Applied Mathematics 225

Unit 4: Finite volume methods

Lecturer: Chris H. Rycroft

Hyperbolic problems

Hyperbolic PDEs appear in a wide variety of situations where wave propagation or advective transport is important.

Many conservation laws (*e.g.* of mass, momentum) are expressed as hyperbolic PDEs.

Finite volume methods are class of discretization methods for hyperbolic PDEs that are based on an integral formulation.

They are often closely related to finite difference methods. However their integral formulation provides many advantages.

Book

We will make use of the following book:

- Randall J. LeVeque, *Finite volume methods for hyperbolic problems*, Cambridge University Press, 2002.

The book has associated code called CLAWPACK (Conservation LAWs PACKage), which is available at `http://www.clawpack.org/`. The code is written in Fortran, with a Python interface.

Hyperbolic conservation laws

We begin by studying a model conservation law. Consider a one dimensional pipe carrying a fluid with velocity $u(x,t)$.

Let $q(x,t)$ be the density of a chemical tracer being carried by the fluid. q has units of mass per unit length.

For a given interval $[x_1, x_2]$, the total amount of chemical in the interval at time t is

$$\int_{x_1}^{x_2} q(x,t)dx.$$

Changes in chemical over time

We now consider how the amount of chemical in the region $[x_1, x_2]$ changes over time. Let $F_i(t)$ be the rate at which the tracer flows past x_i.[1]

Since the total chemical in $[x_1, x_2]$ only changes at the endpoints, we obtain

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = F_1(t) - F_2(t),$$

which is the integral form of the conservation law.

[1] We use the convention that $F_i(x) > 0$ corresponds to tracer flowing to the right.

Flux

For the fluid flow example, the flux is given by the product of density and velocity, so that

$$\text{flux at } (x, t) = u(x_i, t)q(x_i, t)$$

Since u is a known function, we write the flux as $f(q, x, t)$. A particular case that we will study is when the velocity is constant, *i.e.* $u(x, t) = \bar{u}$, and hence $f(q) = \bar{u}q$. This is an example where the flux does not depend on x or t and the equation is called autonomous.

Toward a conservation law PDE

For an autonomous flux, the conservation law becomes

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = f(q(x_1,t)) - f(q(x_2,t)),$$

which can be rewritten as

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = -f(q(x,t))\big|_{x_1}^{x_2}.$$

The right hand side looks similar to the result of an integral. Assuming q is sufficiently smooth,

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(q(x,t))dx.$$

Toward a conservation law PDE

The previous expression can be rewritten as

$$\int_{x_1}^{x_2} \left[\frac{\partial}{\partial t} q(x,t) + \frac{\partial}{\partial x} f(q(x,t)) \right] dx = 0.$$

Since this is true for any x_1 and x_2, it follows that

$$\frac{\partial}{\partial t} q(x,t) + \frac{\partial}{\partial x} f(q(x,t)) = 0,$$

which, using subscript notation for derivatives, becomes

$$q_t + f(q)_x = 0.$$

The advection equation

Let us return now to the case of steady flow, $u(x,t) = \bar{u}$. Then the conservation law becomes

$$q_t + \bar{u} q_x = 0.$$

One can verify that the general solution to this equation is

$$q(x,t) = \tilde{q}(x - \bar{u}t).$$

Hence, if we specify the solution at time $t = 0$ is $\tilde{q}(x)$, then the solution at later times will just translate to the right with constant velocity.

Another viewpoint

Consider the ray $X(t) = x_0 + \bar{u}t$. The solution along this ray has derivative

$$\begin{aligned}\frac{d}{dt} &= q_t(X(t), t) + X'(t) q_x(X(t), t) \\ &= q_t + \bar{u} q_x = 0\end{aligned}$$

and thus q is constant along this ray.

These rays are called characteristics. In general, a characteristic is a curve along which the PDE simplifies in some way.

A more general equation

Suppose now that we consider a spatially dependent velocity $u(x)$. The PDE is

$$q_t + (u(x)q)_x = 0.$$

Define characteristics by $X'(t) = u(X(t))$, so they move with the velocity. Along a characteristic

$$\begin{aligned}
\frac{d}{dt} q(X(t), t) &= q_t(X(t), t) + X'(t) q_x(X(t), t) \\
&= q_t + u(X(t)) q_x \\
&= q_t + (u(X(t))q)_x - u'(X(t))q \\
&= -u'(X(t)) q(X(t), t).
\end{aligned}$$

While q is no longer constant along a characteristic, we obtain an ODE for q along the characteristic.

Solving the PDE reduces to solving a family of ODEs along characteristics.

Finite volume method

A finite volume method is based on subdividing the spatial domain into intervals (known as the finite volumes or grid cells) and keeping track of an approximation to the integral of q over each of these volumes.

Denote the ith grid cell by

$$\mathcal{C}_i = (x_{i-1/2}, x_{i+1/2})$$

The value Q_i^n approximates the average value over the ith interval at time t_n:

$$Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n) dx = \frac{1}{\Delta x} \int_{\mathcal{C}_i} q(x, t_n) dx$$

where $\Delta x = x_{i+1/2} - x_{i-1/2}$. If $q(x, t)$ is a smooth function, then the integral agrees to $O(\Delta x)^2$.

Finite volume method

Taking the time derivative of the integral of q in $\mathcal{C}_i$ yields

$$\frac{d}{dt}\int_{\mathcal{C}_i} q(x,t)dx = f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t)).$$

We aim to develop an explicit timestepping algorithm. Integrating from time t_n to time t_{n+1} yields

$$\int_{\mathcal{C}_i} q(x,t_{n+1})dx - \int_{\mathcal{C}_i} q(x,t_n)dx = \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))dt - \int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))dt.$$

This gives an exact formula for updating the cell average of q:

$$\frac{1}{\Delta x}\int_{\mathcal{C}_i} q(x,t_{n+1})dx = \frac{1}{\Delta x}\int_{\mathcal{C}_i} q(x,t_n)dx - \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))dt - \int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))dt\right].$$

Finite volume method

In general, the time integrals on the right hand side cannot be evaluated exactly. But it does suggest that we should study numerical methods of the form

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

where $F_{i-1/2}^n$ is some approximate to the average flux along $x = x_{i-1/2}$,

$$F_{i-1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2}, t))dt.$$

If we can approximate this average flux based on the values of Q^n, then we will obtain a fully discrete method.

