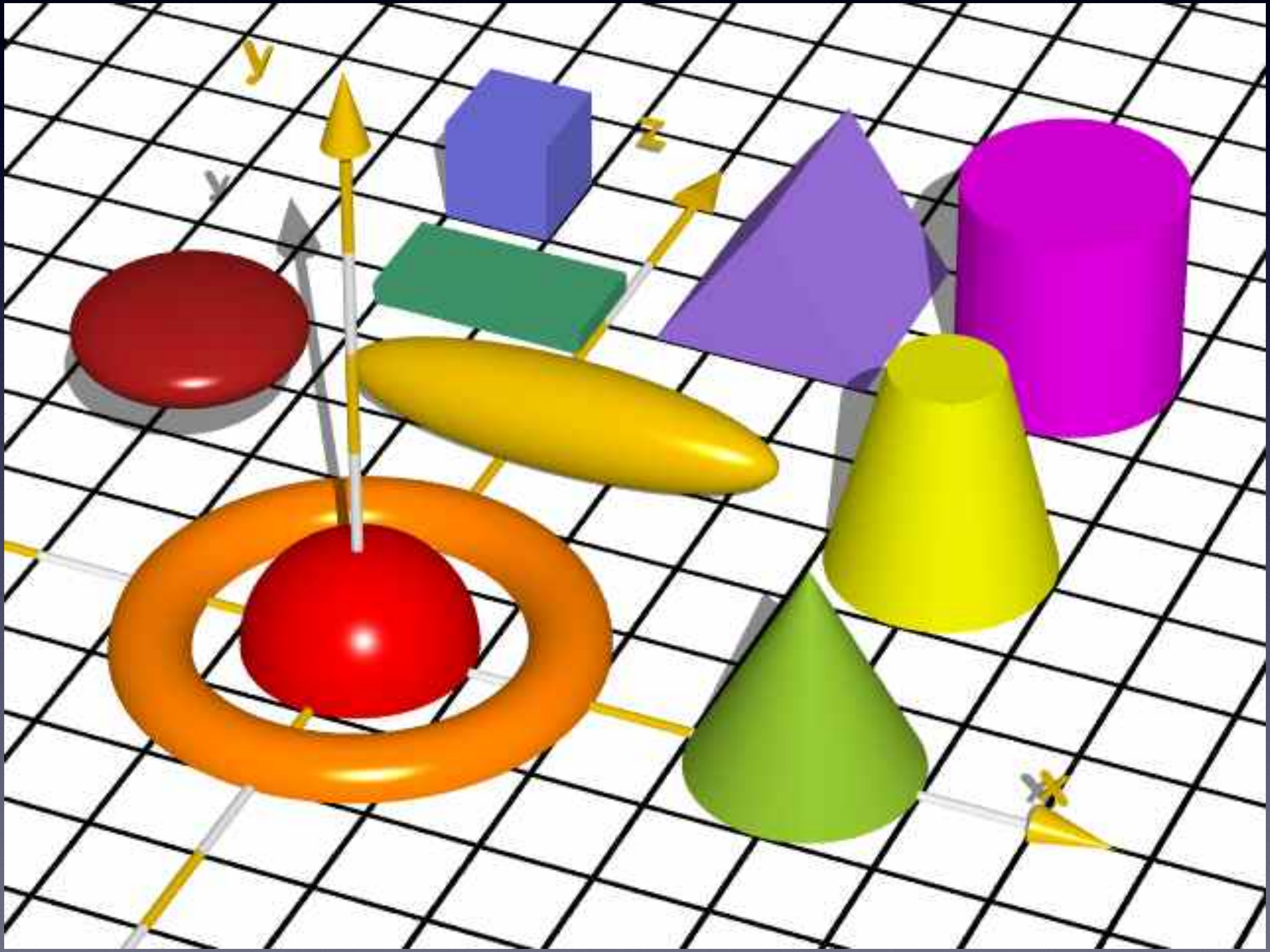
Fractals: Form, Chance, and Dimension (1977)



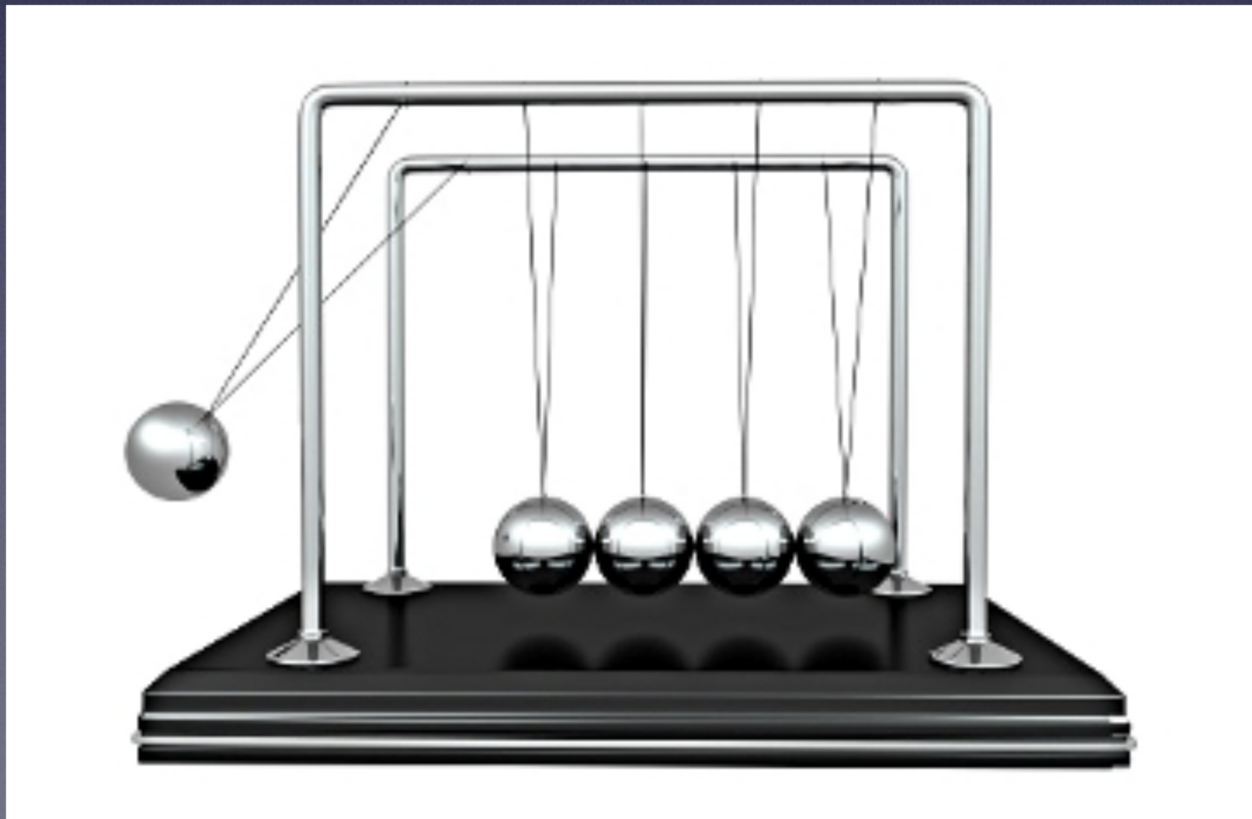
The Fractal Geometry of Nature (1982)

Euclidean geometry

The geometry of smooth, regular objects



Human shapes



Natural shapes

First key idea – self-similarity

If you zoom in or zoom out, the object looks similar

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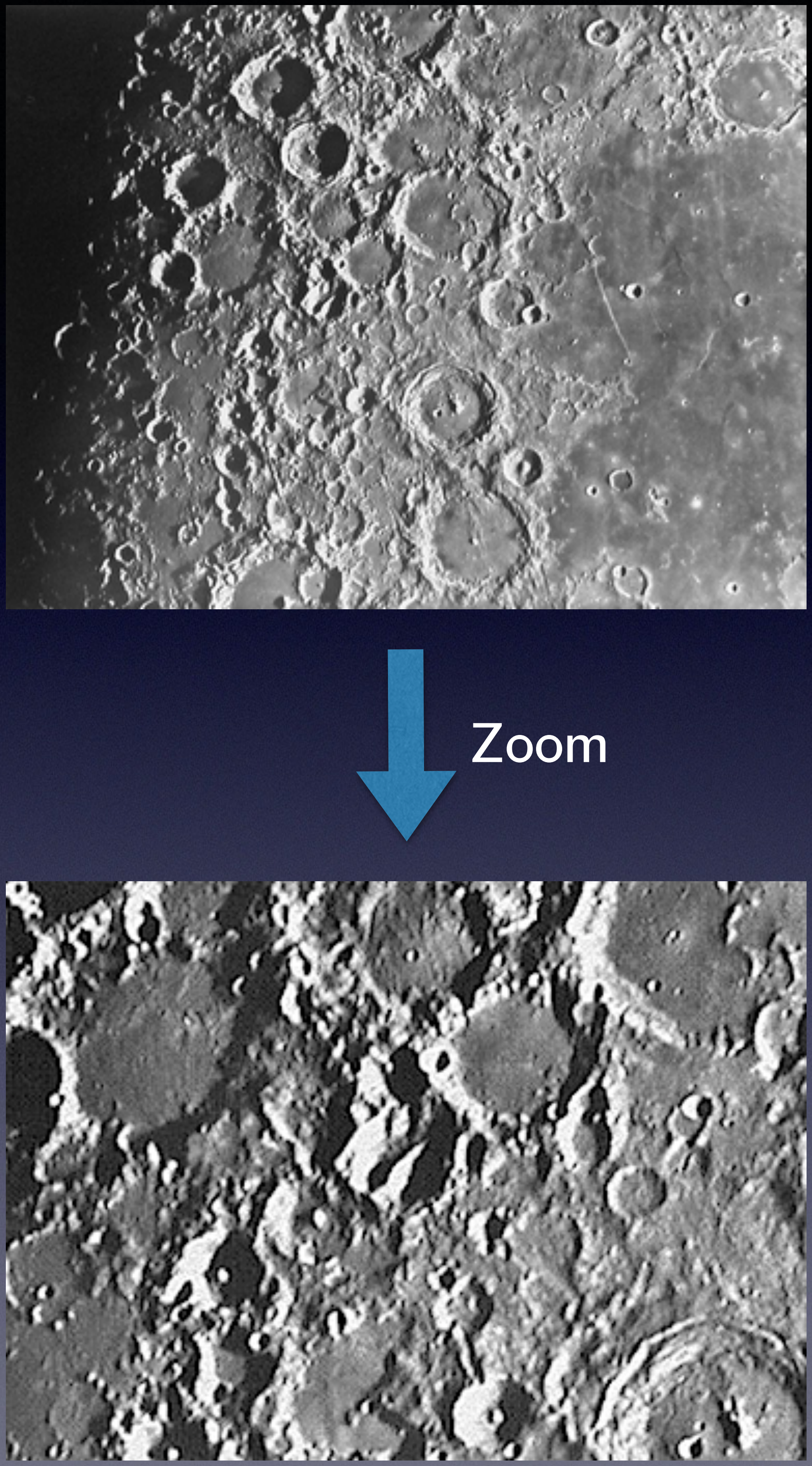
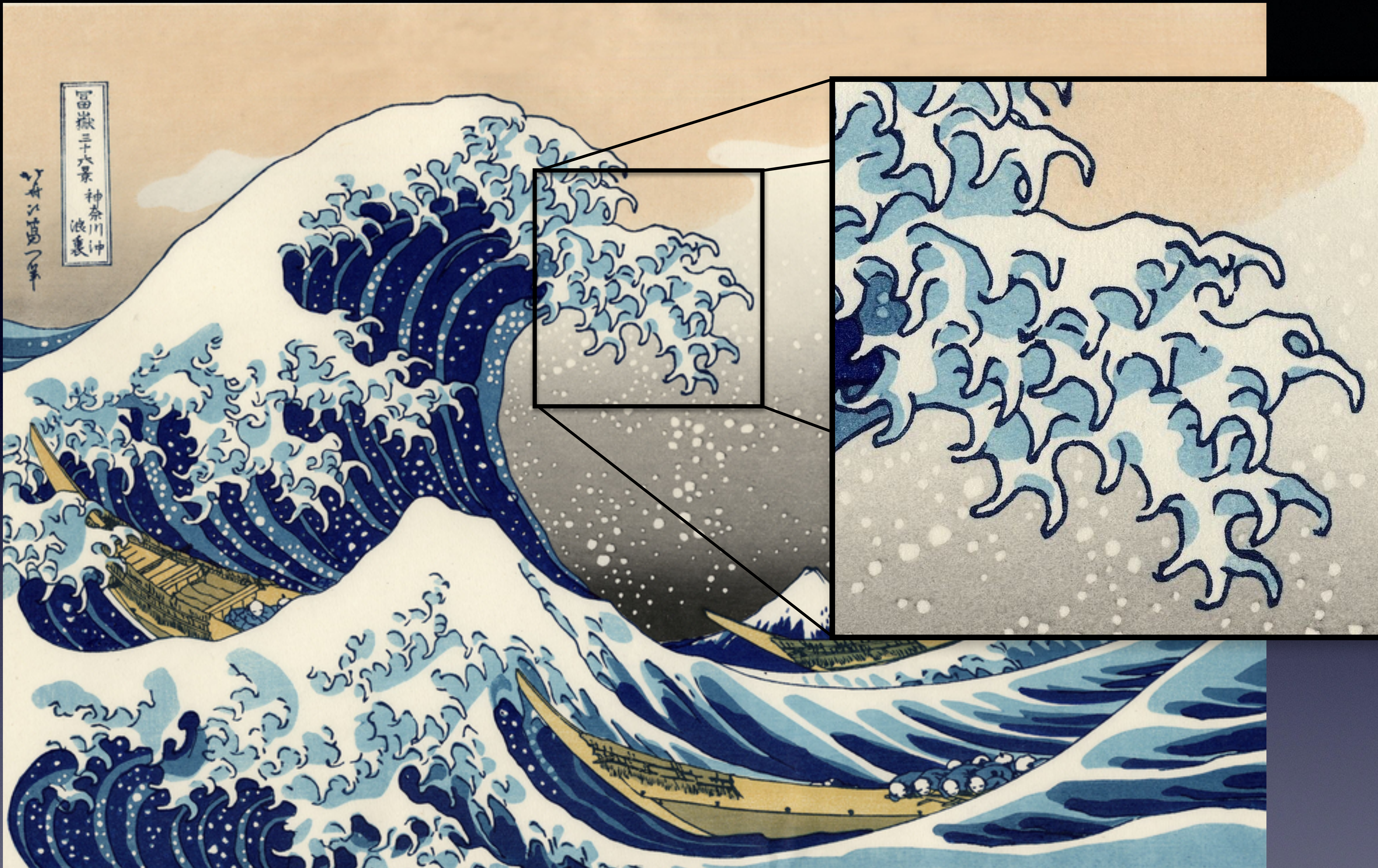
If you zoom in or zoom out, the object looks similar





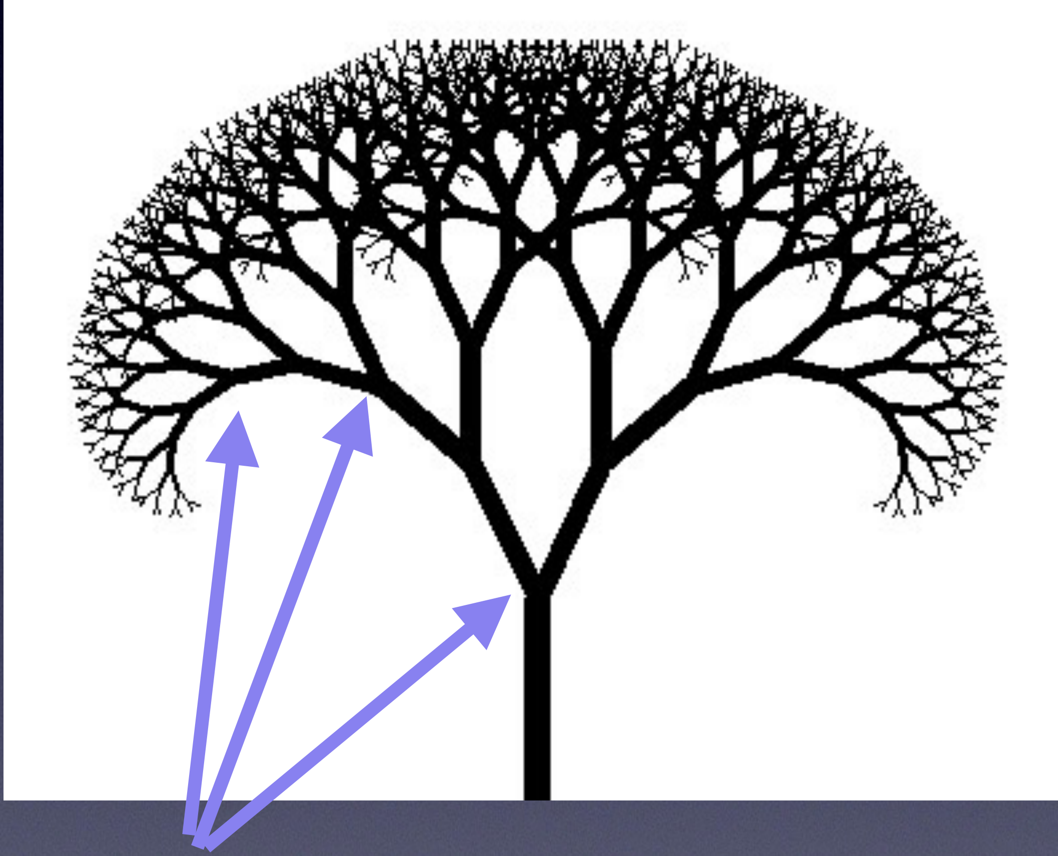
The Great Wave off Kanagawa

by Hokusai, *c.* 1830

Book of Kells (c. 800 AD)

Second key idea – simple rules

- On the surface, you see complexity
- Don't think about what is seen, but instead focus on what it took to produce what you see
- Simple rules and physical principles underpin natural shapes
- A tree's DNA does not need encode every single twig, it just needs to encode **how to branch**

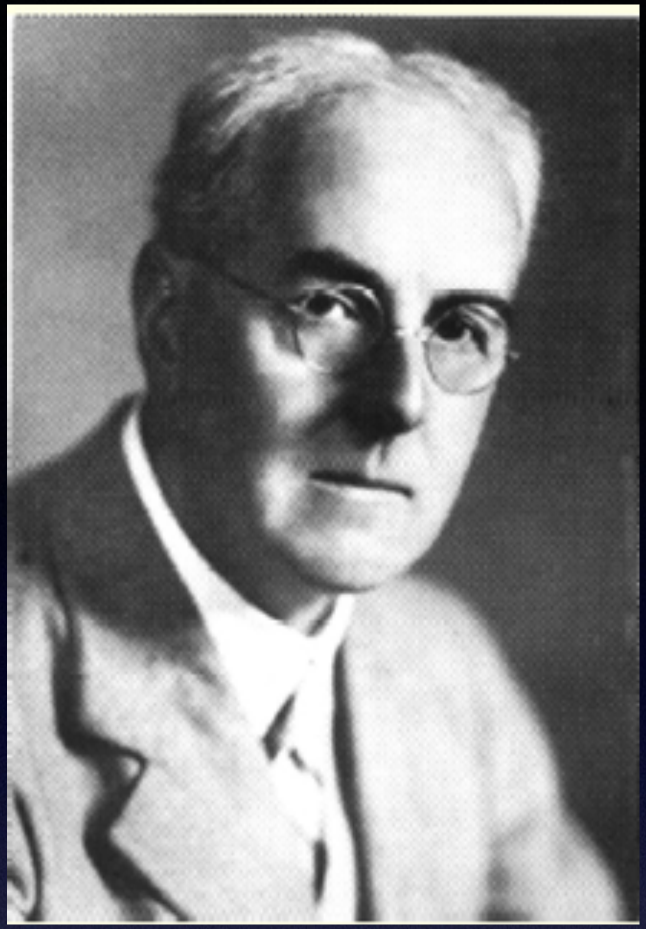


How long is the coastline of the British Isles?

How long is the coastline of the British Isles?