Numerical flux function

In a hyperbolic problem information propagates at a finite speed, so it is reasonable to suppose that $F^n_{i-1/2}$ only depends on Q^n_{i-1} and Q^n_i. Then we could use

$$F^n_{i-1/2} = \mathcal{F}(Q^n_{i-1}, Q^n_i)$$

where $\mathcal{F}$ is a numerical flux function. Then the method becomes

$$Q^{n+1}_i = Q^n_i - \frac{\Delta t}{\Delta x}\left(\mathcal{F}(Q^n_i, Q^n_{i+1}) - \mathcal{F}(Q^n_{i-1}, Q^n_i)\right).$$

Numerical conservation

This solution method is said to be in conservation form since it mimics the conservation property of the mathematical equation.

Summing the values of $\Delta x\, Q_i^{n+1}$ over any set of cells gives

$$\Delta x \sum_{i=I}^{J} Q_i^{n+1} = \Delta x \sum_{i=I}^{J} Q_i^{n} - \Delta t \left(F_{J+1/2}^{n} - F_{I+1/2}^{n} \right)$$

and so the only change in this sum are due to fluxes in and out at the extreme edges.

Example: diffusion

While the derivation above assumes that the flux $f(q)$ only depends on q, it can more generally depend on derivatives of q. The diffusion equation has a flux

$$f(q_x, t) = -\beta(x) q_x.$$

Given two cell averages Q_{i-1} and Q_i, a natural definition for the numerical flux at the cell interface is

$$\mathcal{F}(Q_{i-1}, Q_i) = -\beta_{i-1/2} \left(\frac{Q_i - Q_{i-1}}{\Delta x} \right)$$

where $\beta_{i-1/2} \approx \beta(x_{i-1/2})$.

Example: diffusion

Using this numerical flux yields

$$Q_i^{n+1} = Q_i^n + \frac{\Delta t}{\Delta x^2} \left(\beta_{i+1/2}(Q_{i+1}^n - Q_i^n) - \beta_{i-1/2}(Q_i^n - Q_{i-1}^n) \right). \tag{1}$$

For β constant this becomes

$$Q_i^{n+1} = Q_i^n + \frac{\Delta t}{\Delta x^2}\beta(Q_{i-1}^n - 2Q_i^n + Q_{i-1}^n),$$

which matches an explicit finite difference scheme.

Example: diffusion

It is worth comparing this numerical scheme to a finite-difference formula. Expanding the x-derivative yields

$$q_t = [\beta(x) q_x]_x = \beta_x q_x + \beta q_{xx},$$

which could be implemented using finite differences as

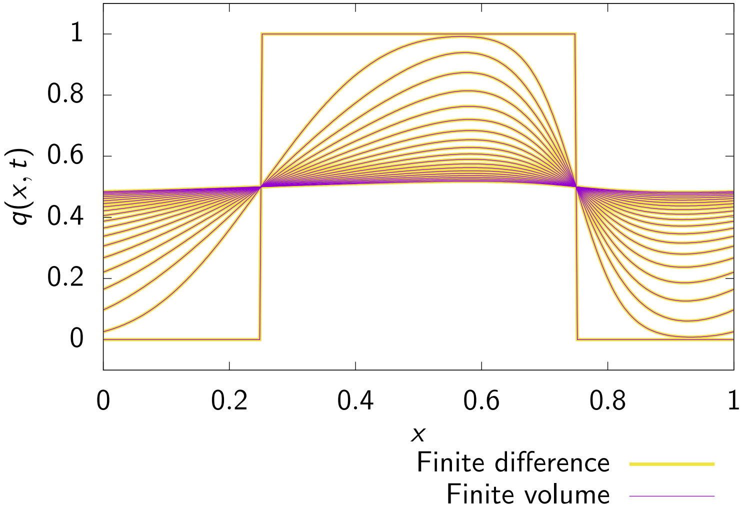
$$Q_i^{n+1} = Q_i^n + \frac{\beta_{i+1} - \beta_{i-1}}{2\Delta x} \frac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x} + \beta_i \frac{Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n}{\Delta x^2}. \tag{2}$$

The example code `4a_f_volume/d_solve.cc` will solve this diffusion equation using the two schemes in Eqs. 1 & 2. It uses $\beta(x) = 0.12 + 0.08 \sin 2\pi x$ and

$$q(x,0) = \begin{cases} 1 & \text{if } 1/4 < x < 3/4, \\ 0 & \text{otherwise.} \end{cases}$$

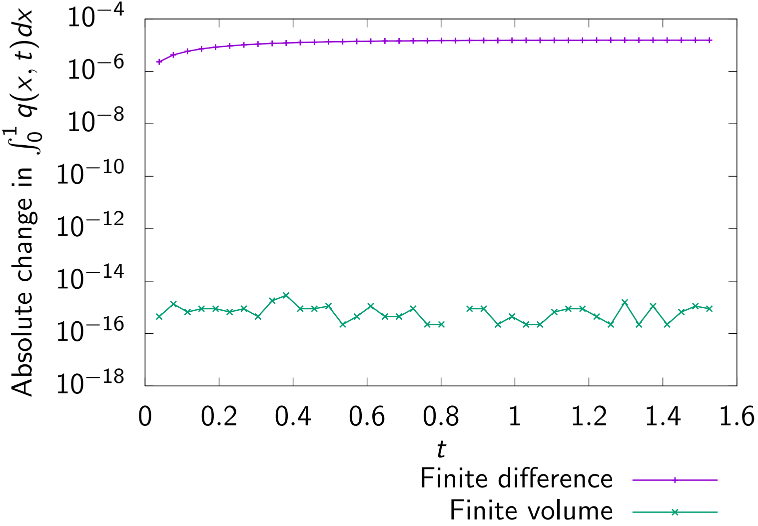
Method comparison

Showing the two approaches are near-identical . . .



Numerical conservation

. . . but the finite volume method numerically conserves q



Convergence

As for finite difference methods, we want to study numerical methods that converge to the true solution, so that when $\Delta x \to 0$ and $\Delta t \to 0$ the true solution is recovered.

Convergence generally requires two conditions

- The method must be consistent with the differential equation, so that it approximates it well locally.[2]
- The method must be stable in some appropriate sense, so that small errors made at each timestep do not grow too fast at later timesteps.

For more details, see Applied Math 205, Unit 3

[2] Truncation error tends to zero in the limit.