- Originally observed by British meteorologist and mathematician Lewis Fry Richardson
- The measurement of coastline depends on the size of your measuring stick
- Specifically for stick length G, the measured length roughly follows

$$\text{Coastline length} = MG^{1-D}$$



Lewis Fry Richardson (1881–1953)

How long is the coast of Britain: statistical self-similarity and fractional dimension (Science magazine, 1967)

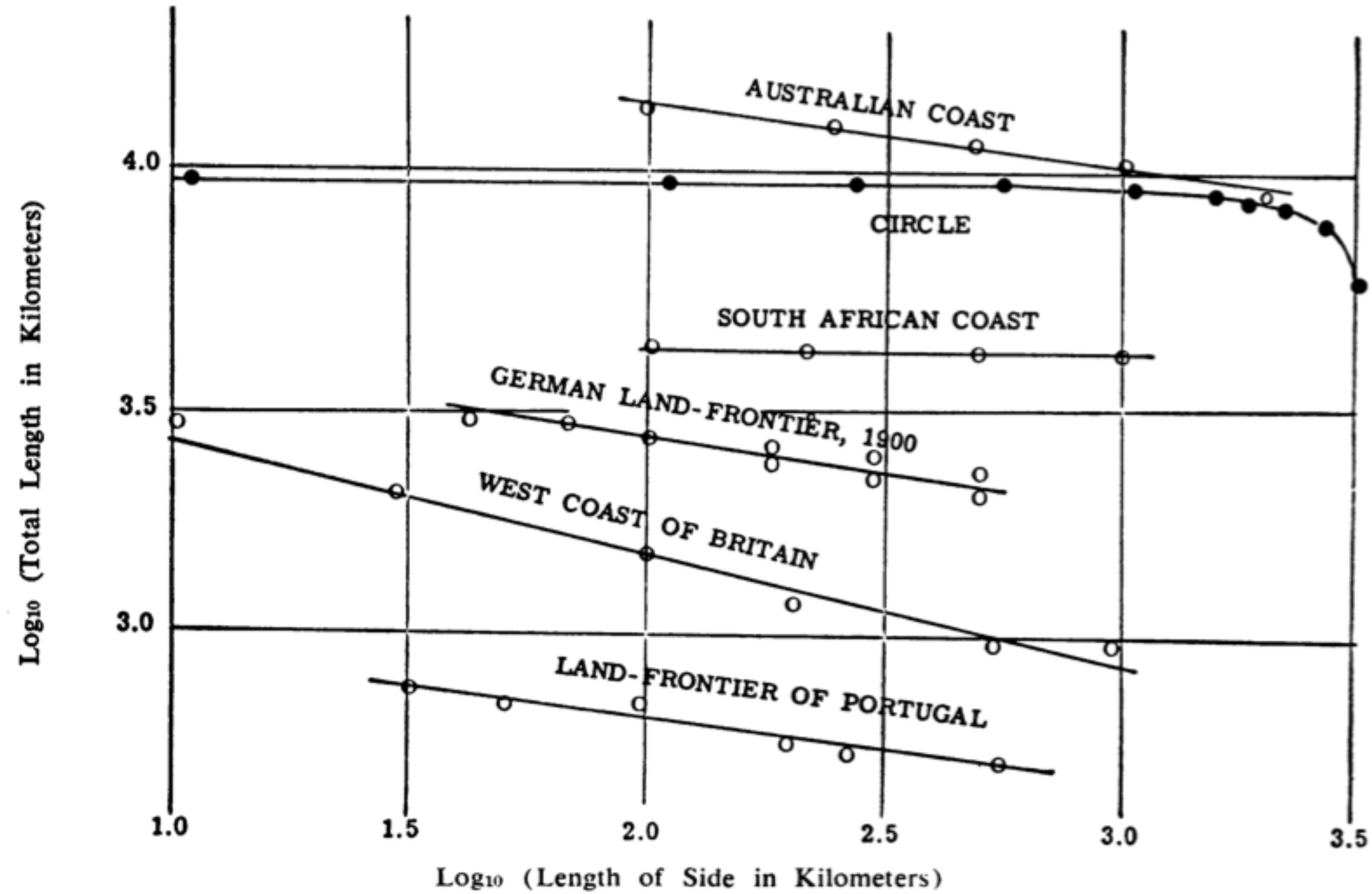
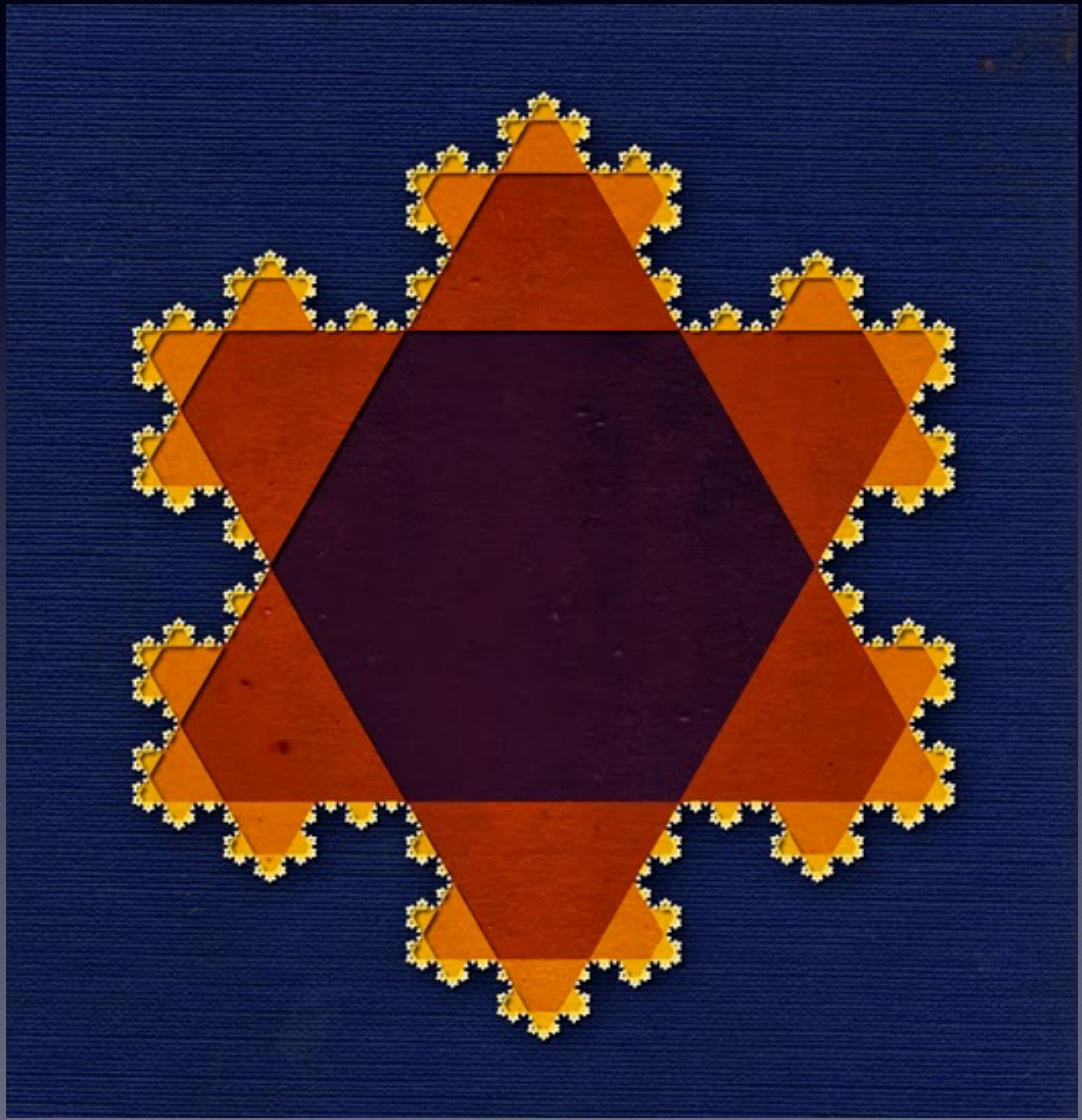


Fig. 1. Richardson's data concerning measurements of geographical curves by way of polygons which have equal sides and have their corners on the curve. For the circle, the total length tends to a limit as the side goes to zero. In all other cases, it increases as the side becomes shorter, the slope of the doubly logarithmic graph being in absolute value equal to D–1. (Reproduced from *2*, Fig. 17, by permission.)

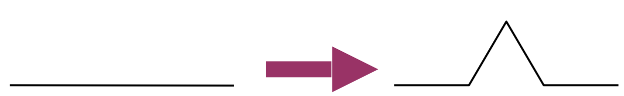
The "Monster curves"

- Mandelbrot was familiar with an obscure branch of mathematics originating in the 1930's
- Mathematicians had described strange geometrical objects created through *iteration*
- Referred to as "Monster curves" because they were too difficult to analyze at the time
- But Mandelbrot had the power of the computer



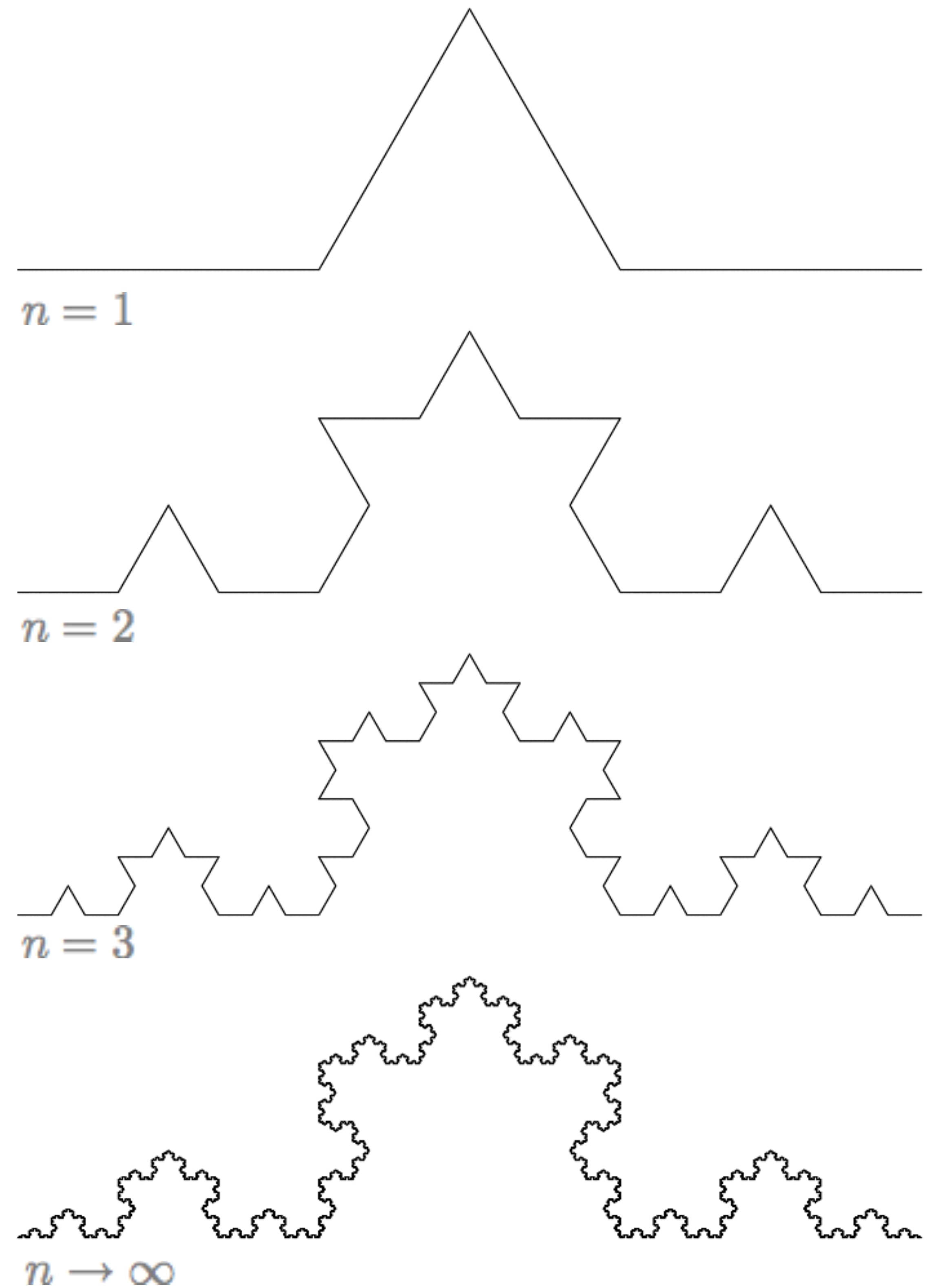
A mathematical model: the Koch curve

- Originally described by Helge von Koch (1870–1924)
- Start with a straight line
- Replace



- Perimeter after n steps

$$p_n = p_0 \left(\frac{4}{3}\right)^n \to \infty$$

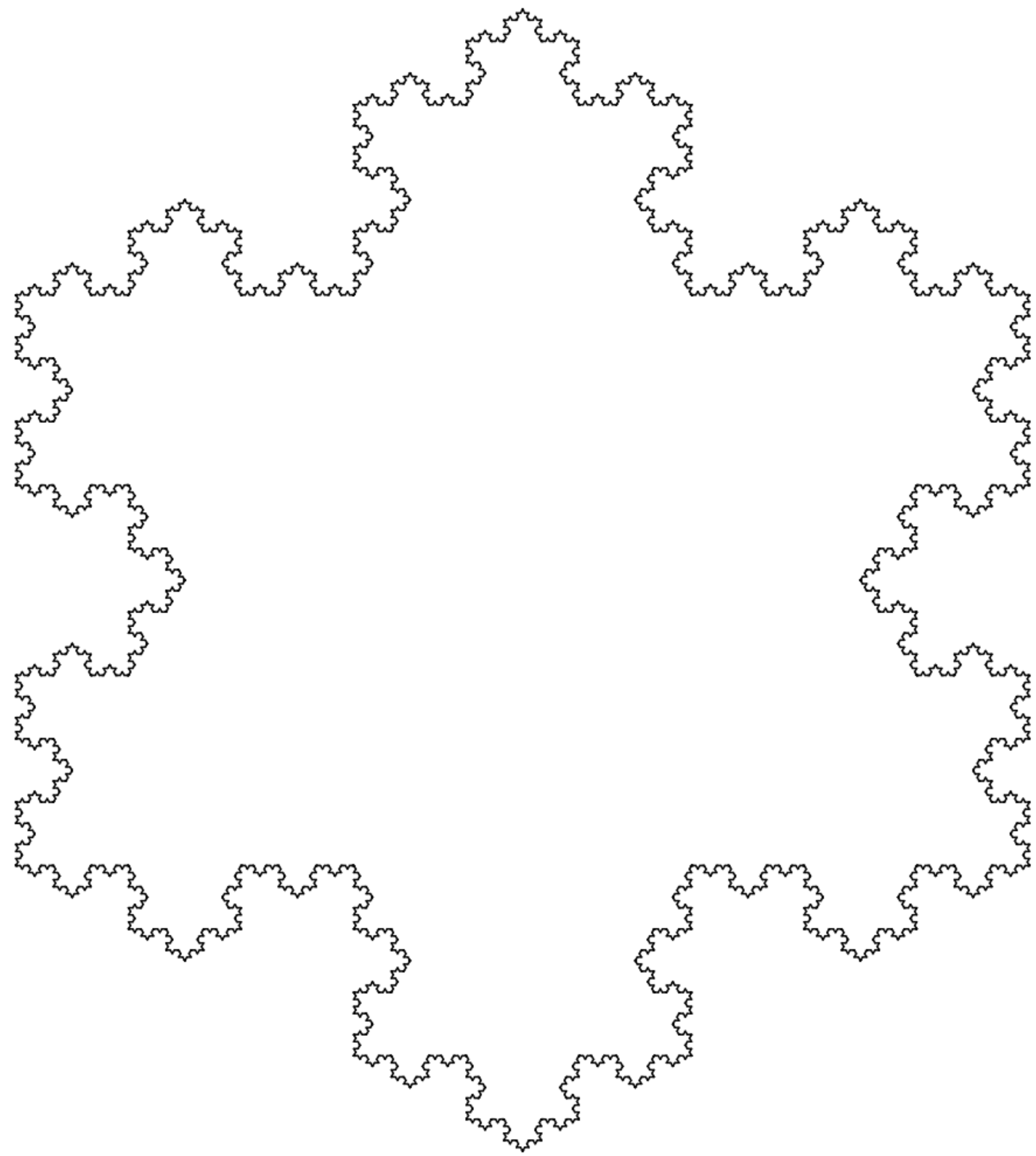


Koch snowflake

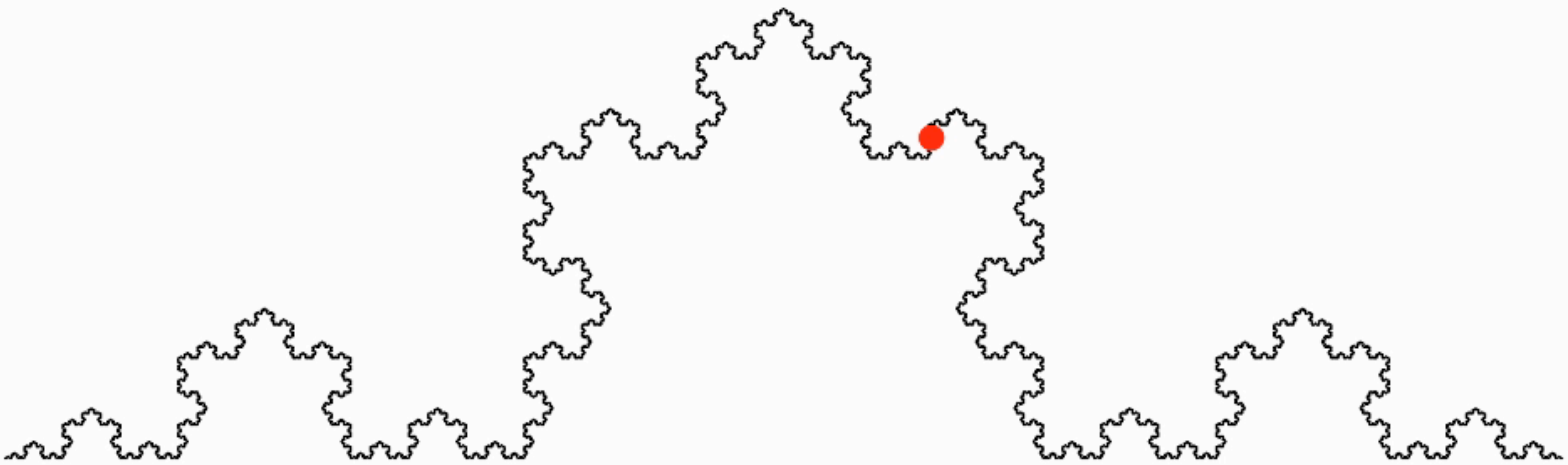
- Start with equilateral triangle and apply the iterative process
- Perimeter

$$p_n = p_0 \left(\frac{4}{3}\right)^n \to \infty$$

- But the enclosed area is finite!
- Same behavior as coastline

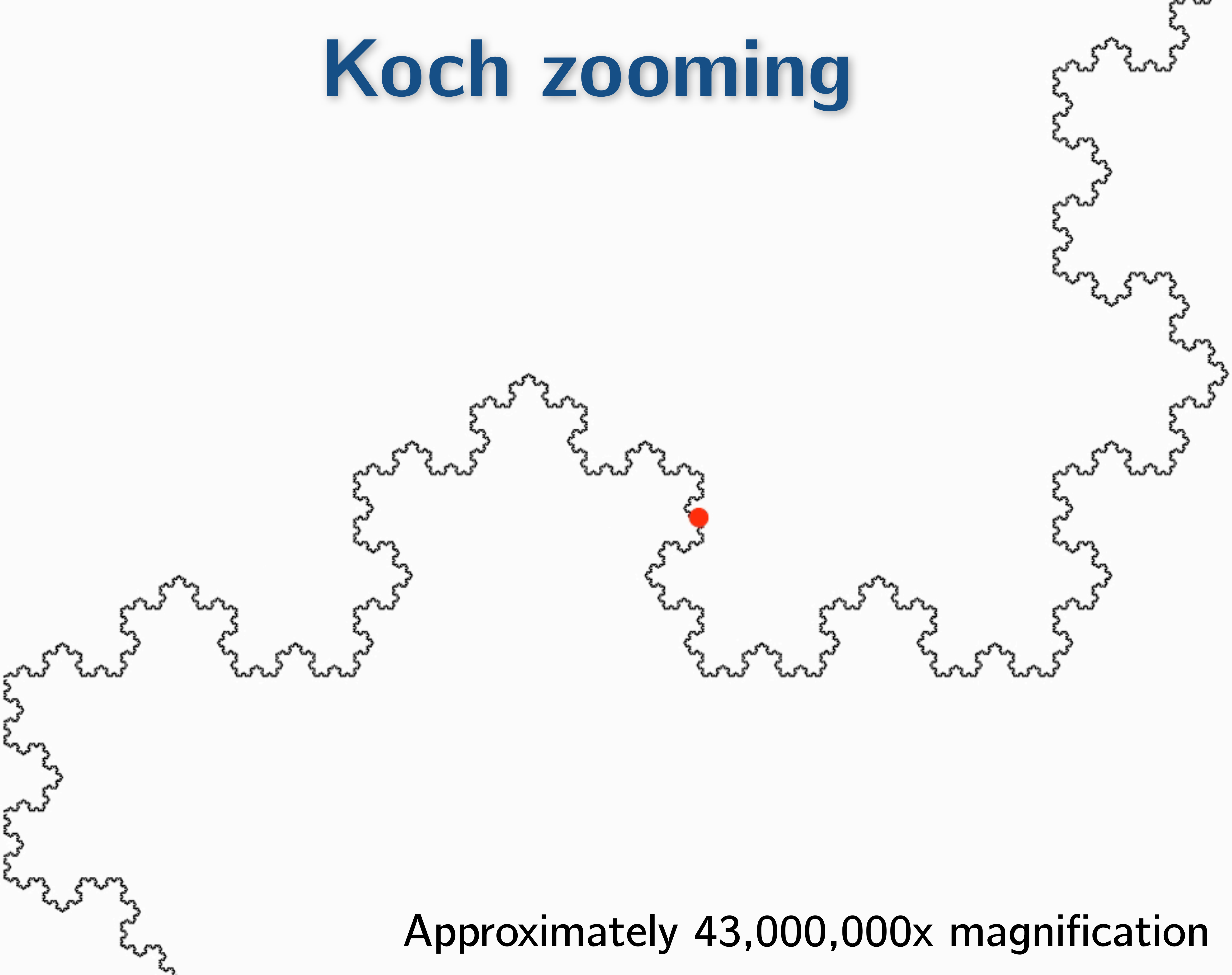


Koch zooming



Approximately 43,000,000x magnification

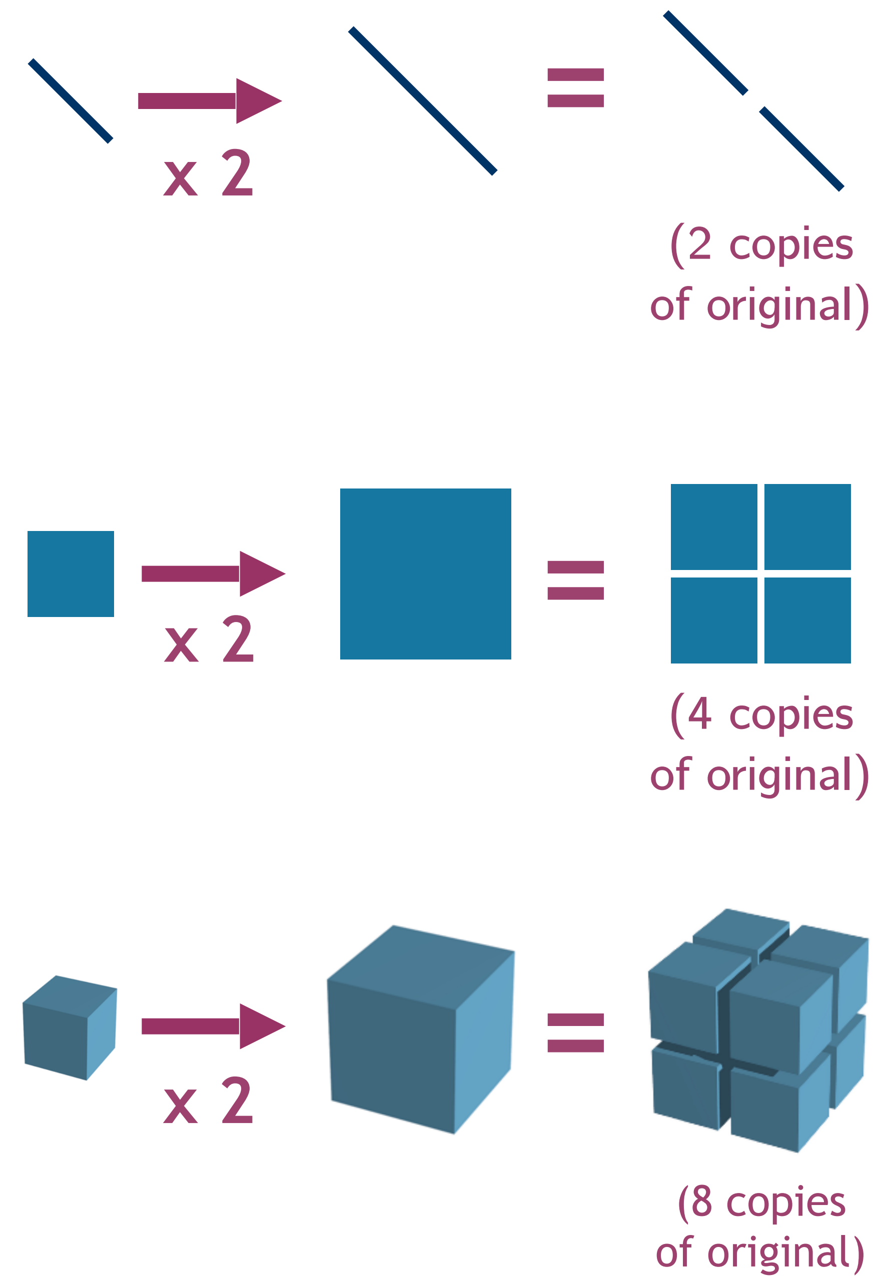
Koch zooming



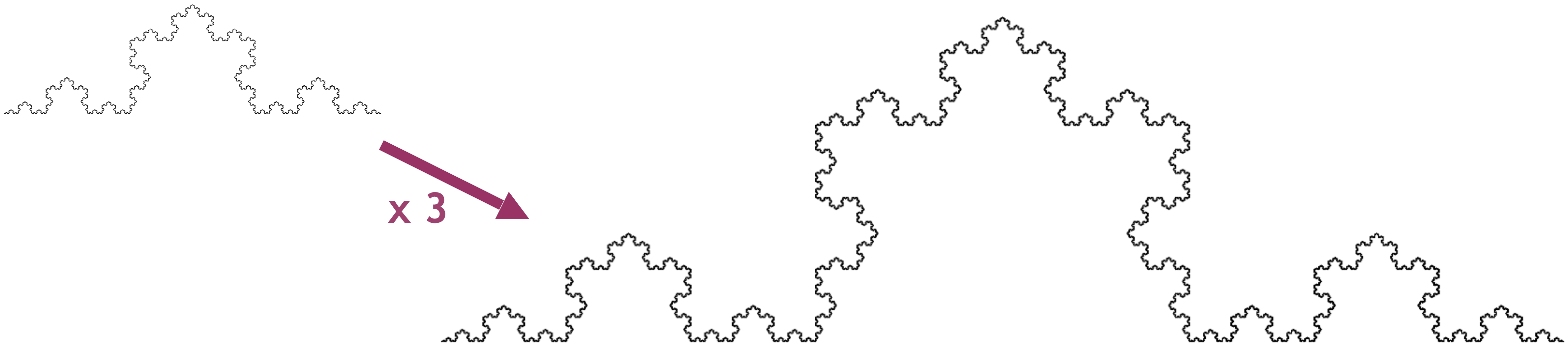
Approximately 43,000,000x magnification

A signature of dimension

- Consider scaling an object by a factor of 2
- Need 2^d copies of the original object to cover the scaled object
- Should be true for fractals as well



Fractal dimension of the Koch curve

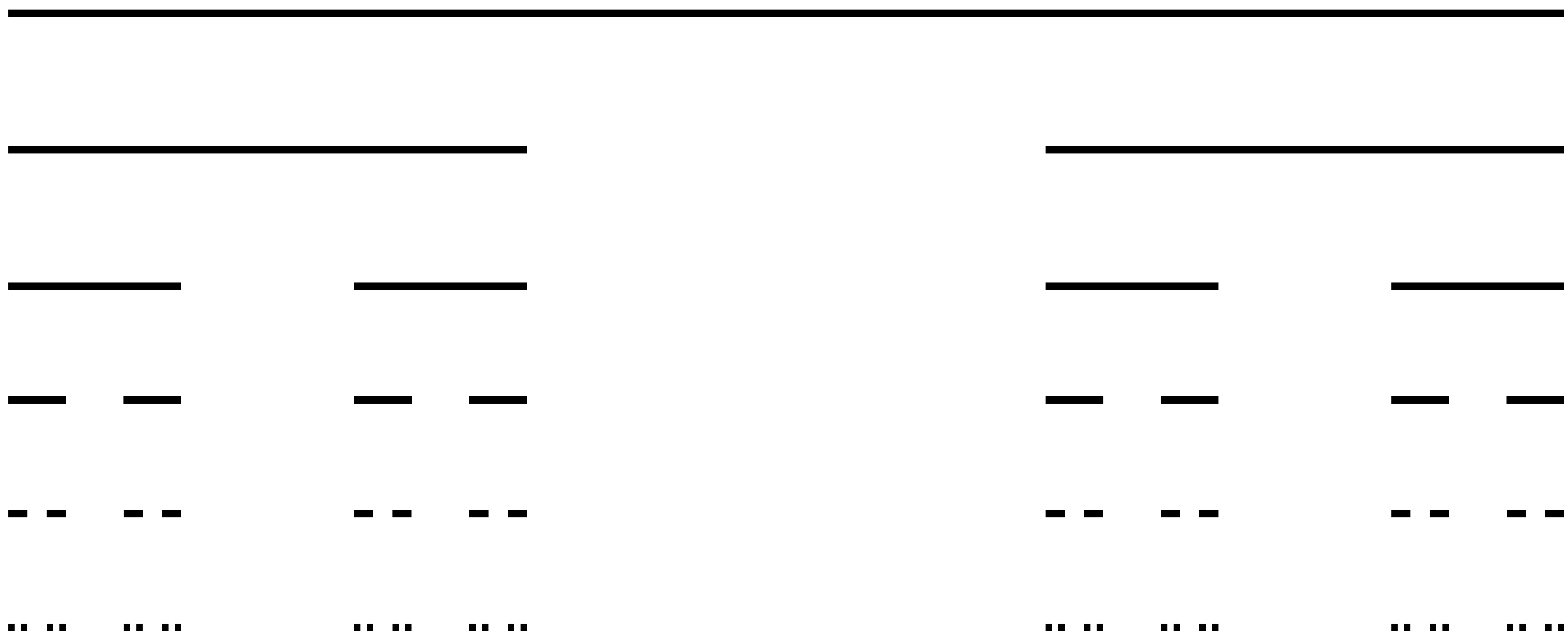


- Get four copies of the original shape

$$3^d = 4 \qquad d = \frac{\log 4}{\log 3} = 1.262$$

- "More than a line, but less than a plane"

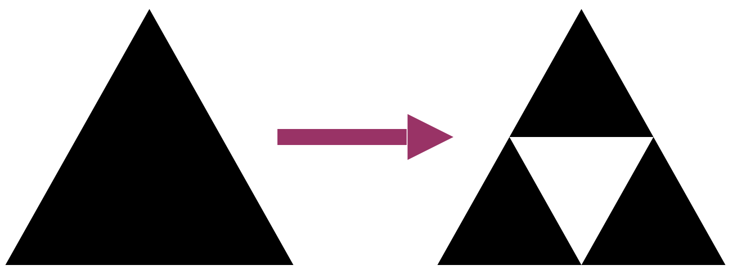
Cantor set



- Repeatedly delete the middle third of a straight line
- Fractal dimension is

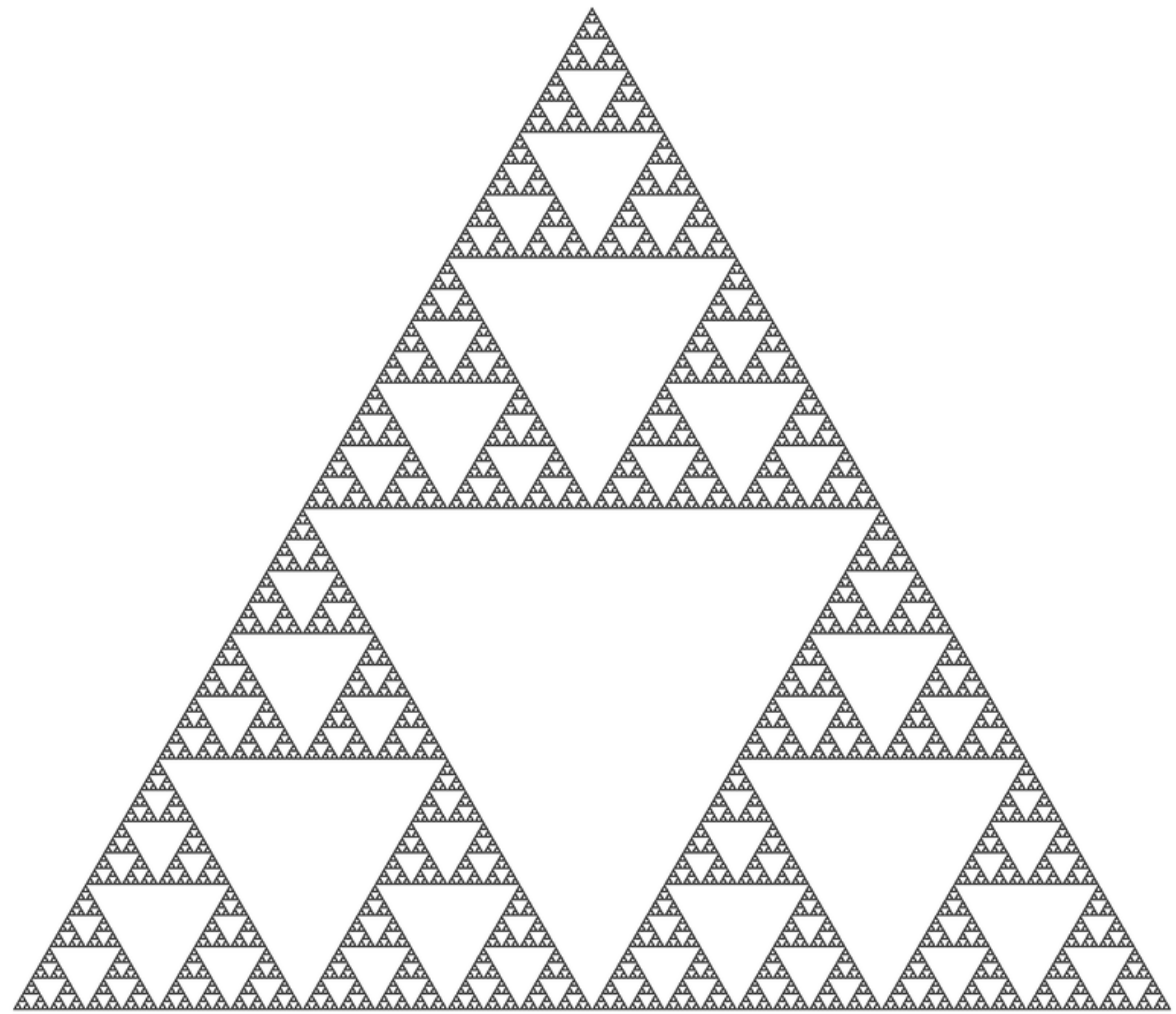
$$2^d = 3 \qquad d = \frac{\log 2}{\log 3} = 0.631$$