Courant–Friedrichs–Lewy (CFL) condition

The CFL condition is a necessary condition for stability:

> *A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as Δt and Δx go to zero.*

An unstable flux choice

Not all schemes that satisfy the CFL condition are stable. Consider

$$F_{i-1/2}^n = \mathcal{F}(Q_{i-1}^n, Q_i^n) = \frac{f(Q_{i-1}^n) + f(Q_i^n)}{2}.$$

This gives the update rule

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}\left(f(Q_{i+1}^n - f(Q_{i-1}^n)\right),$$

which is equivalent to a centered difference derivative of f, and is generally unstable.

The Lax–Friedrichs method

The Lax–Friedrichs (LxF) method has the form

$$Q_i^{n+1} = \frac{Q_{i-1}^n + Q_{i+1}^n}{2} - \frac{\Delta t}{2\Delta x}\left(f(Q_{i+1}^n) - f(Q_{i-1}^n)\right).$$

This is similar to the unstable method, except that Q_i^{n+1} is replaced with $(Q_{i-1}^n + Q_{i+1}^n)/2$. It can be written in conservative form with the choice of numerical flux

$$\mathcal{F}(Q_{i-1}^n, Q_i^n) = \frac{f(Q_{i-1}^n) + f(Q_i^n)}{2} - \frac{\Delta x}{2\Delta t}\left(Q_i^n - Q_{i-1}^n\right).$$

The Lax–Friedrichs method

The additional term looks like a diffusive flux. We could interpret this method as solving

$$q_t + f(q)_x = \beta q_{xx}$$

where $\beta = \frac{\Delta x^2}{2\Delta t}$. Note that if $\Delta x \to 0$ and $\Delta x/\Delta t$ is kept constant, then the diffusive term will vanish in the limit, recovering the original equation.

This term can be interpreted as numerical diffusion which is enough to stablize the method.

Upwind method

Consider the constant-coefficient advection equation $q_t + \bar{u} q_x = 0$, where $\bar{u} > 0$.

Consider $\mathcal{C}_i$. Since the left edge of the cell only contains characteristics originating from $\mathcal{C}_{i-1}$, a possible choice of numerical flux is

$$F^n_{i-1/2} = \bar{u} Q^n_{i-1}.$$

This leads to the first-order upwind method,

$$Q^{n+1}_i = Q^n_i - \frac{\bar{u}\Delta t}{\Delta x}(Q^n_i - Q^n_{i-1}).$$

This exactly matches a first-order finite-difference scheme.

Multiple components

So far, we have considered conservation laws with a single scalar field. However, many problems of interest involve vector fields with multiple components.

Let us return to the one-dimensional pipe flow model. Rather than consider the transport of small passive chemical tracer, let us instead look at the density of fluid $\rho(x, t)$ itself.

Liquids are usually well-approximated as incompressible, so ρ is constant. Let us instead consider a gas, which can undergo substantial compression.

Conservation of mass

Let the gas velocity be $u(x,t)$. Then conservation of mass gives

$$\rho_t + (\rho u)_x = 0.$$

This sometimes called the continuity equation.

Since both u and ρ are time-evolving, we require another equation.

Conservation of momentum

The momentum in an interval $[x_1, x_2]$ is

$$\int_{x_1}^{x_2} \rho(x,t)u(x,t)dx$$

Consider the flux of momentum at x_1. There are two terms:

- Macroscopic convective flux: this is the momentum carried along with the fluid, and has the form of ρu multiplied by the fluid velocity u, giving ρu^2.
- Microscopic momentum flux: the fluid transfers momentum due to pressure p.

Hence the change in momentum in an interval is

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)u(x,t)dx = -\left[\rho u^2 + p\right]_{x_1}^{x_2}.$$

Conservation of momentum

Assuming the fields are smooth, we arrive at the equation

$$(\rho u)_t + (\rho u^2 + p)_x = 0,$$

which expresses conservation of momentum.

We now have two equations, but we have introduced pressure p as an additional unknown. Under the assumption of constant entropy, we can propose an equation of state that p is a function of ρ only. A typical form is

$$p = P(\rho) = \tilde{\kappa}\rho^\gamma$$

for two constants $\tilde{\kappa}$ and γ. For air, $\gamma \approx 1.4$.

Coupled system

We therefore end up with the coupled nonlinear system

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x &= 0. \end{aligned}$$

This becomes

$$q_t + f(q)_x = 0$$

if $q = (a, b) = (\rho, \rho u)$ and

$$f(q) = \begin{pmatrix} \rho u \\ \rho u^2 + P(\rho) \end{pmatrix} = \begin{pmatrix} b \\ b^2/a + P(a) \end{pmatrix}.$$

A general multicomponent system would have $q : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ and a function $f : \mathbb{R}^m \to \mathbb{R}^m$.

Quasilinear form

Assuming q is smooth, the conservation law can be rewritten as

$$q_t + f'(q)q_x = 0,$$

which is referred to as the quasilinear form, since it resembles the linear case

$$q_t + Aq_x = 0.$$

In general, one can always obtain a linear system from a nonlinear equation by linearizing around some state.

Linearization of the gas dynamics example

Suppose we now look at small perturbations in velocity and density in the gas dynamics example. We write

$$q(x,t) = q_0 + \tilde{q}(x,t)$$

where q_0 is a background state and $\tilde{q}$ is a small perturbation. The linearized equation is

$$\tilde{q}_t + f'(q_0)\tilde{q}_x = 0$$

where

$$\begin{aligned} f'(q) &= \begin{pmatrix} \partial f^1/\partial a & \partial f^1/\partial b \\ \partial f^2/\partial a & \partial f^2/\partial b \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -b^2/a^2 + P'(a) & 2b/a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -u^2 + P'(\rho) & 2u \end{pmatrix}. \end{aligned}$$

Linear acoustics equations

After some algebraic manipulations, and expressing the system in terms of $\tilde{p}$[3] and $\tilde{u}$, we obtain the linear acoustics equations

$$\begin{aligned} \tilde{p}_t + u_0 \tilde{p}_x + K_0 \tilde{u}_x &= 0, \\ \rho_0 \tilde{u}_t + \tilde{p}_x + \rho_0 u_0 \tilde{u}_x &= 0 \end{aligned}$$

where $K_0 = \rho_0 P'(\rho_0)$. Since we now study this in its own right, we drop the tildes and write $q = (p, u)$. In matrix form

$$\begin{pmatrix} p \\ u \end{pmatrix}_t + \begin{pmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix}_x = 0,$$

or alternatively $q_t + Aq_x = 0$ in matrix form.