Sierpinski triangle

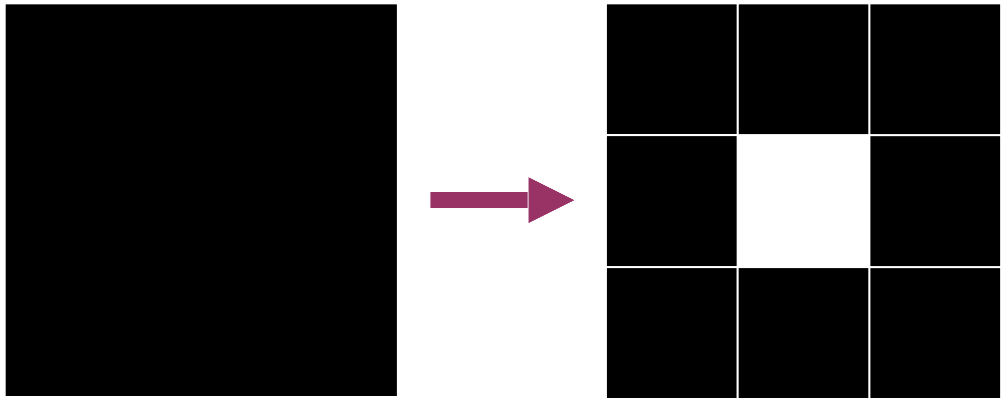


$$d = \frac{\log 3}{\log 2} = 1.585$$

- Infinite perimeter, but with zero area

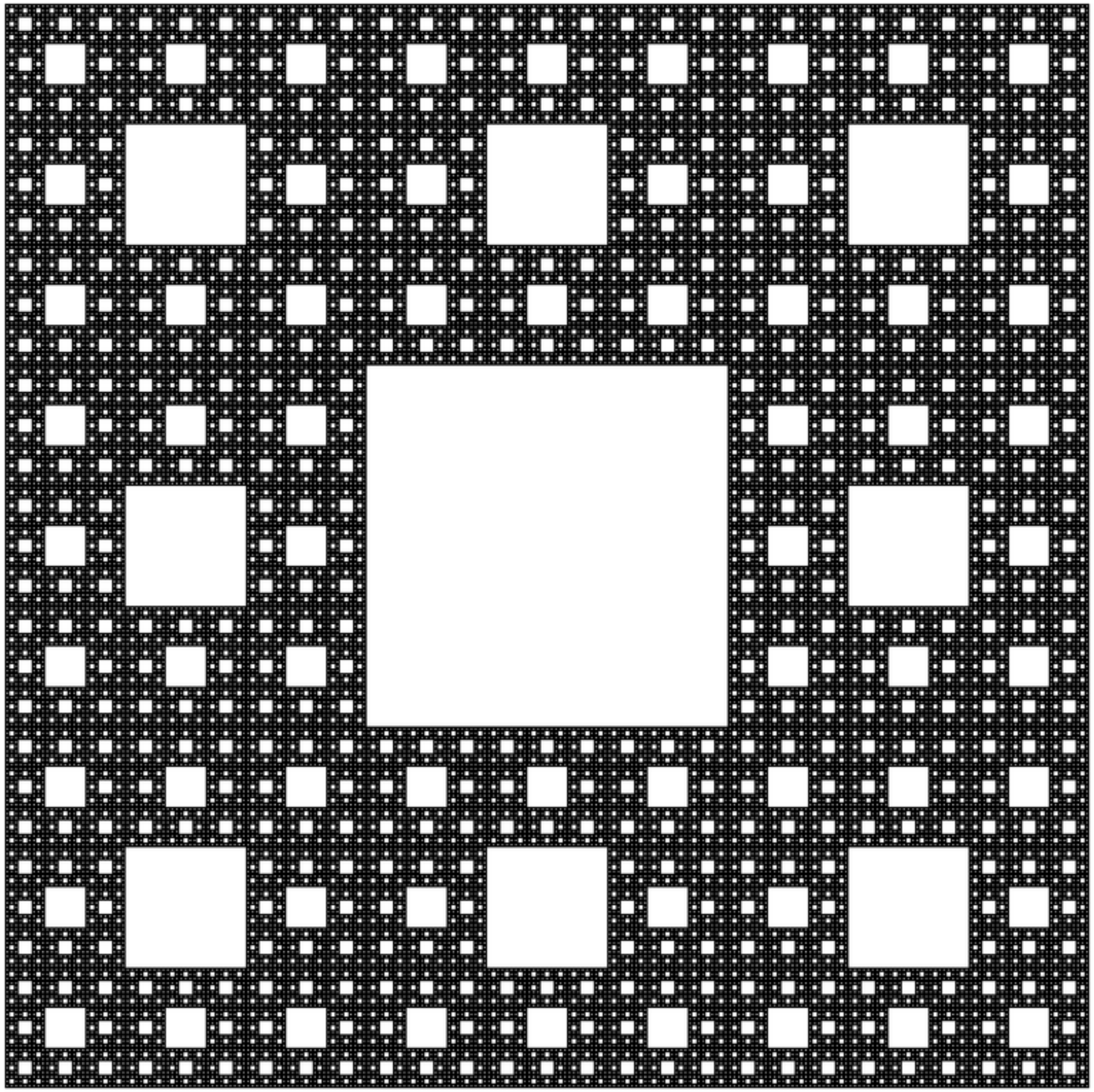


Sierpinski carpet

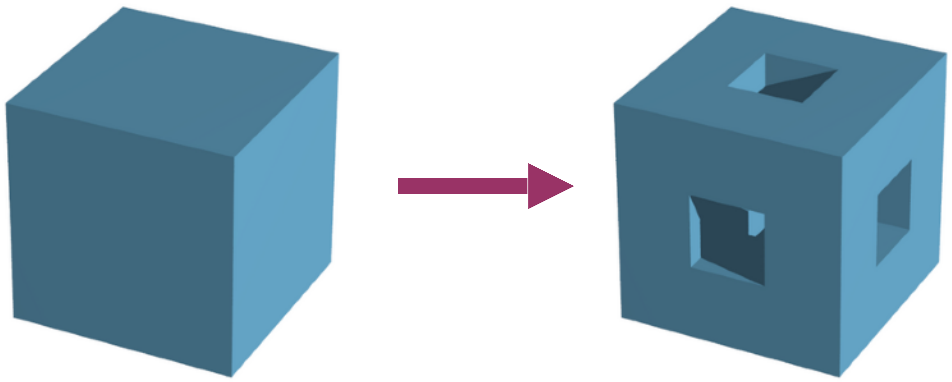


$$d = \frac{\log 8}{\log 3} = 1.893$$

- Infinite perimeter, zero area

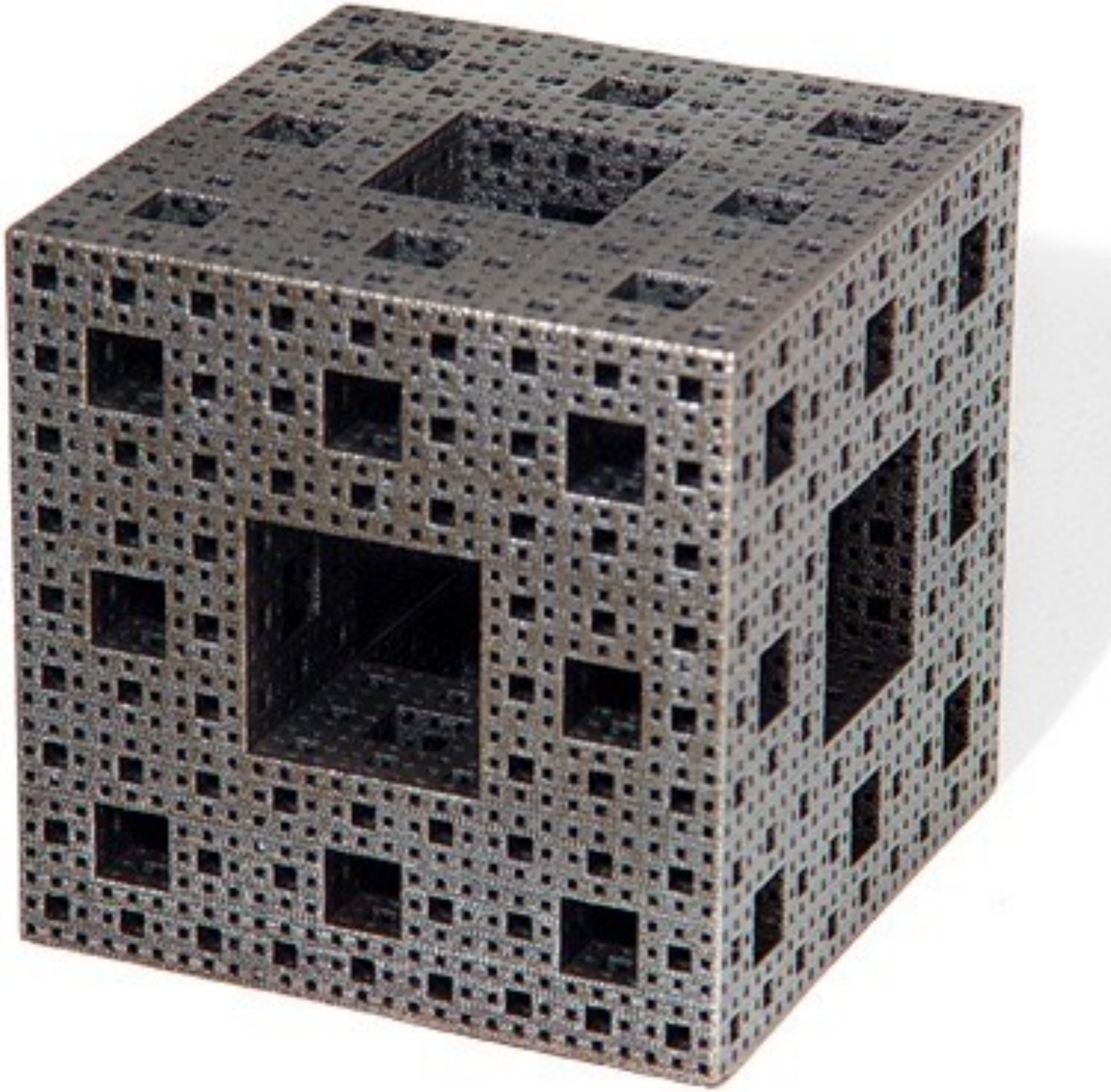


Menger sponge

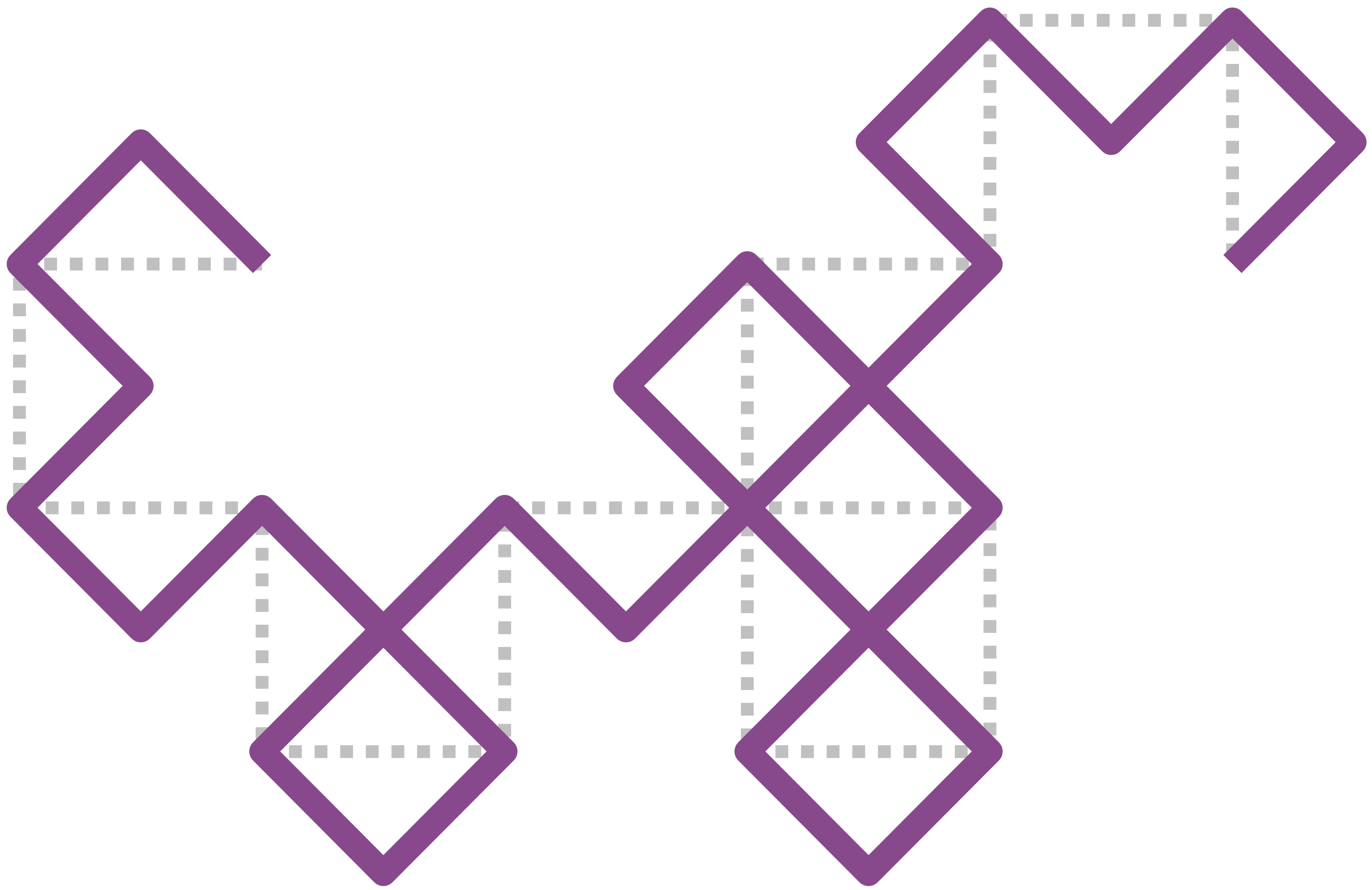


$$d = \frac{\log 20}{\log 3} = 2.727$$

- Zero volume and infinite surface area

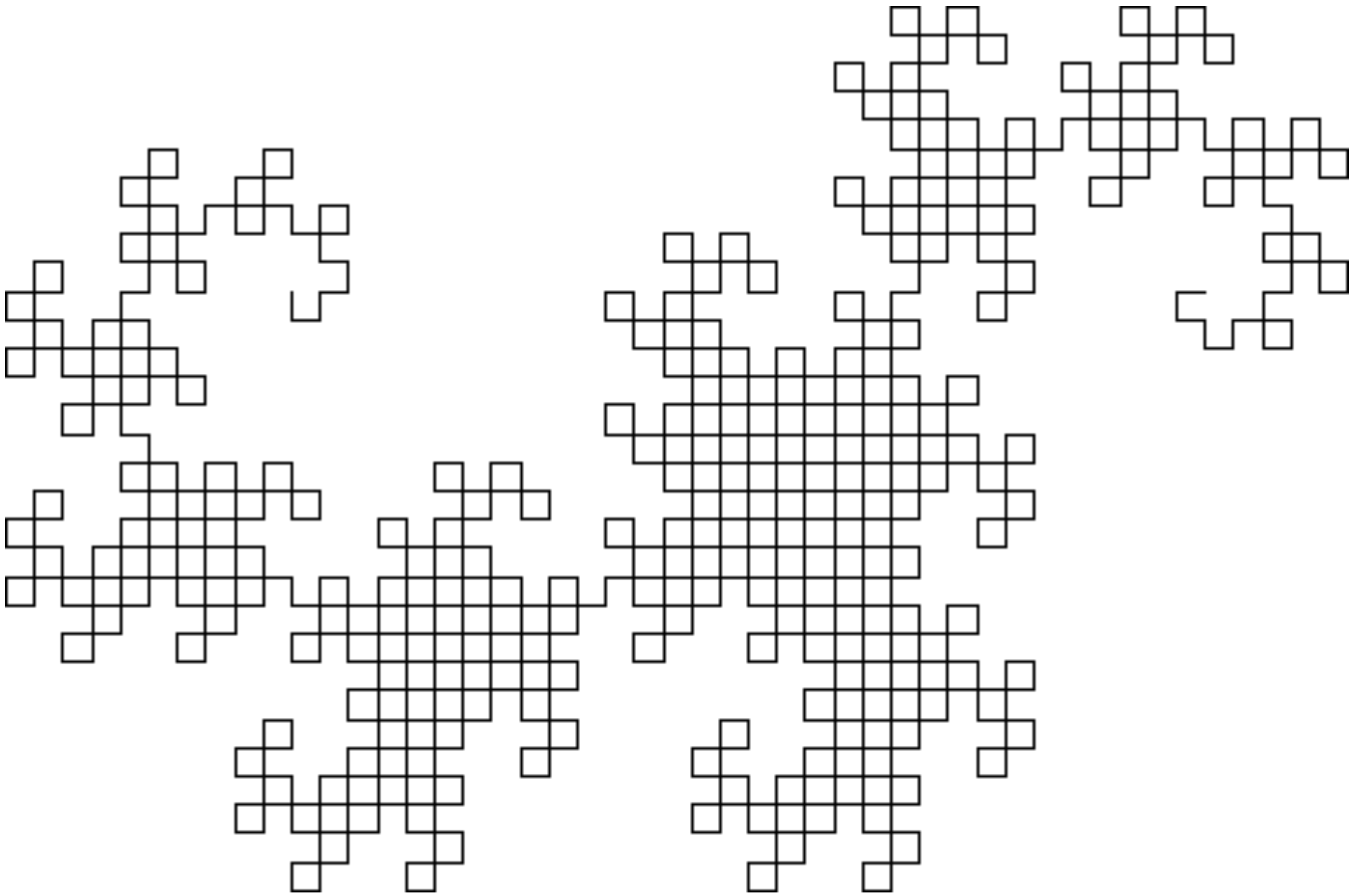


Paper folding



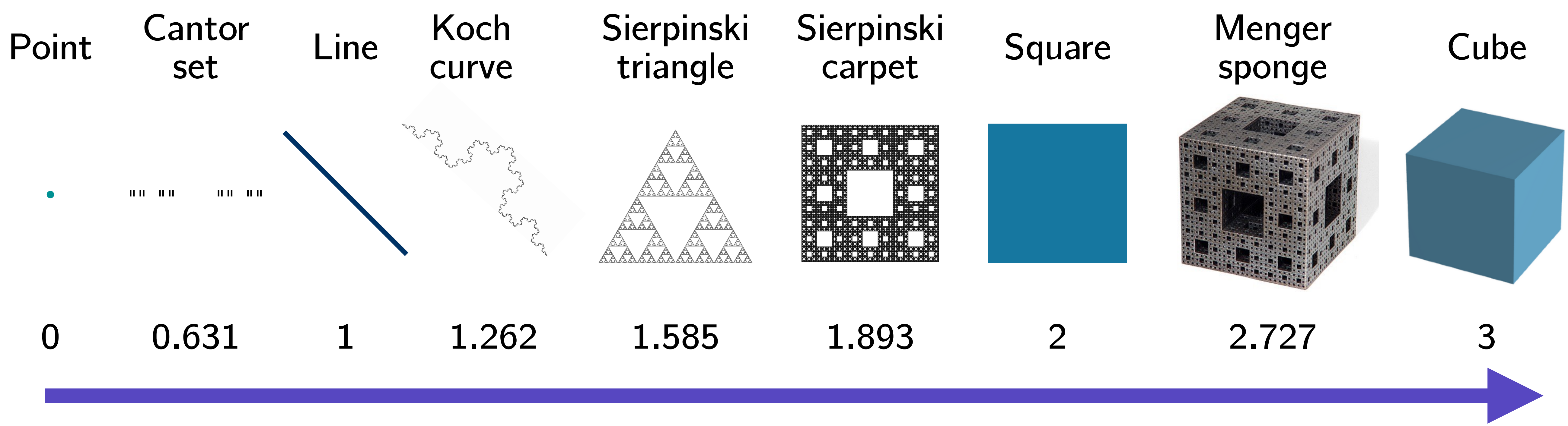
- Replace straight line by bends, alternating left and right

From the demo: the dragon curve



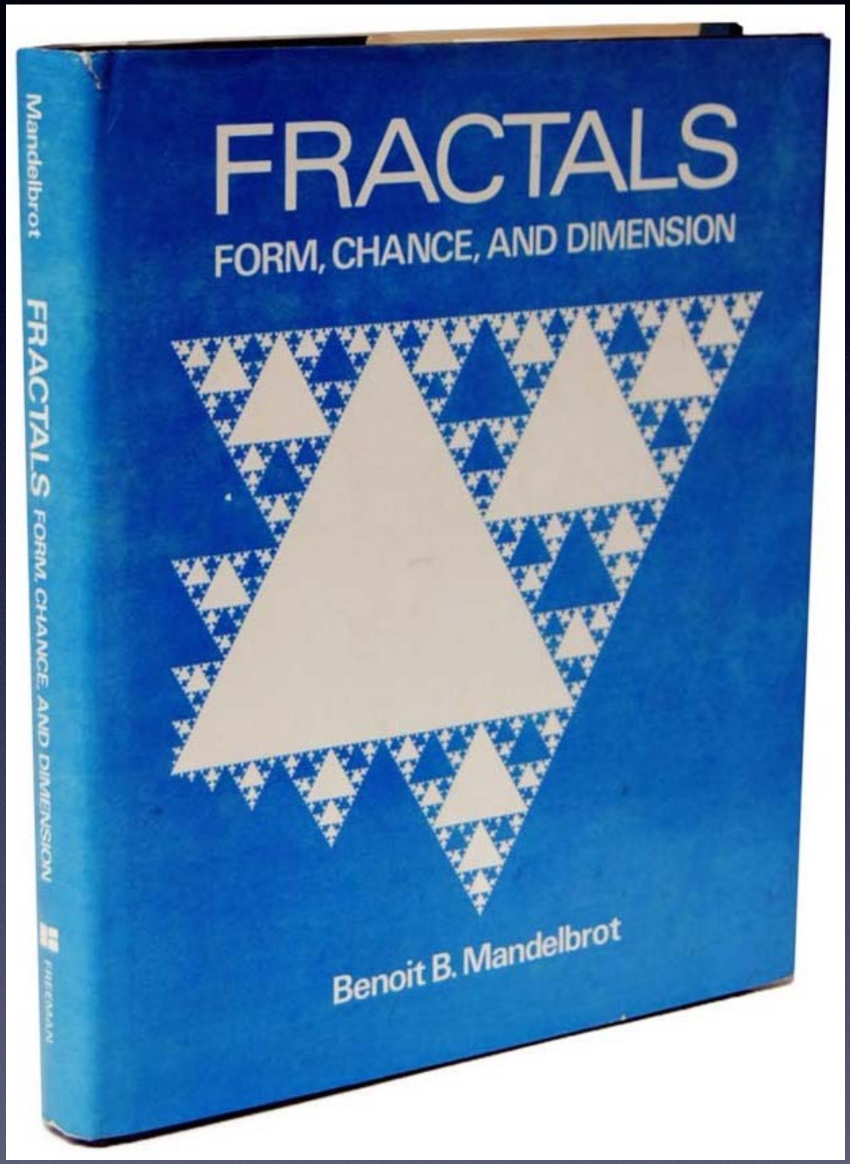
- After ten iterations a fractal shape appears
- This is a continuous, non-intersecting curve