[3] As for velocity and density, we define $p = p_0 + \tilde{p}$ where $\tilde{p}$ is a small perturbation.

Sound wave solutions

Consider searching for traveling wave solutions

$$q(x,t) = \bar{q}(x - st)$$

for some velocity s. Substitutng into the general equation $q_t + Aq_x = 0$ yields

$$A\bar{q}'(x - st) = s\bar{q}'(x - st).$$

This is only possible if $\bar{q}$ is an eigenvector of A and s is an eigenvalue. For the given problem, we obtain eigenvalues[4]

$$\lambda^1 = u_0 - c_0, \qquad \lambda^2 = u_0 + c_0$$

where $c_0 = \sqrt{K_0/\rho_0} = \sqrt{P'(\rho_0)}$. We therefore obtain left- and right-moving waves of speed c_0 with respect to the background velocity u_0.

[4] Note that λ^p is the pth eigenvalue. The p does not mean a power here.

Linear acoustics: general solution

Write the corresponding eigenvectors as

$$r^1 = \begin{pmatrix} -\rho_0 c_0 \\ 1 \end{pmatrix}, \qquad r^2 = \begin{pmatrix} \rho_0 c_0 \\ 1 \end{pmatrix}.$$

Then when $u_0 = 0$, a general solution is

$$q(x,t) = \bar{w}^1(x + c_0 t) r^1 + \bar{w}^2(x - c_0 t) r^2$$

where $\bar{w}^1$ and $\bar{w}^2$ are scalar functions, which could be determined from given initial and/or boundary conditions.

Multiple components: general approach

Consider the linear hyperbolic system

$$q_t + Aq_x = 0$$

where $q(x, t)$ is an m-vector and $A \in \mathbb{R}^{m\times m}$. The problem is hyperbolic if A is diagonalizable with real eigenvalues, so

$$A = R\Lambda R^{-1}.$$

Introduce new variables $w = R^{-1}q$. Then

$$w_t + \Lambda w_x = 0,$$

which is a set of m decoupled advection equations.

Mathematical solution to the Cauchy problem

For the Cauchy problem, we are given initial data

$$q(x, 0) = \mathring{q}(x) \qquad \text{for } x \in \mathbb{R}.$$

We compute initial data

$$\mathring{w}(x) = R^{-1}\mathring{q}(x).$$

The pth component is

$$w_t^p + \lambda^p w_x^p = 0$$

with solution

$$w^p(x, t) = \mathring{w}^p(x - \lambda^p t),$$

and hence $q(x, t) = Rw(x, t)$.

Superposition of waves

Alternatively

$$q(x,t) = \sum_{p=1}^{m} w^p(x,t) r^p.$$

Let $\ell^1, \ldots, \ell^n$ be the left eigenvectors of the matrix A, *i.e.* the rows of the matrix $L = R^{-1}$. Then

$$q(x,t) = \sum_{p=1}^{m} \left[\ell^p \mathring{q}(x - \lambda^p t)\right] r^p.$$

Riemann problem

While classical solutions to differential equations can be smooth, the previous formula can be used even if the initial data $\mathring{q}$ is discontinuous.

The Riemann problem consists of special initial data with a single jump discontinuity,

$$\mathring{q}(x) = \begin{cases} q_l & \text{if } x < 0, \\ q_r & \text{if } x > 0. \end{cases}$$

Write

$$q_l = \sum_{p=1}^{m} w_l^p r^p, \qquad q_r = \sum_{p=1}^{m} w_r^p r^p.$$

Then

$$w^p(x,t) = \begin{cases} w_l^p & \text{if } x - \lambda^p t < 0, \\ w_r^p & \text{if } x - \lambda^p t > 0. \end{cases}$$

Riemann problem

If $P(x,t)$ is the maximum value of p for which $x - \lambda^p t > 0$, then

$$q(x,t) = \sum_{p=1}^{P(x,t)} w_r^p r^p + \sum_{p=P(x,t)+1}^{m} w_l^p r^p.$$

Across the pth characteristic, the solution q jumps by

$$(w_r^p - w_l^p) r^p = \alpha^p r^p.$$

Godunov's method for linear systems

The upwind method is a special case of the following reconstruct–evolve–average approach, also referred to as the REA algorithm:

1. Reconstruct a piecewise polynomial function $\tilde{q}(x, t_n)$ defined for all x, from the cell averages Q_i^n. In the simplest case this is a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \qquad \text{for all } x \in \mathcal{C}_i.$$

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt
3. Average this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1}) dx$$

Numerical flux function for Godunov's method

Instead of finding Q_i^{n+1} by directly computing the average of $\tilde{q}$ in $\mathcal{C}_i$, we can define a numerical flux function that is consistent with it. Recall that

$$F_{i-1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2}, t))dt.$$

We can compute this integral exactly when q is replaced by $\tilde{q}^n$ from the Riemann problem at time t_n.

The Riemann problem centered at $x_{i-1/2}$ has a similarity solution that is constant along rays $(x - x_{i-1/2})/(t - t_n) = \text{constant}$. Looking at the value along $(x - x_{i-1/2})/(t - t_n) = 0$ gives the value of $\tilde{q}^n(x_{i-1/2}, t)$.

Numerical flux function for Godunov's method

Denote this by $Q^\vee_{i-1/2} = q^\vee(Q^n_{i-1}, Q^n_i)$. Then we define the numerical flux function as

$$F^n_{i-1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q^\vee(Q^n_{i-1}, Q^n_i))dt = f(q^\vee(Q^n_{i-1}, Q^n_i)).$$

Thus Godunov's method can be expressed as

- Solve the Riemann problem at $x_{i-1/2}$ to find $q^\vee(Q^n_{i-1}, Q^n_i)$.
- Define the numerical flux $F^n_{i-1/2} = \mathcal{F}(Q^n_{i-1}, Q^n_i)$.
- Apply the flux-differencing formula.

Wave propagation form of Godunov's method

It is useful to examine Godunov's method applied to the linear advection equation. Consider for $m = 3$ components with $\lambda^1 < \lambda^2 < 0 < \lambda^3$.

Consider a step from t_n to t_{n+1} using a piecewise constant reconstruction. The function $\tilde{q}(x, t_{n+1})$ will usually have three discontinuities in cell $\mathcal{C}_i$:

- At $x_{i-1/2} + \lambda^3 \Delta t$, propagating from the left edge,
- At $x_{i+1/2} - \lambda^1 \Delta t$ and $x_{x+1/2} - \lambda^2 \Delta t$, propagating from the right edge.