Fractal dimension – a tool for measuring shape

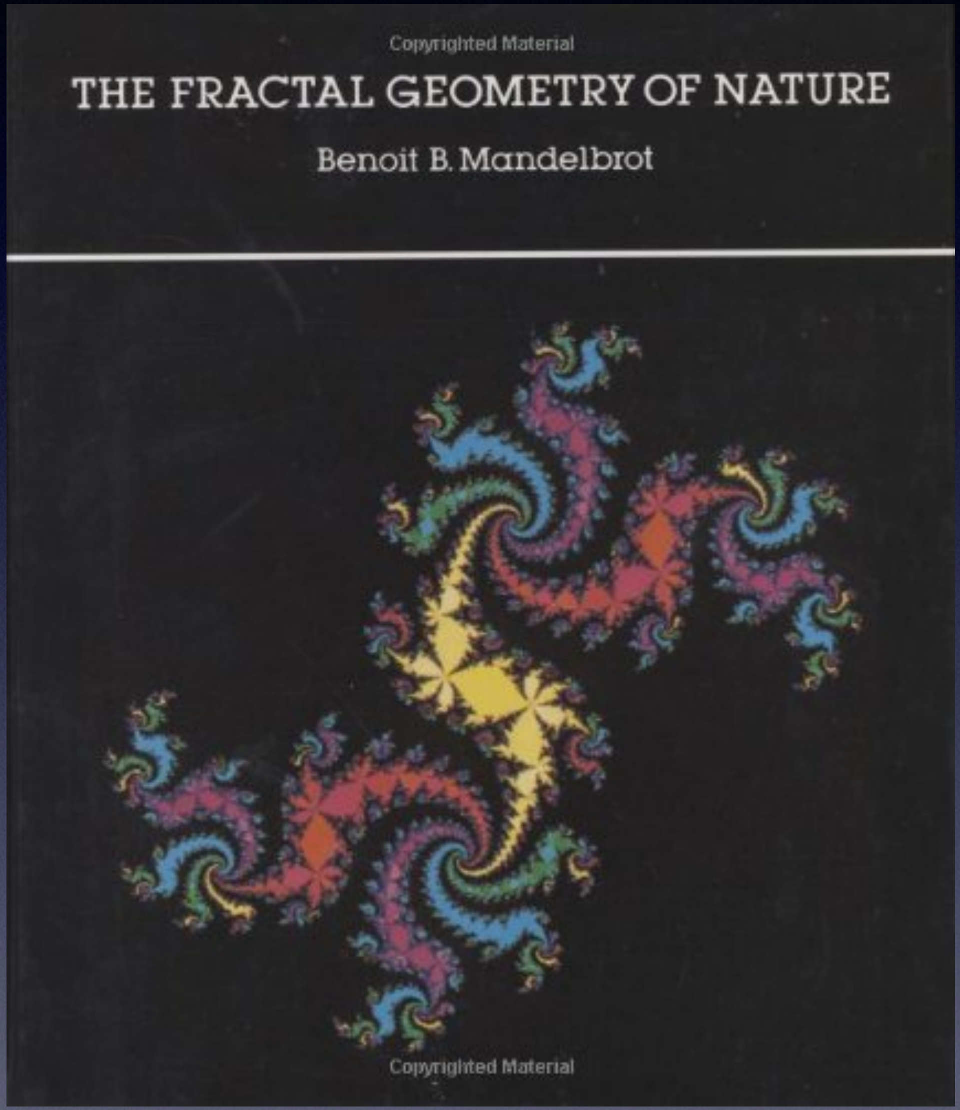


- Provides general way to compare fractal shapes
- People started seeing fractals everywhere

Mandelbrot's books



Fractals: Form, Chance, and Dimension (1977)



The Fractal Geometry of Nature (1982)

Quotes from Mandelbrot's books

I conceived and developed ...

I confirmed ...

In my travels through newly opened or newly settled territory, I was often moved to exert the right of naming its landmarks ...

Quotes from Gleick

Many scientists failed to appreciate this kind of style. nor were they mollified that Mandelbrot was equally copious with his references to predecessors, some thoroughly obscure. (And all, as his detractors noticed, quite safely deceased.) They thought it was just his way of trying to position himself squarely in the center, setting himself up like the Pope, casting his benedictions from one side of the field to the other ... They also ... resented the way he moved in and out of different disciplines, making his claims and conjectures and leaving the real work of proving them to others.

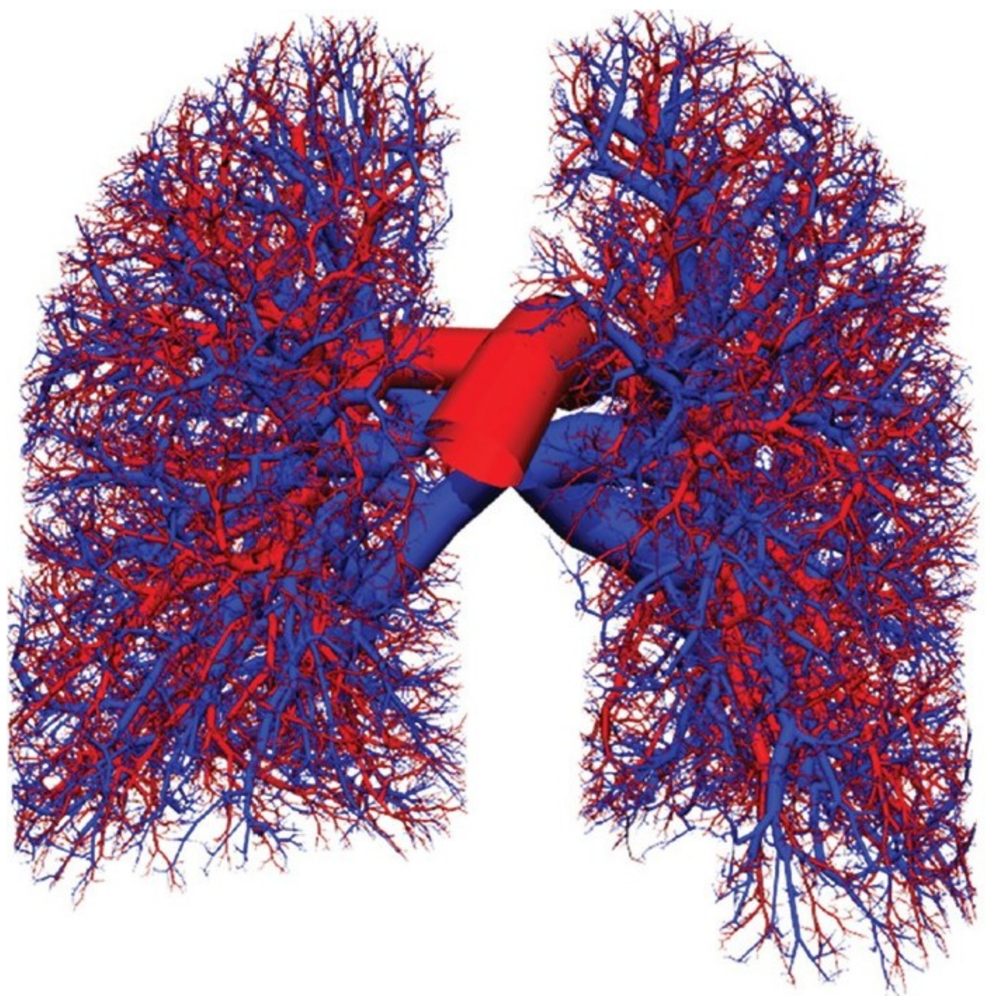
Quotes from Gleick

Mandelbrot's book was wide-ranging and stuffed with the minutiae of mathematical history. Wherever chaos led, Mandelbrot had some basis to claim that he had been there first. Little did it matter that most readers found his reference obscure or even useless. They had to acknowledge his extraordinary intuition for the direction of advances in fields he had never actually studied, from seismology to physiology. It was sometimes uncanny, sometimes irritating. Even an admirer would cry with exasperation, "Mandelbrot didn't have ***everybody's*** *thoughts before they did."*

Fractal dimension: a highly useful geometrical tool

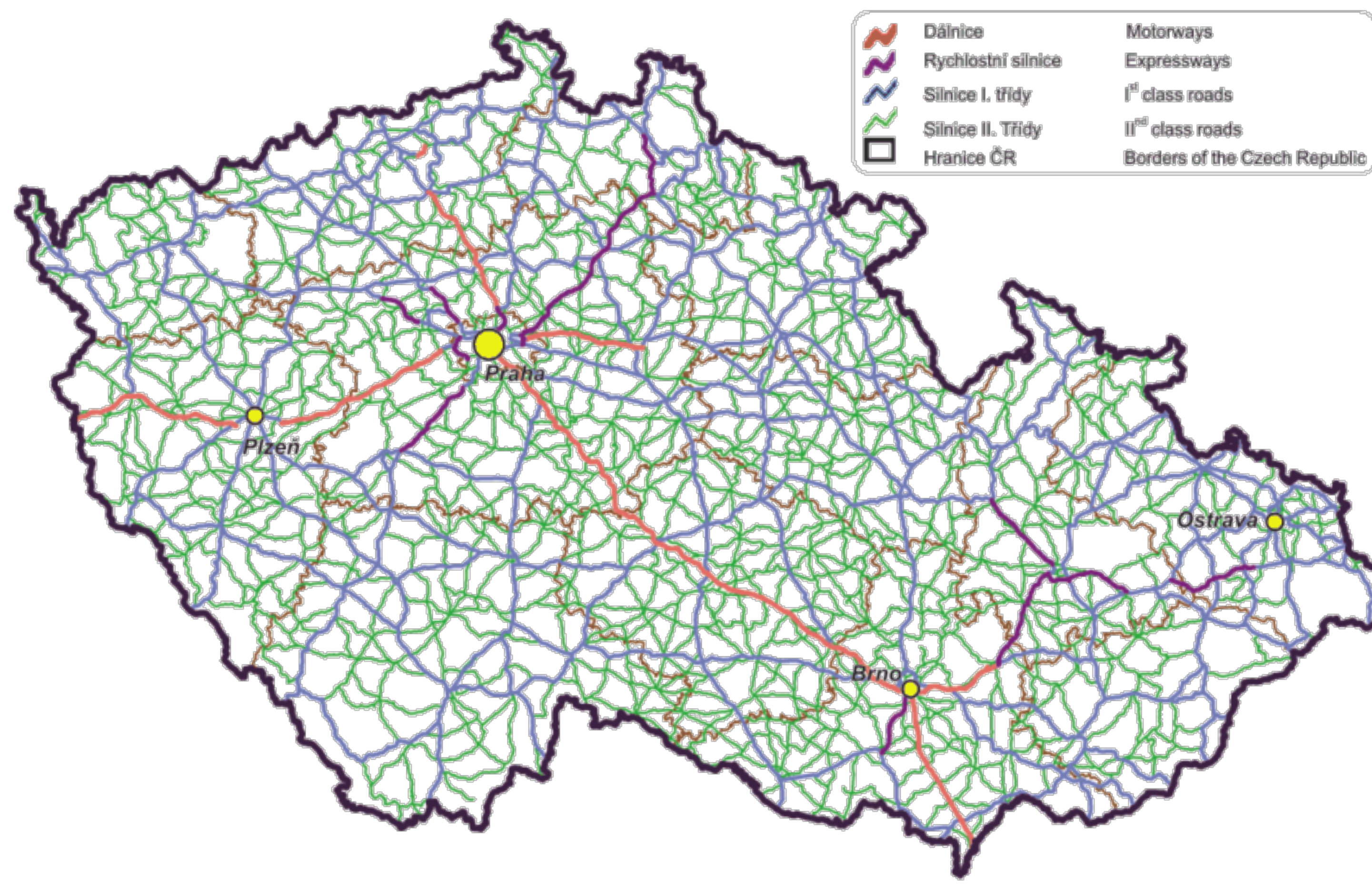
- Trees have complicated shapes but can be generated by very simple branching processes
- Fractal dimension d of the tree branches is in the range $2 < d < 3$
- More than a plane, but less than a cube
- Occupies large space without filling the space—simple and efficient biological design

Fractal networks



Blood vessel networks: branching pattern provides access to all volume, without occupying the volume

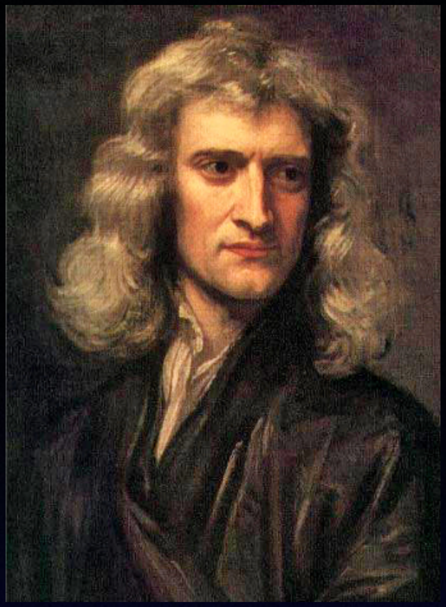
$2 < d < 3$



Road networks: typically no completely centralized planning, and follow simple rules connecting small roads to larger ones

$1 < d < 2$

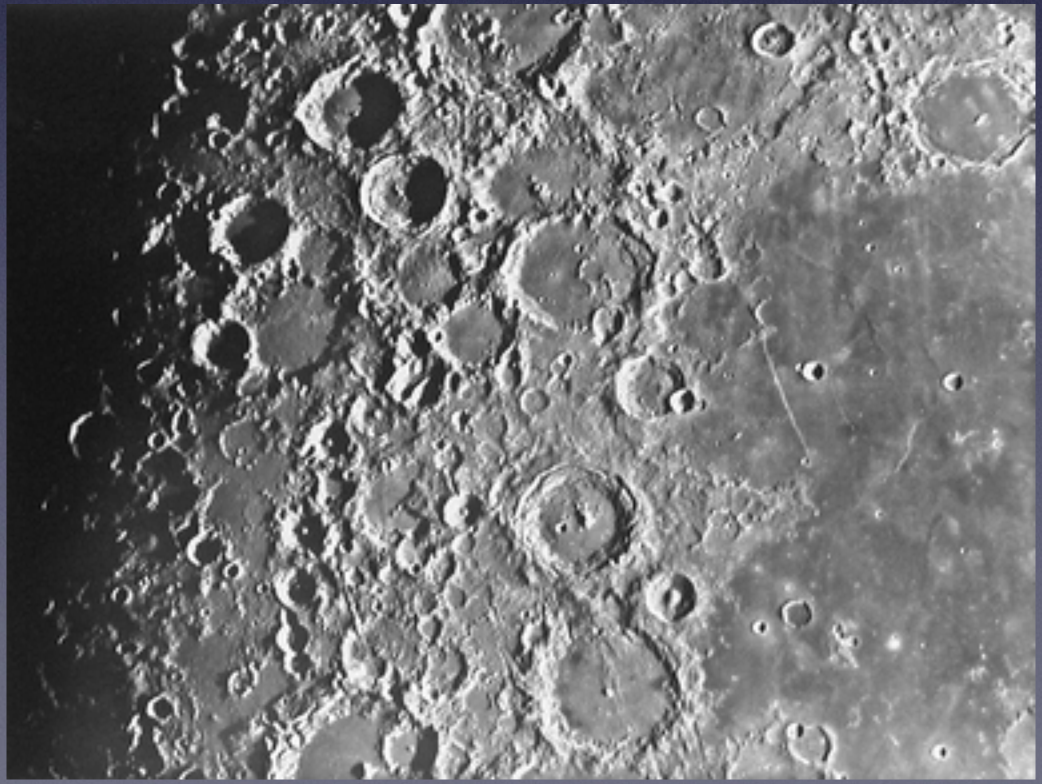
Why natural patterns are fractal

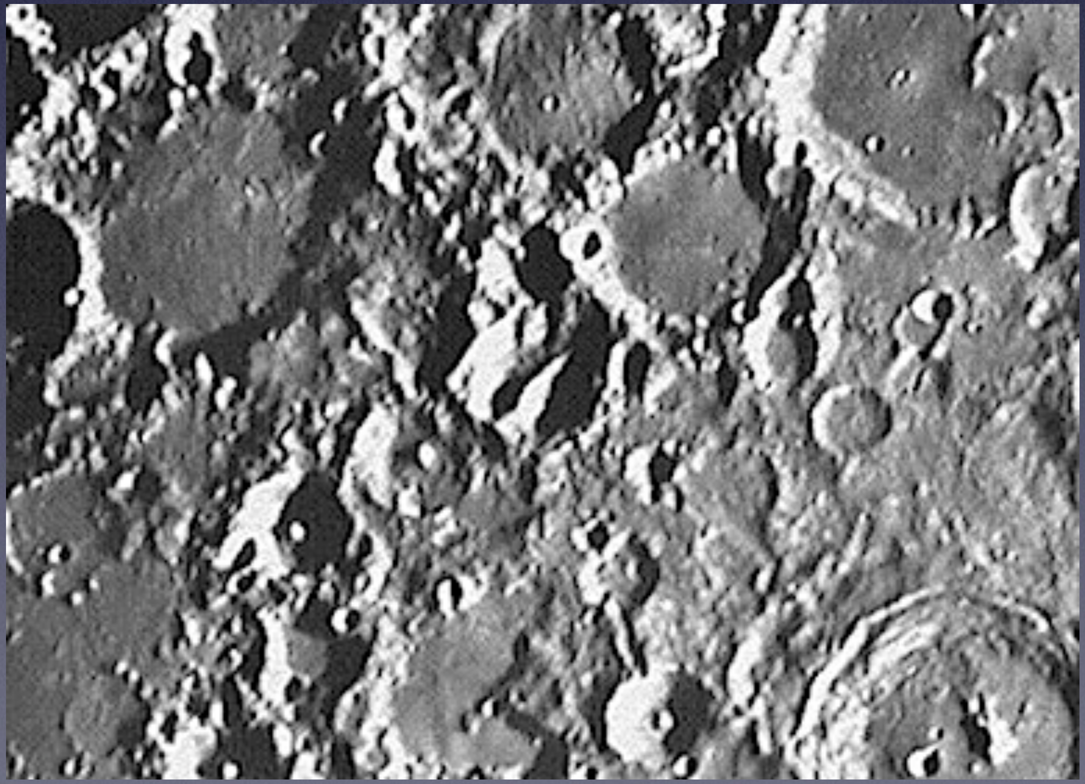


The laws of nature are independent of any physical scale

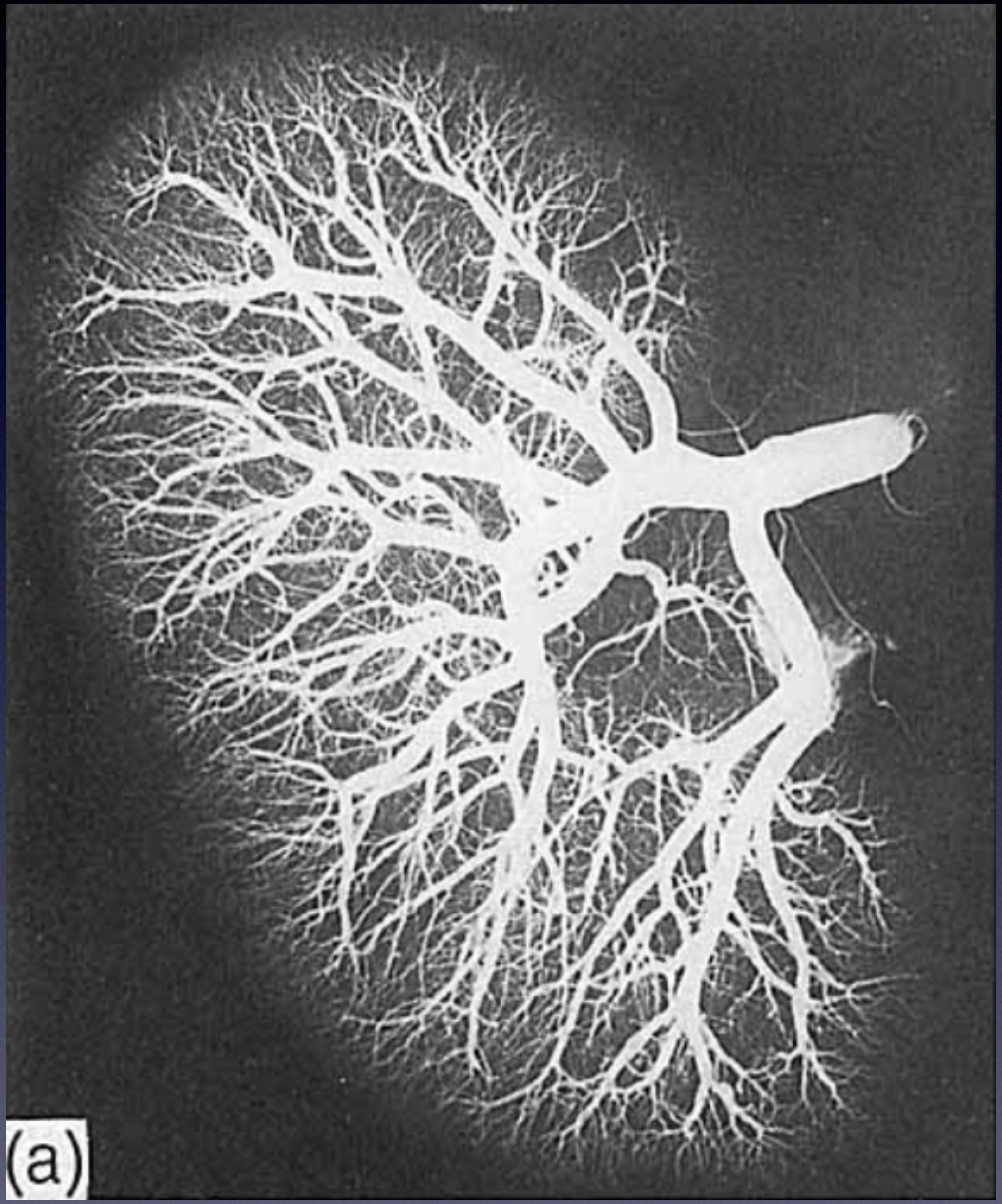
$$F = m \, a$$

Force | Mass | Acceleration

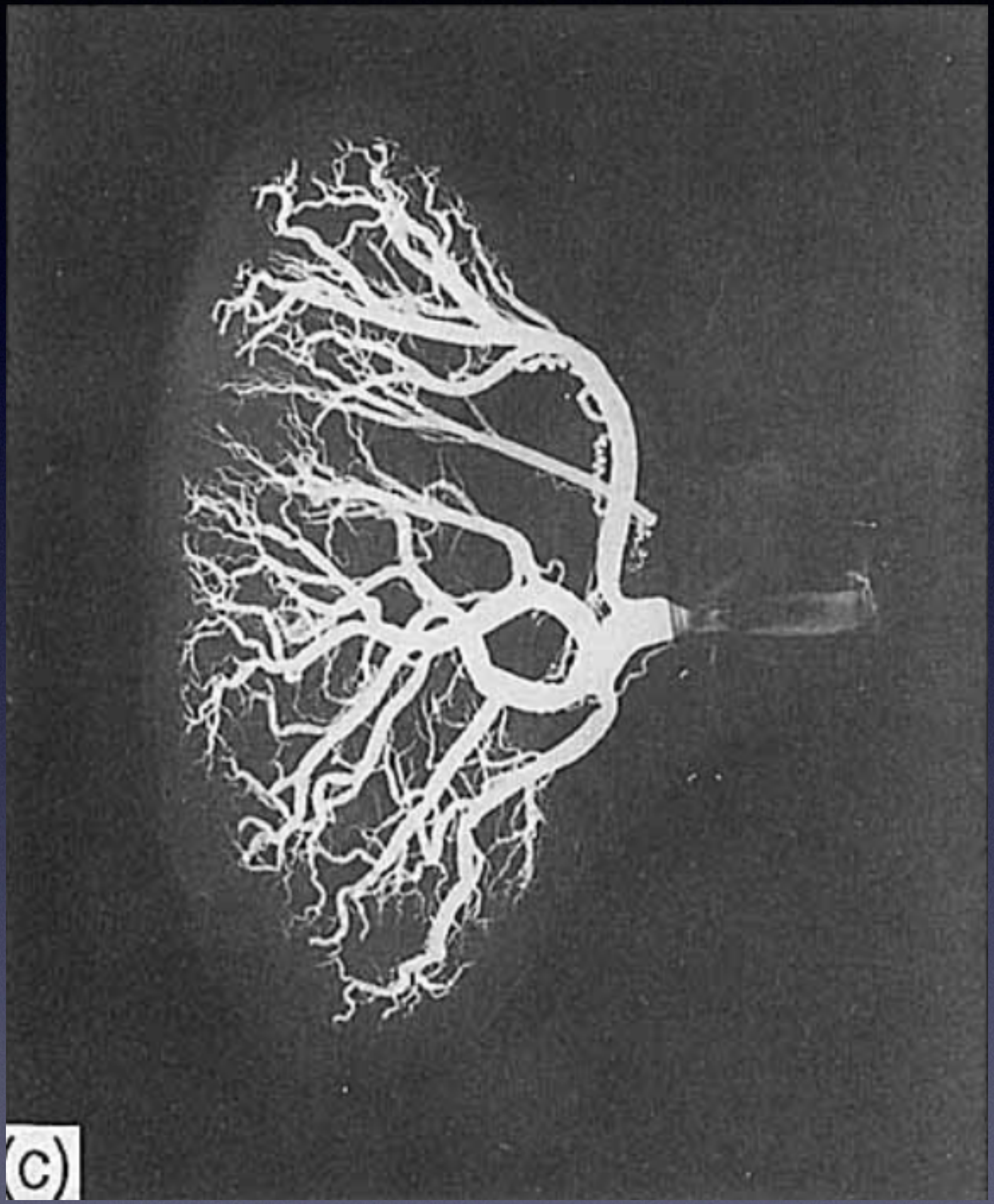




Practical applications



Normal kidney
Fractal dimension 1.60



Kidney with renal artery stenosis
Fractal dimension 1.50

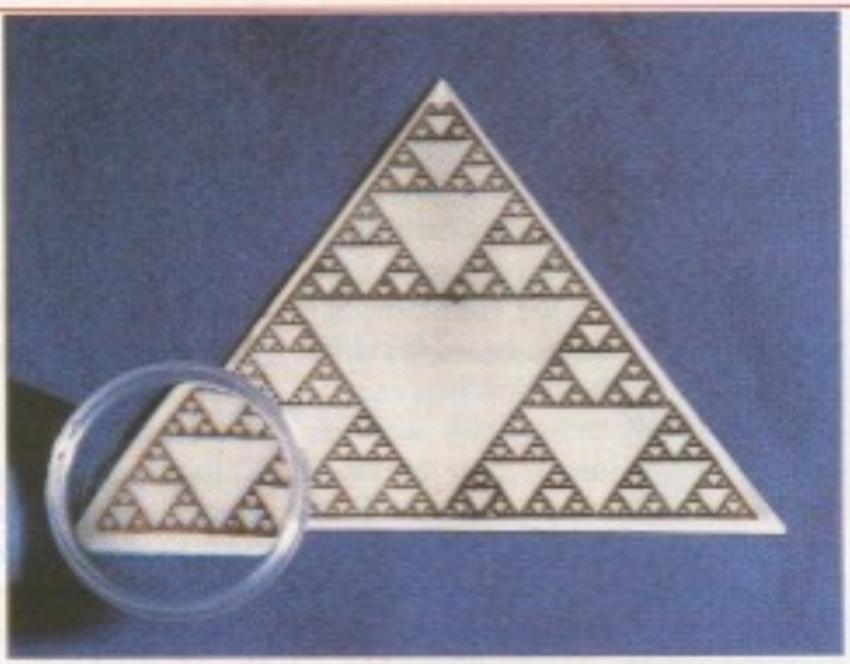
Simon S. Cross *et al.*, Journal of Pathology **170**, 479–484 (1993).

Practical Fractals

Fractals have become one of the unifying principles of science, but apart from computer graphics, technological applications of these geometric forms have been slow in coming. Over the past decade, however, researchers have begun applying fractals to a notoriously tricky subject: antenna design.

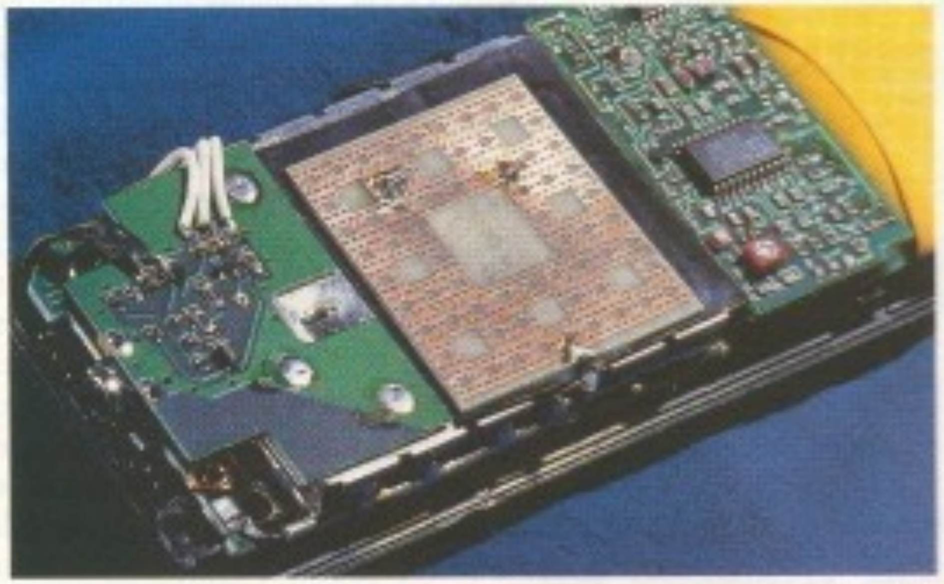
Antennas seem simple enough, but the theory behind them, based on Maxwell's equations of electromagnetism, is almost impenetrable. As a result, antenna engineers are reduced to trial and error—mostly the latter. Even the highest-tech receivers often depend on a scraggly wire no better than what Guglielmo Marconi used in the first radio a century ago.

Fractals help in two ways. First, they can improve the performance of antenna arrays. Many antennas that look like a single unit, including most radar antennas, are actually arrays of up to thousands of small antennas. Traditionally, the individual antennas are either randomly scattered or regularly spaced. But Dwight Jaggard of the University of Pennsylvania, Douglas Werner of Pennsylvania State University and others have discovered that a fractal arrangement can combine the robustness of a random array and the efficiency of a regular array—with a quarter of the number of elements. "Fractals bridge the gap," Jaggard says. "They have short-range disorder and long-range order."



FRACTAL TRIANGLE can act as a miniaturized antenna.

Second, even isolated antennas benefit from having a fractal shape. Nathan Cohen, a radio astronomer at Boston University, has experimented with wires bent into fractals known as Koch curves or fashioned into so-called Sierpinski triangles (*above*). Not only can crinkling an antenna pack the same length into a sixth of the area, but the jagged shape also generates electrical capacitance and inductance, thereby eliminating the need for external components to tune the antenna or broaden the range of frequencies to which it responds.



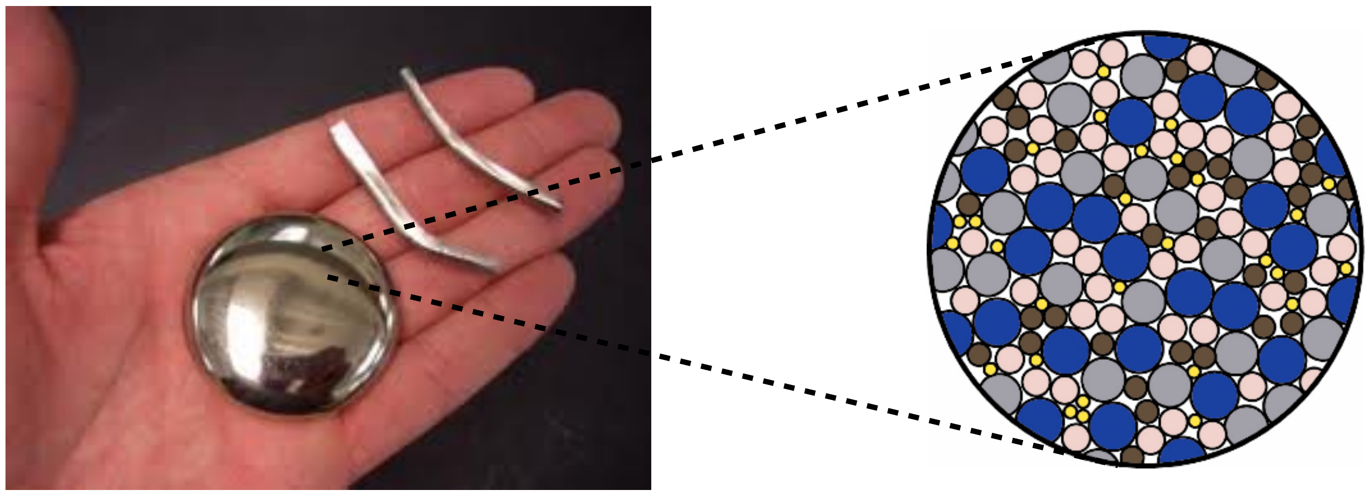
HIDDEN INSIDE a cordless phone, a square fractal antenna (center board) *replaces the usual rubbery stalk.*

Cohen, who founded Fractal Antenna Systems four years ago, is now working with T&M Antennas, which makes cellular phone antennas for Motorola. T&M engineer John Chenoweth says that the fractal antennas are 25 percent more efficient than the rubbery "stubby" found on most phones. In addition, they are cheaper to manufacture, operate on multiple bands—allowing, for example, a Global Positioning System receiver to be built into the phone—and can be tucked inside the phone body (*left*).

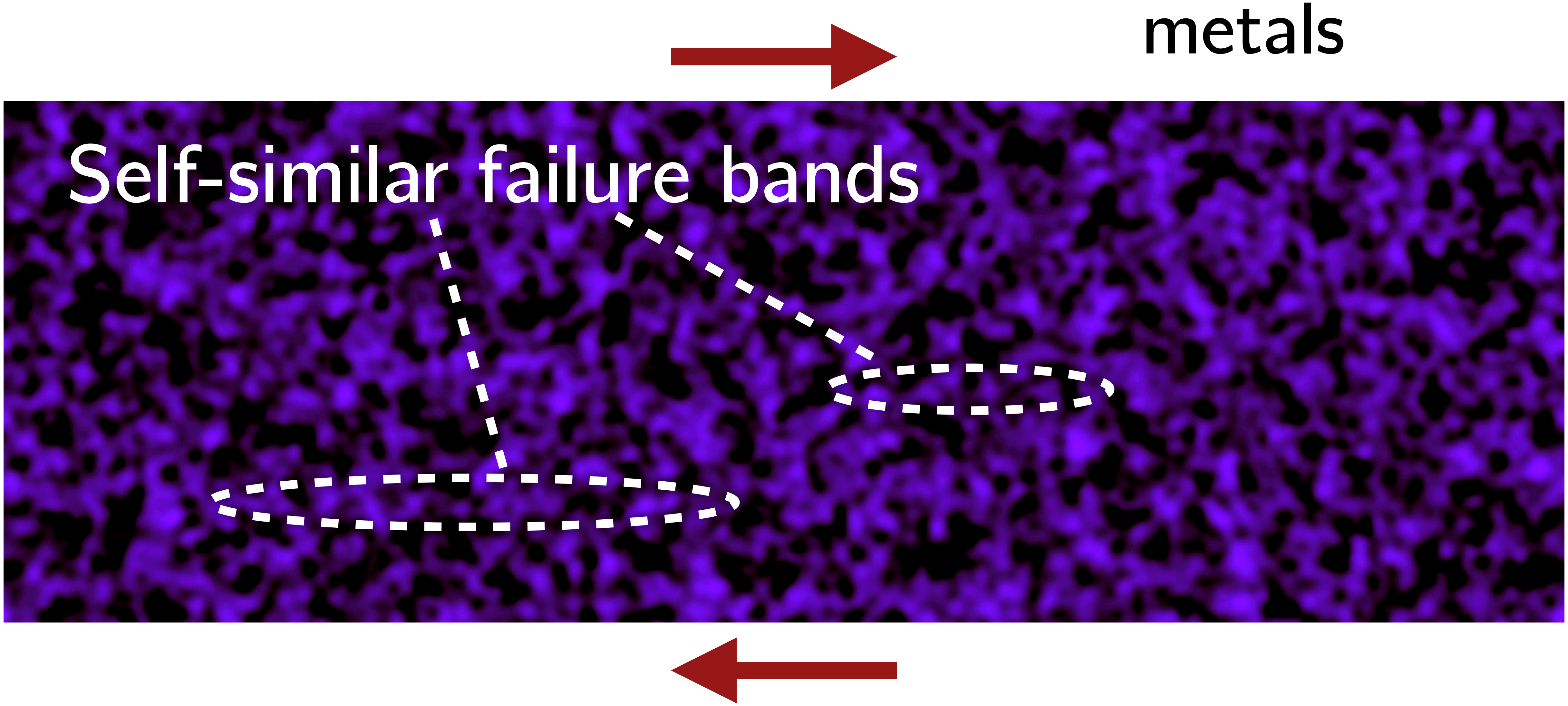
Just why these fractal antennas work so well was answered in part in the March issue of the journal *Fractals*. Cohen and his colleague Robert Hohlfeld proved mathematically that for an antenna to work equally well at all frequencies, it must satisfy two criteria. It must be symmetrical about a point. And it must be self-similar, having the same basic appearance at every scale—that is, it has to be fractal. —*George Musser*

PHOTOGRAPHS BY ANDY FREAL; ANTENNAS COURTESY OF FRACTAL ANTENNA SYSTEMS

Metallic glass modeling

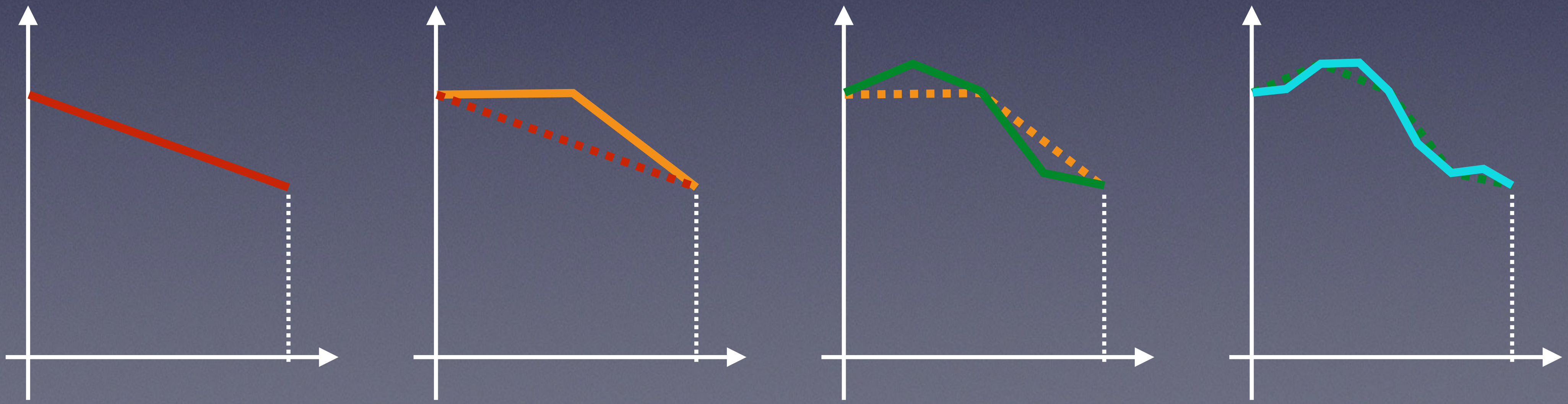


Alloys developed since the 1970's with a random atomic structure, unlike most metals