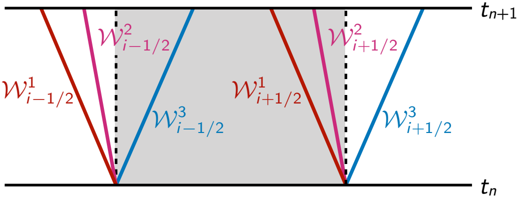
From the Riemann problem solution, we know that

$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha_{i-1/2}^p r^p = \sum_{p=1}^{m} \mathcal{W}_{i-1/2}^p,$$

where $\mathcal{W}_{i-1/2}^p = \alpha_{i-1/2}^p r^p$ is the change in solution due passing the discontinuity with velocity λ^p.

Wave propagation form of Godunov's method

We obtain the following picture



The effect on the cell average in $\mathcal{C}_i$ from wave 3 is

$$-\frac{\lambda^3 \Delta t}{\Delta x}\mathcal{W}^3_{i-1/2},$$

and considering all waves gives the Godunov update

$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \frac{\lambda^3 \Delta t}{\Delta x}\mathcal{W}^3_{i-1/2} - \frac{\lambda^1 \Delta t}{\Delta x}\mathcal{W}^1_{i+1/2} - \frac{\lambda^2 \Delta t}{\Delta x}\mathcal{W}^2_{i+1/2} \\ &= Q_i^n - \frac{\Delta t}{\Delta x}\left(\lambda^3\mathcal{W}^3_{i-1/2} + \lambda^1\mathcal{W}^1_{i+1/2} + \lambda^2\mathcal{W}^2_{i+1/2}\right). \end{aligned}$$

Wave propagation form of Godunov's method

To obtain a general update rule, it is useful to define

$$\lambda^+ = \max(\lambda, 0), \qquad \lambda^- = \min(\lambda, 0).$$

Then the general update formula is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}_{i+1/2}^p\right].$$

Making the definitions

$$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}_{i-1/2}^p,$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}_{i-1/2}^p.$$

allows the update formula to be simplified to

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right).$$

Wave propagation form of Godunov's method

Furthermore, if the matrices

$$\Lambda^+ = \begin{pmatrix} (\lambda^1)^+ & & & \\ & (\lambda^2)^+ & & \\ & & \ddots & \\ & & & (\lambda^m)^+ \end{pmatrix}, \quad \Lambda^- = \begin{pmatrix} (\lambda^1)^- & & & \\ & (\lambda^2)^- & & \\ & & \ddots & \\ & & & (\lambda^m)^- \end{pmatrix}.$$

are defined, then A can be separated into left-moving and right-moving parts,

$$A^+ = R\Lambda^+ R^{-1}, \qquad A^- = R\Lambda^- R^{-1}.$$

Note that if $\Delta Q_{i-1/2} = Q_i - Q_{i-1}$, then

$$\begin{aligned} A^+ \Delta Q_{i-1/2} &= R\Lambda^+ R^{-1}(Q_i - Q_{i-1}) = R\Lambda^+ \alpha_{i-1/2} \\ &= \sum_{p=1}^{m} (\lambda^p)^+ \alpha^p_{i-1/2} r^p = \mathcal{A}^+ \Delta Q_{i-1/2} \end{aligned}$$

and therefore the matrix product matches the definition of $\mathcal{A}^+ \Delta Q_{i-1/2}$ that was previously introduced.

Roe's method

Roe's method is based on

$$|\Lambda| = \begin{pmatrix} |\lambda^1| & & & \\ & |\lambda^2| & & \\ & & \ddots & \\ & & & |\lambda^m| \end{pmatrix}$$

and defining $|A| = R|\Lambda|R^{-1}$. Then the update law becomes

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A \left(Q_{i+1}^n - Q_{i-1}^n \right) - \frac{\Delta t}{2\Delta x} \sum_{p=1}^{m} \left(|\lambda^p| \mathcal{W}_{i+1/2}^p - |\lambda^p| \mathcal{W}_{i-1/2}^p \right)$$

This separates the update into an unstable centered-difference term, plus a regularizing diffusive term. For linear advection Roe's method is equivalent to the wave propagation form of Godunov's method. However, it is a useful basis for extending to some nonlinear problems.

Numerical test of Godunov's method

Consider the scalar advection equation $q_t = \bar{u} q_x$. Godunov's method reduces to

$$Q_i^{n+1} = Q_i^n + \frac{\bar{u}\Delta x}{\Delta t}\left(Q_{i-1}^n - Q_i^n\right),$$

which is equivalent a first-order one-sided finite-difference method.

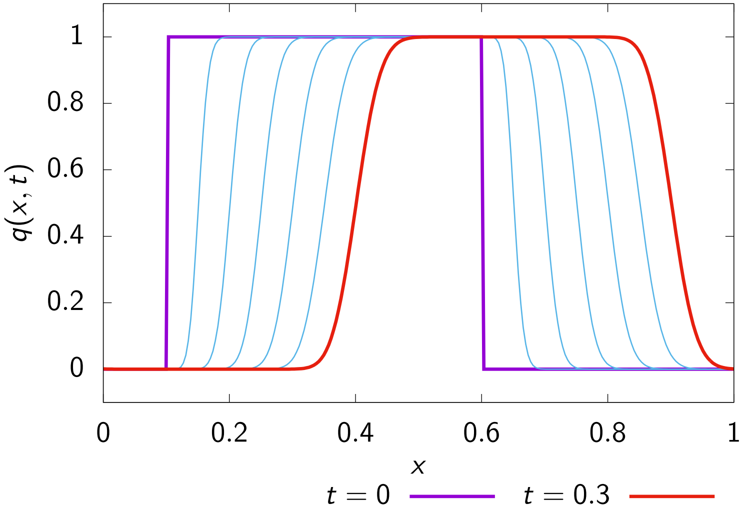
To test this method and others that follow, we use the periodic interval $[0, 1)$ with initial condition

$$q(x, 0) = \begin{cases} 1 & \text{if } 1/10 < x < 3/5, \\ 0 & \text{otherwise.} \end{cases}$$

We use 256 gridpoints, $\bar{u} = 1$, and $\Delta t = \frac{\Delta x}{5\bar{u}}$.