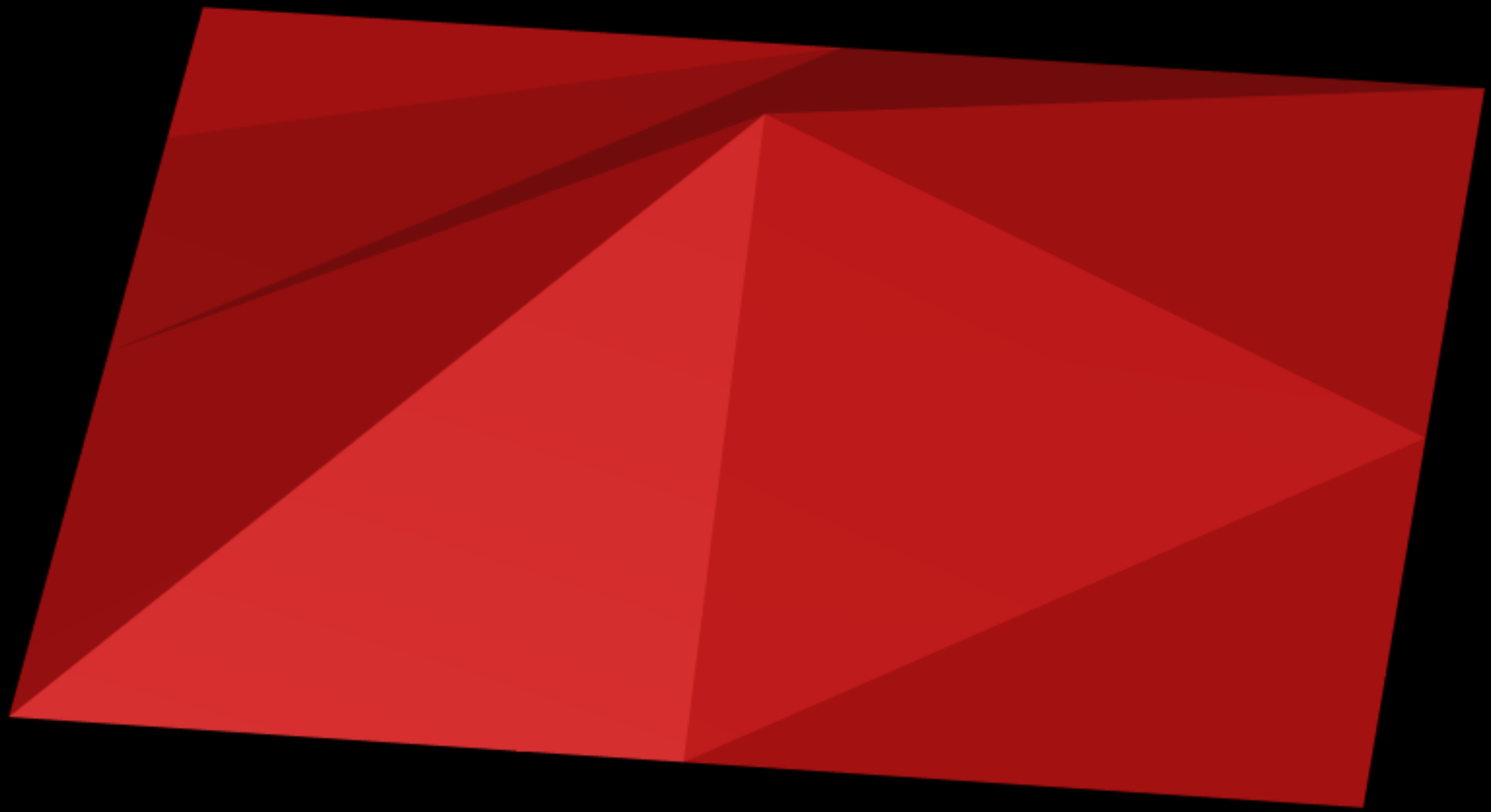
Fractal landscapes

- Loren Carpenter (*b.* 1947): a computer programmer who worked for Boeing
- Read *The Fractal Geometry of Nature* and devised algorithm to make fractal landscapes to put behind airplane images



Method: repeatedly subdivide and add random variations

Creating a landscape

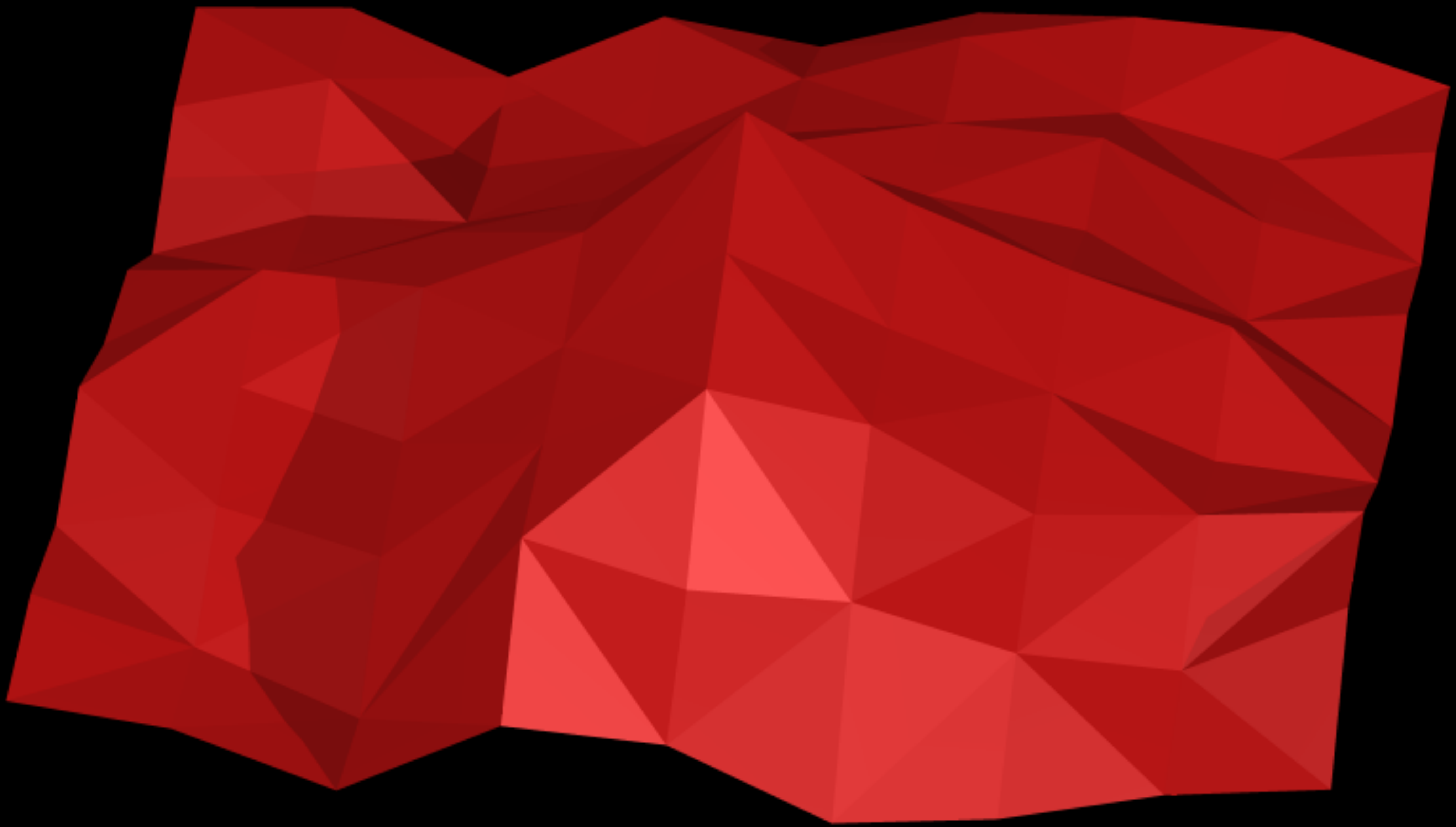


Start with 3 by 3 grid of points
connected with triangles

Creating a landscape

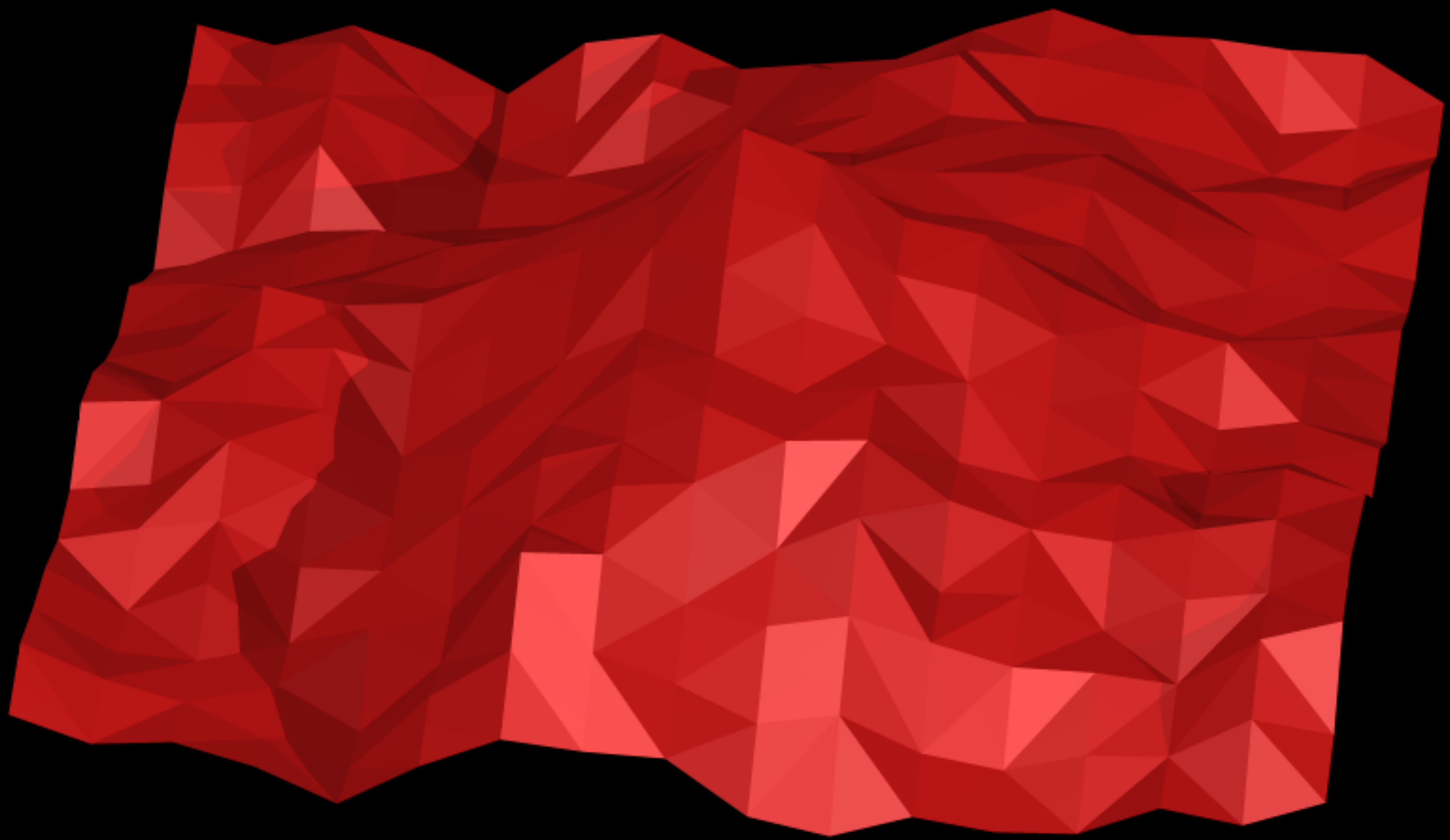
Subdivide into a 5 by 5 grid

Creating a landscape



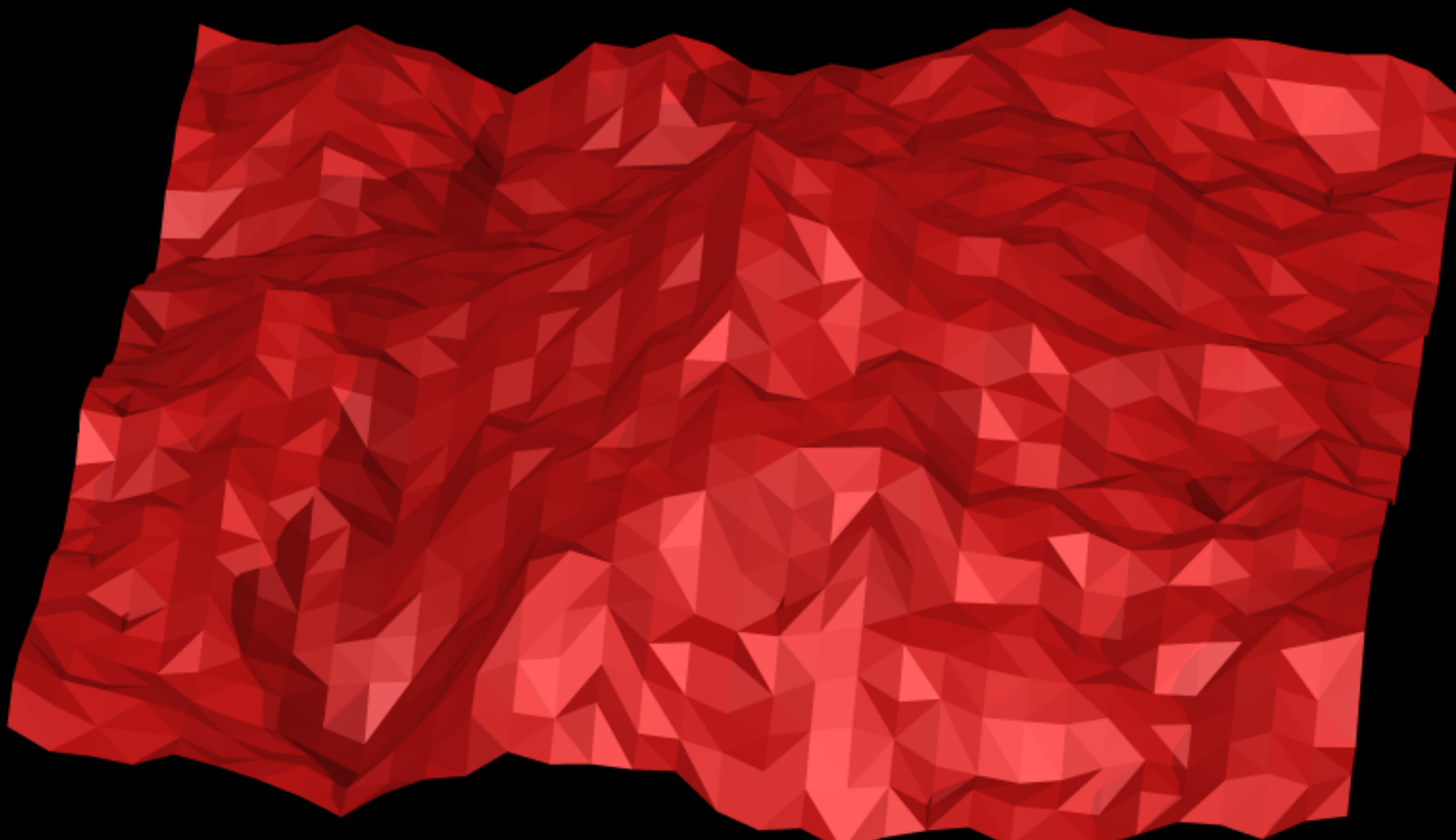
Subdivide into a 9 by 9 grid

Creating a landscape



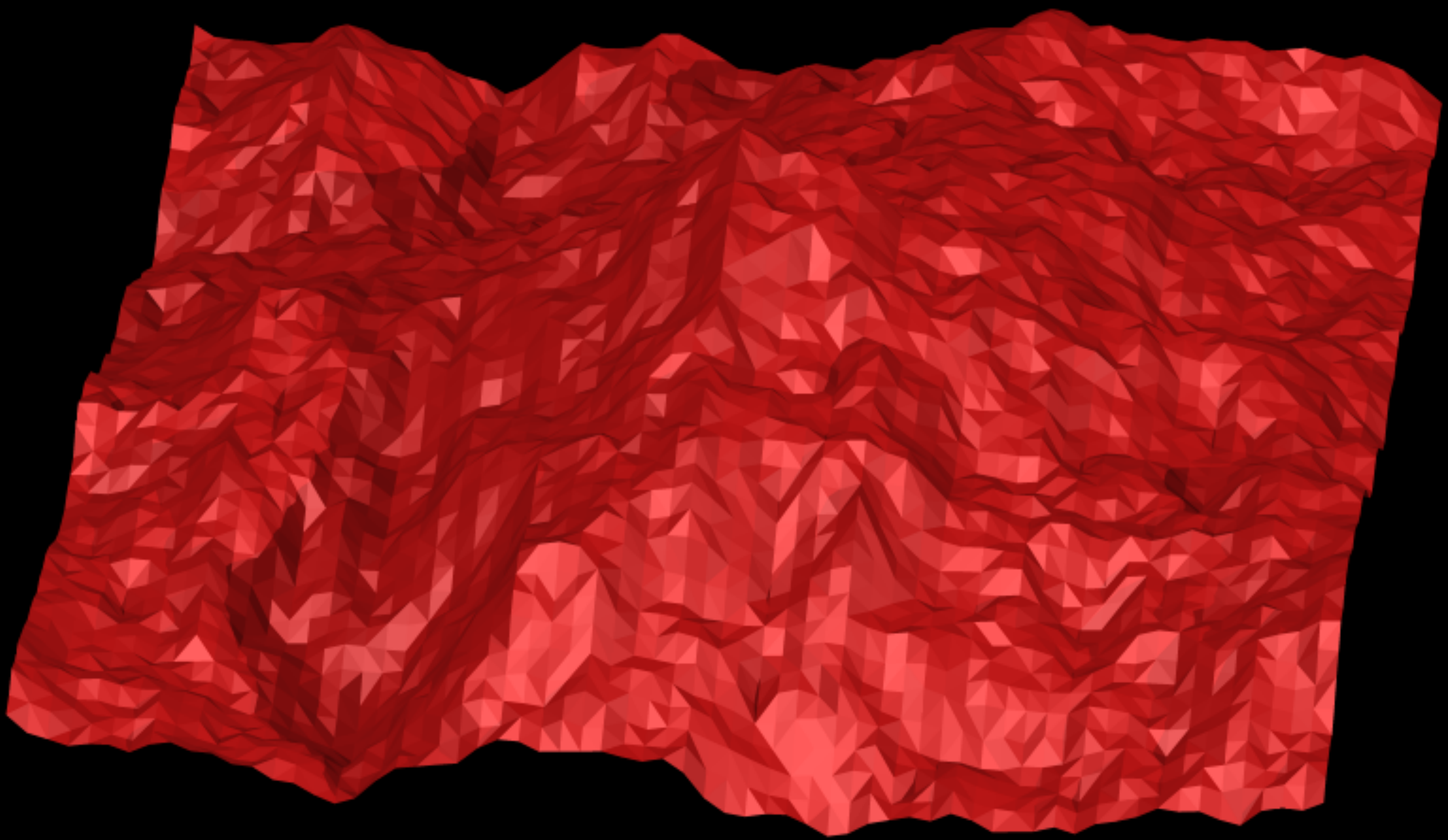
Subdivide into a 17 by 17 grid

Creating a landscape



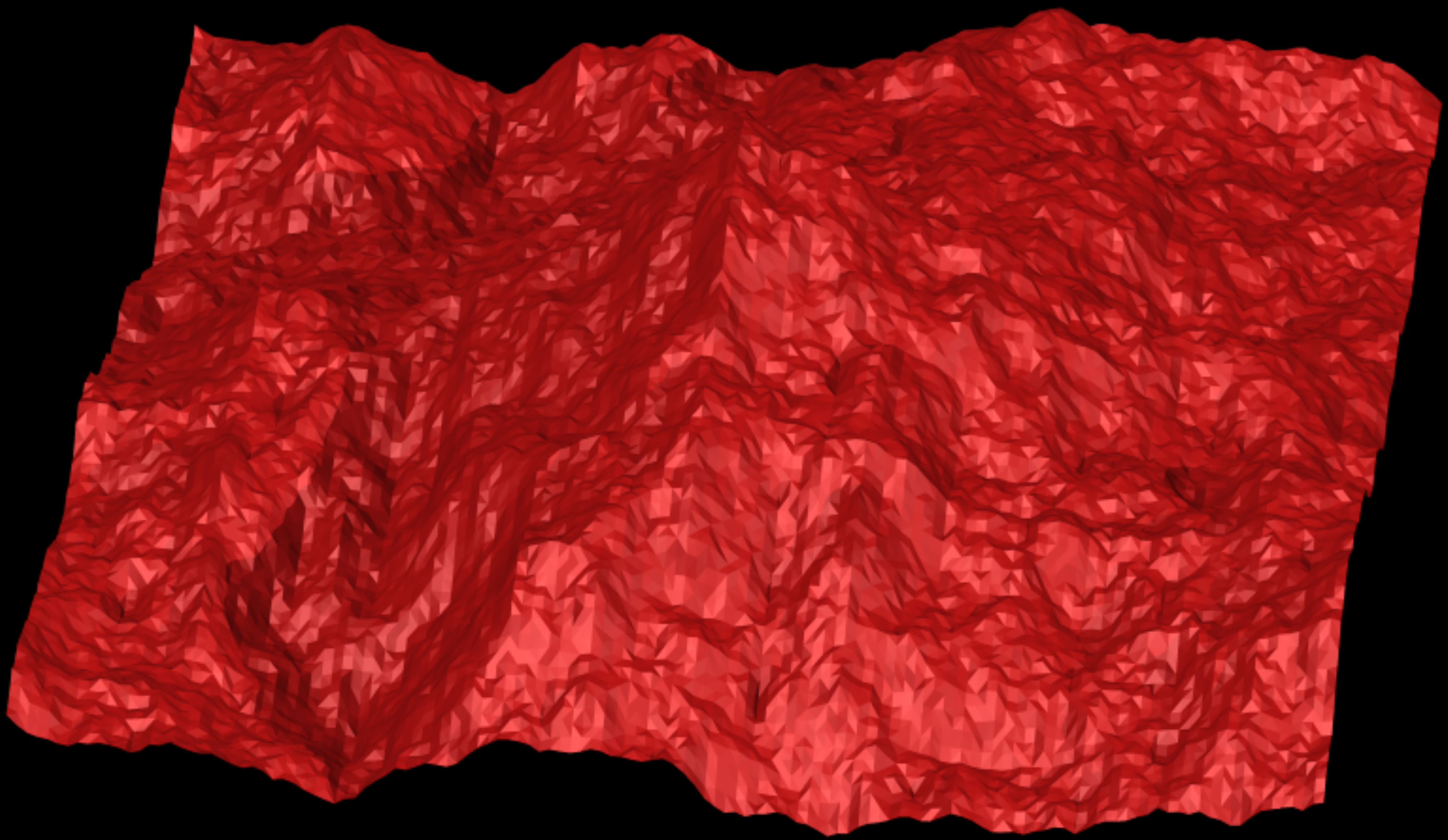
Subdivide into a 33 by 33 grid

Creating a landscape



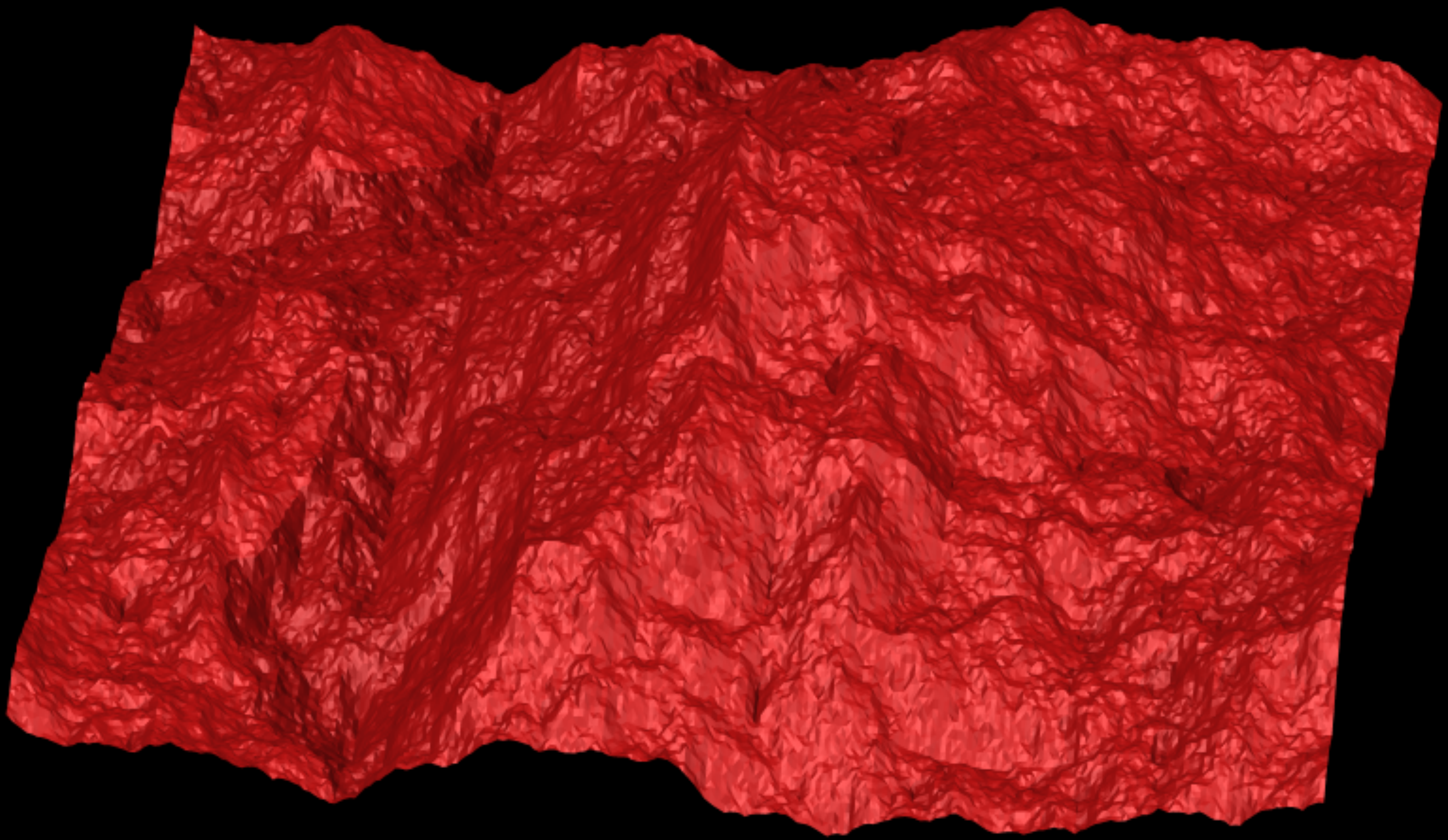
Subdivide into a 65 by 65 grid

Creating a landscape



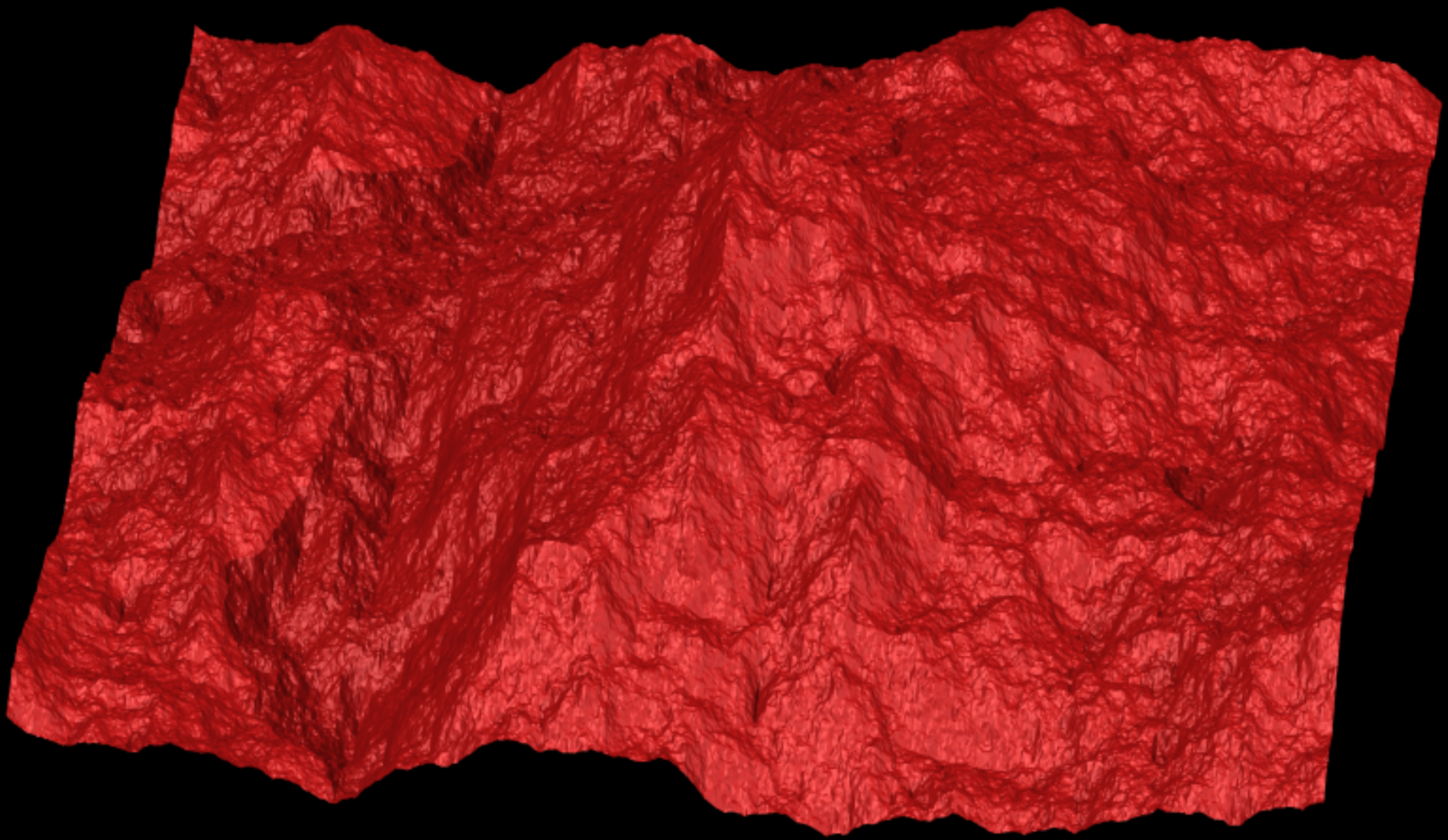
Subdivide into a 129 by 129 grid

Creating a landscape



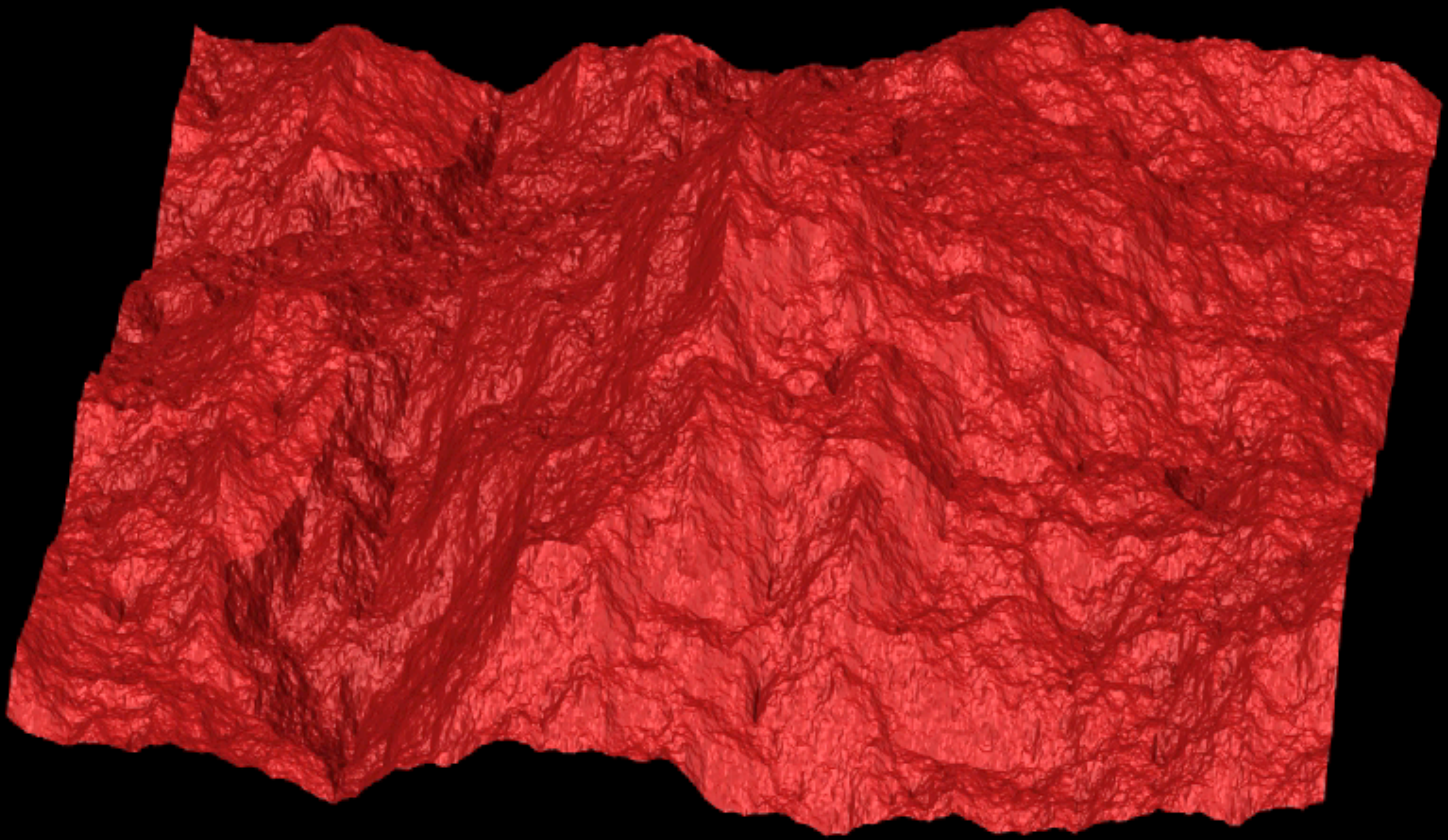
Subdivide into a 257 by 257 grid

Creating a landscape



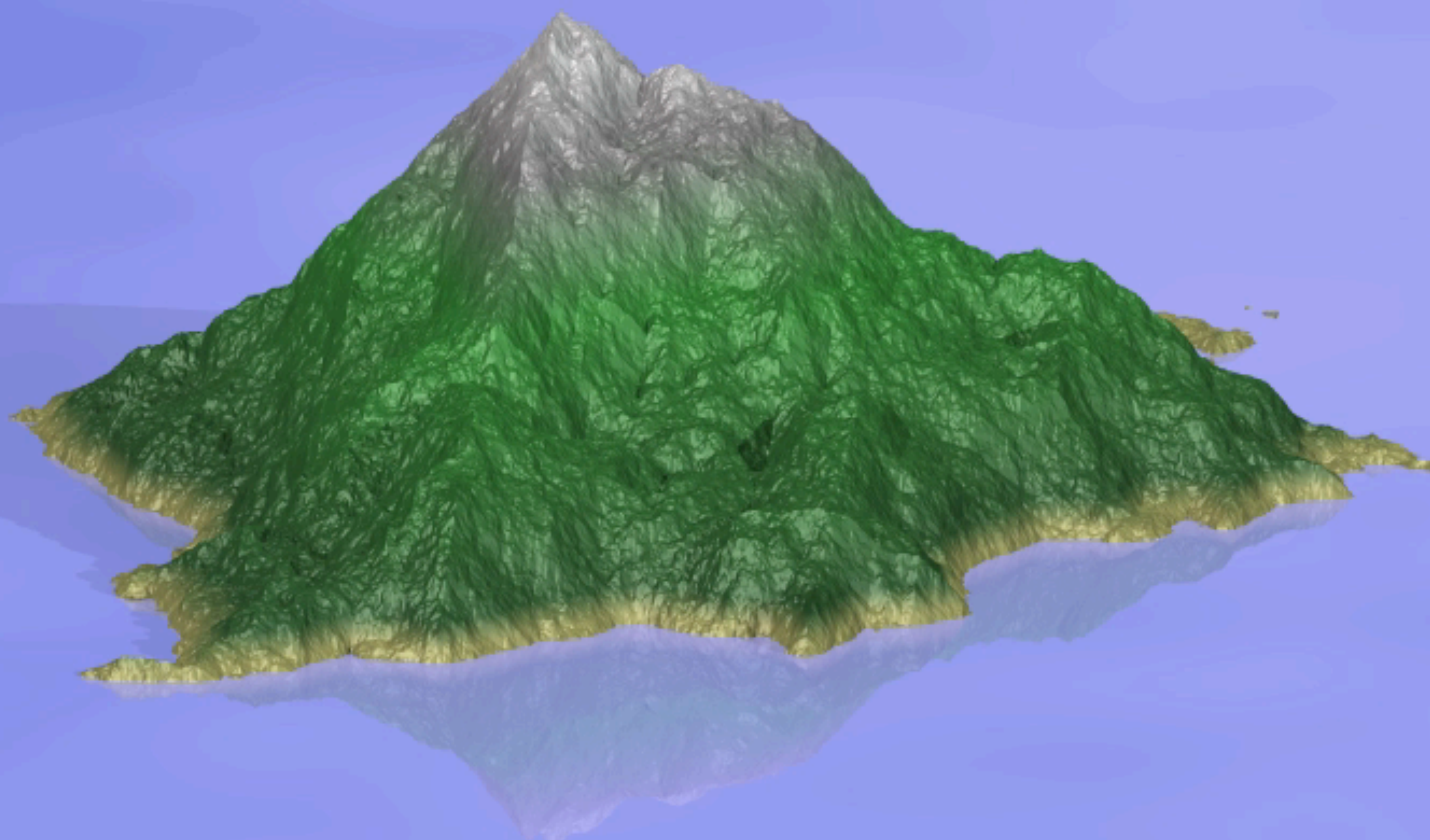
Subdivide into a 513 by 513 grid

Creating a landscape



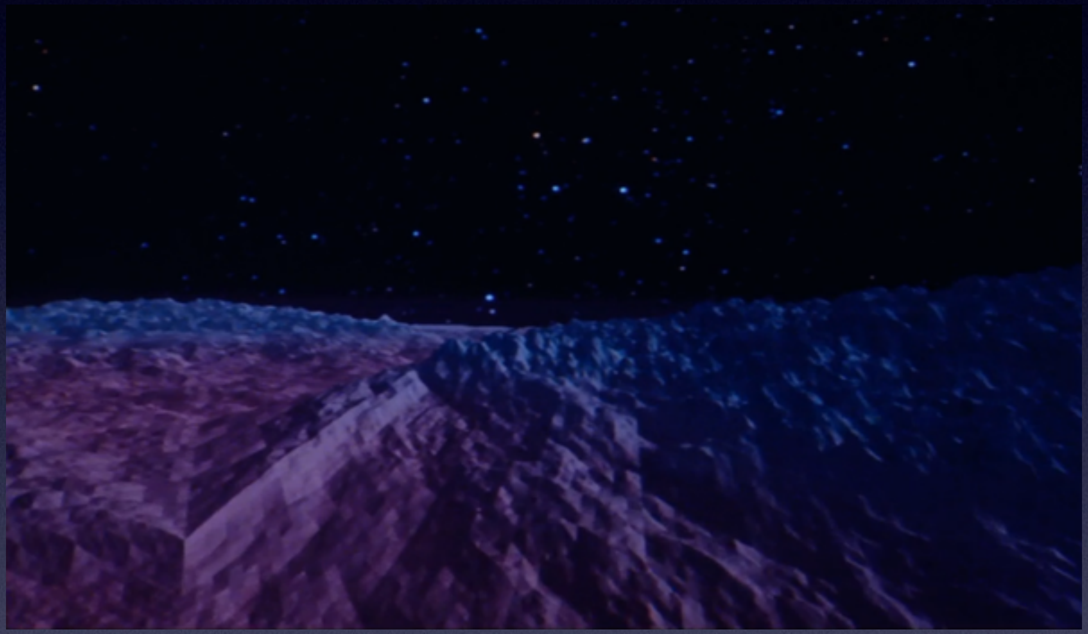
Creating a landscape

Creating a landscape



The Wrath of Khan

Carpenter on the make the first fully animated sequence in a movie of the “Genesis Device” in *Star Trek II: The Wrath of Khan* (1982)



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- Steven Strogatz, *Nonlinear Dynamics and Chaos*, Westview Press, 2001.

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Slides will be posted at http://seas.harvard.edu/~chr/present/amherst_fractals.pdf