First-order Godunov method

Showing a substantial blurring of the discontinuities



Lax–Wendroff method

We now consider higher-order methods. Consider the equation $q_t + Aq_x = 0$ where q is a vector with m components, and A is a matrix. The Lax–Wendroff method is based on the second-order Taylor series expansion

$$q(x, t_{n+1}) = q(x, t_n) + \Delta t\, q_t(x, t_n) + \frac{(\Delta t)^2}{2} q_{tt}(x, t_n) + \dots .$$

From the equation,

$$q_{tt} = -Aq_{xt} = A^2 q_{xx}$$

and hence

$$q(x, t_{n+1}) = q(x, t_n) + \Delta t\, Aq_x(x, t_n) + \frac{(\Delta t)^2}{2} A^2 q_{xx}(x, t_n) + \dots . \tag{3}$$

Lax–Wendroff method

Converting the x derivatives in Eq. 3 into centered finite differences yields

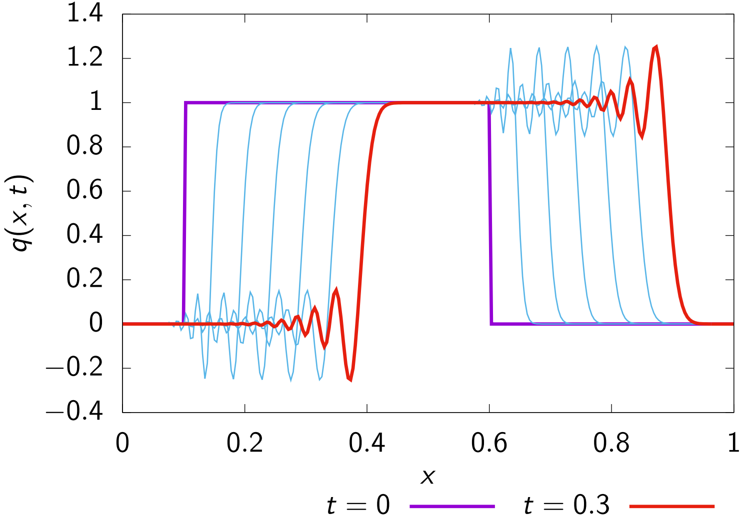
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n).$$

This can be rewritten in conservative form with fluxes

$$F_{i-1/2}^n = \frac{1}{2} A(Q_{i-1}^n + Q_i^n) - \frac{\Delta t}{2\Delta x} A^2(Q_i^n - Q_{i-1}^n).$$

Numerical test of Lax–Wendroff

Discontinuities are sharper, but oscillations are visible



Beam–Warming method

Suppose that all the eigenvalues of A are positive. Then we could use one-sided derivatives instead:

$$q_x(x_i, t_n) = \frac{1}{2\Delta x}\left[3q(x_i, t_n) - 4q(x_{i-1}, t_n) + q(x_{i-2}, t_n)\right] + O(\Delta x^2),$$
$$q_{xx}(x_i, t_n) = \frac{1}{\Delta x^2}\left[q(x_i, t_n) - 2q(x_{i-1}, t_n) + q(x_{i-2}, t_n)\right] + O(\Delta x).$$

This yields the numerical method

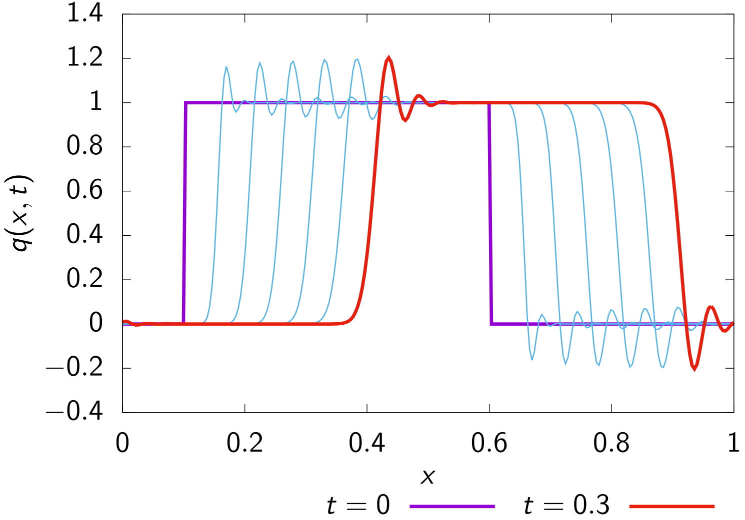
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}A(3Q_i^n - 4Q_{i-1}^n + Q_{i-2}^n) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2(Q_i^n - 2Q_{i-1}^n + Q_{i-2}^n),$$

with corresponding fluxes

$$F_{i-1/2}^n = AQ_{i-1}^n + \frac{1}{2}A\left(1 - \frac{\Delta t}{\Delta x}A\right)(Q_{i-1}^n - Q_{i-2}^n).$$

Numerical test of Beam–Warming

Oscillations are also visible, but are upwind of the discontinuities



High resolution methods

As seen in the examples, the Lax–Wendroff and Beam–Warming methods do better when applied to smooth initial data, but do worse when applied to discontinuous data.

For discontinuous data, the simple upwind method does better.

We want to devise a high resolution method that combines the best of both worlds. It should achieve second-order accuracy where possible, but not insist on it in places where the solution is not behaving smoothly.

High resolution methods

To begin, let us rewrite the Lax–Wendroff flux as

$$F^n_{i-1/2} = (A^- Q^n_i + A^+ Q^n_{i-1}) + \frac{1}{2}|A| \left(I - \frac{\Delta t}{\Delta x}|A| \right) (Q^n_i - Q^n_{i-1})$$

This has the form of an upwind flux, plus a correction term.

The correction looks like a diffusive flux but the coefficient is positive when the CFL condition is satisfied. Hence this term is antidiffusive and helps correct the overly diffusive upwind approximation.

The basic idea is to apply some sort of limiter that changes the magnitude of correction actually used, depending on how the solution behaves.

Limiters

The solution to the hyperbolic system consists of a superposition of waves, some of which may be smooth and some of which may be discontinuous. Ideally we would like apply the limiters to each component separately.

We return to the scalar case. Once we have developed techniques for the scalar case, they can be extended to the multicomponent case.

REA algorithm for a piecewise linear construction

To achieve better than first-order accuracy with the REA algorithm, we need to use a better reconstruction than piecewise constant. One option is to use a piecewise linear construction,

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n (x - x_i),$$

where $x_i = (x_{i-1/2} + x_{i+1/2})/2 = x_{i-1/2} + \Delta x/2$. Note, importantly, that the average of $\tilde{q}$ is Q_i^n. We still need to specify the slope σ_i^n.

For the scalar advection equation $q_t + \bar{u} q_x$ with $\bar{u} > 0$, the REA algorithm yields

$$Q_i^{n+1} = Q_i^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{1}{2}\frac{\bar{u}\Delta t}{\Delta x}(\Delta x - \bar{u}\Delta t)(\sigma_i^n - \sigma_{i-1}^n).$$

This is only valid when the CFL condition is satisfied: $\bar{u}\Delta t \leq \Delta x$.

Choice of slopes

There are several obvious choices for the slope calculation, some of which yield methods that we have seen before:

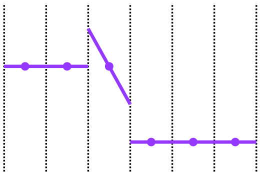
- Centered slope: $\sigma_i^n = (Q_{i+1}^n - Q_{i-1}^n)/(2\Delta x)$ (Fromm)
- Upwind slope: $\sigma_i^n = (Q_i^n - Q_{i-1}^n)/\Delta x$ (Beam–Warming)
- Downwind slope: $\sigma_i^n = (Q_{i+1}^n - Q_i^n)/\Delta x$ (Lax–Wendroff)

Fromm's method results in the stencil

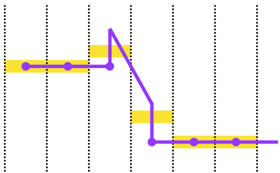
$$Q_i^{n+1} = Q_i^n = \frac{\bar{u}\Delta t}{4\Delta x}\left(Q_{i+1}^n + 3Q_i^n - 5Q_{i-1}^n + Q_{i-2}^n\right) \\ + \frac{\bar{u}^2\Delta t^2}{4\Delta x^2}\left(Q_{i+1}^n - Q_i^n - Q_{i-1}^n + Q_{i-2}^n\right).$$

Oscillations

While the linear reconstruction achieves higher accuracy, it can introduce spurious oscillations, as seen the in the Lax–Wendroff and Beam–Warming examples. To see why, consider a downwind slope reconstruction for discontinuous data:



Time t_n



Time t_{n+1}

Here $\Delta t = \Delta x / 2\bar{u}$. Purple lines show $\tilde{q}$. Yellow lines show the average of $\tilde{q}$ at t_{n+1}.

The method causes some values of Q^{n+1} to overshoot their original values, resulting in the oscillations. We would like to avoid this by limiting the slope.

Total variation

To see how much we should limit the slope, it is useful to define total variation TV. For a discretized function

$$\mathrm{TV}(Q) = \sum_i |Q_i - Q_{i-1}|.$$

For a function

$$\mathrm{TV}(Q) = \lim_{\epsilon \to 0} \sup \frac{1}{\epsilon} \int_{-\infty}^{\infty} |q(x) - q(x-\epsilon)| dx,$$

which reduces to

$$\mathrm{TV}(Q) = \int_{-\infty}^{\infty} |q'(x)| dx$$

when q is differentiable.

TVD methods

A method is called total variation diminishing (TVD) if for any set of data Q^n, the values Q^{n+1} satisfy

$$\mathrm{TV}(Q^{n+1}) \leq \mathrm{TV}(Q^n).$$

One method to devise a TVD numerical scheme is to find a reconstruction such that

$$\mathrm{TV}(\tilde{q}^n(\cdot, t_n)) \leq \mathrm{TV}(Q^n).$$

Then the method will be TVD because the evolving and averaging steps cannot possibly increase the total variation.

Slope-limiter methods

Several procedures for constructing slopes result in TVD methods. In the minmod method, the slopes are given by

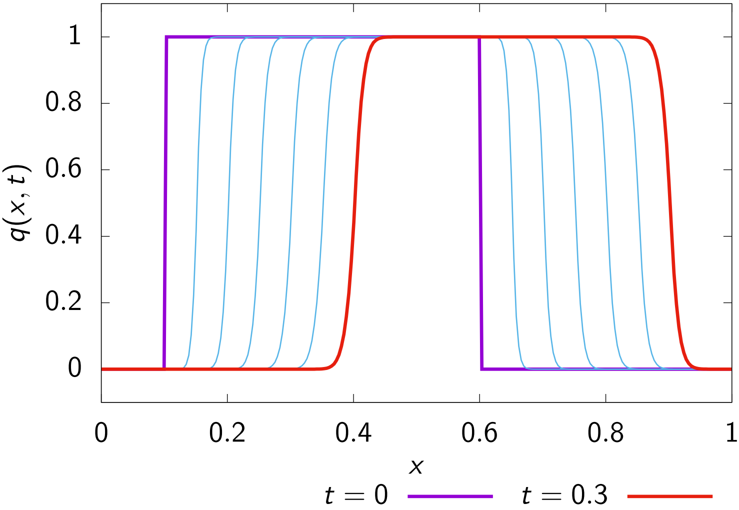
$$\sigma_i^n = \operatorname{minmod}\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}, \frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)$$

where

$$\operatorname{minmod}(a, b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0, \\ b & \text{if } |b| < |a| \text{ and } ab > 0, \\ 0 & \text{if } ab \leq 0. \end{cases}$$

Numerical test of minmod limiter

Oscillations are removed and discontinuities remain sharper



Superbee limiter

In the superbee method

$$\sigma_i^n = \text{maxmod}(\sigma_i^{(1)}, \sigma_i^{(2)})$$

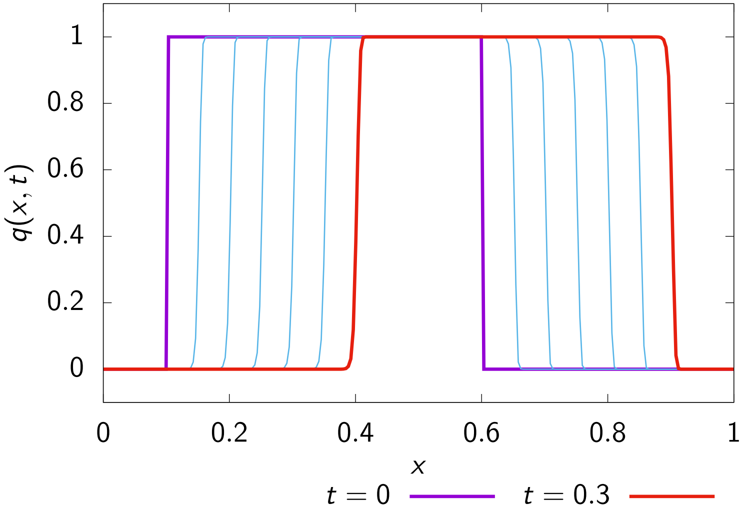
where

$$\sigma_i^{(1)} = \text{minmod}\left(\left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right), 2\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right)\right),$$

$$\sigma_i^{(2)} = \text{minmod}\left(\left(2\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right), \left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right)\right).$$

Numerical test of superbee limiter

The discontinuities remain even sharper



Essentially non-oscillatory (ENO) method

Another useful approach for maintaining sharp discontinuities is the essentially non-oscillatory (ENO) method. Here we consider one example of an ENO scheme. Define the second derivative via a centered-difference formula,

$$[q_{xx}]_i = \frac{Q_{i+1} - 2Q_i + Q_{i-1}}{\Delta x^2}.$$

Now consider solving the standard equation $q_t + \bar{u} q_x = 0$. The spatial derivative is discretized as

$$[q_x]_i = \begin{cases} \frac{Q_{i+1} - Q_{i-1}}{2\Delta x} & \text{if } |[q_{xx}]_i| \le |[q_{xx}]_{i-1}|, \\ \frac{3Q_i - 4Q_{i-1} + Q_{i-2}}{2\Delta x} & \text{if } |[q_{xx}]_i| > |[q_{xx}]_{i-1}|, \end{cases}$$

Thus the method switches between a centered difference and a one-sided derivative depending on which set of three gridpoints is more colinear.

Essentially non-oscillatory (ENO) method

From here, the update formula is given by

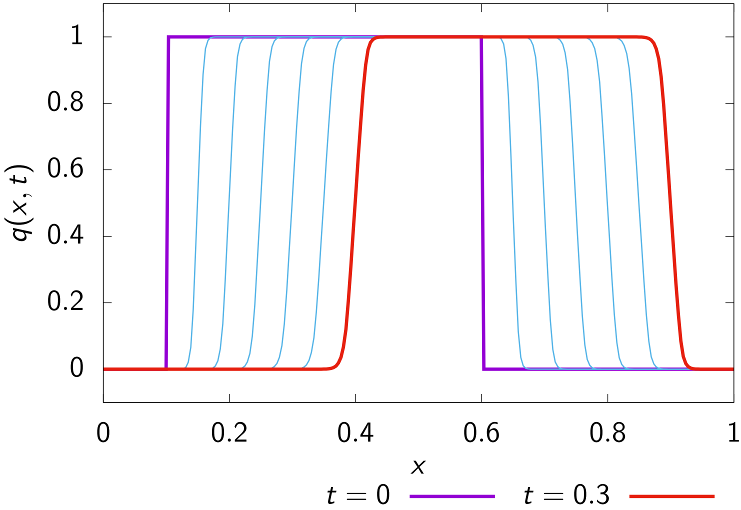
$$Q_i^{n+1} = Q_i^n - \Delta t \, \bar{u}[q_x]_i^n$$

where the spatial derivative is computed using the ENO stencil.
The ENO method handles discontinuities well.

Question: is it a conservative scheme?

Numerical test of ENO2 method

Has a similar sharpness to the superbee method



Nonlinear equations: motivation

Many hyperbolic equations that we encounter have nonlinearities. For example, consider the Navier–Stokes equations for fluid flow,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure, ρ is the fluid density and ν is the kinematic viscosity.

The blue term represents convection and has a nonlinearity.

Traffic equation

To study nonlinear equations, we consider a simple hyperbolic PDE for traffic flow.

This model captures many of the important features of nonlinear hyperbolic equations, and is physically interpretable.

It captures many features of real traffic, such as how the density of cars on a highway can sometimes fluctuate by a large amount for no obvious reasons.

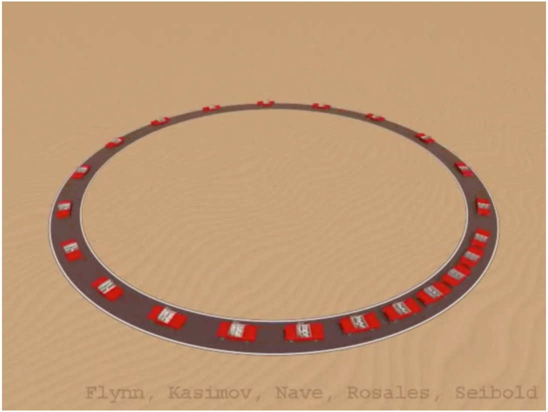
Traffic circle experiment

From initially even density, density shockwaves form

https://www.youtube.com/watch?v=Suugn-p5C1M

Traffic circle model

A computer simulation that predicts the same behavior



https://www.youtube.com/watch?v=Q78Kb4uLAdA

Off-topic but cool: the Magic Roundabout in Swindon, UK

Voted one of the "10 Scariest Junctions in the United Kingdom"[5]

https://www.youtube.com/watch?v=Kafx_GGHqVg

[5] http://news.bbc.co.uk/2/hi/uk_news/england/london/7140892.stm

A model for traffic flow

Let $q(x,t)$ be the density of cars on a one-dimensional highway. Assume that the cars are moving in the positive x direction only.

Let the speed of cars be $u(x,t)$. The car flux uq satisfies the conservation law

$$q_t + (uq)_x = 0.$$

We assume that the speed of the cars is a function $U(q)$ of the density only, so that

$$q_t + (qU(q))_x = 0.$$

Equivalently

$$q_t + (f(q))_x = 0 \tag{4}$$

for a flux $f(q)$.

Question: What is a good model for $U(q)$?

Car speed and flux

If there is no traffic, then a car can achieve a maximum speed u_{max}. As the traffic density increases, we expect that the car's speed will go down. A simple model proposes that the car's speed will decrease linearly to zero at some maximum density. By rescaling this maximum density to one, we obtain

$$U(q) = u_{\text{max}}(1 - q)$$

and thus the flux is

$$f(q) = u_{\text{max}} q(1 - q).$$

Note that $f''(q) < 0$ for $0 \leq q \leq 1$ and so f is a concave function. Having convex or concave functions f will be a useful property later on. In either case, it is referred to as a convex flux.

Discrete car trajectories

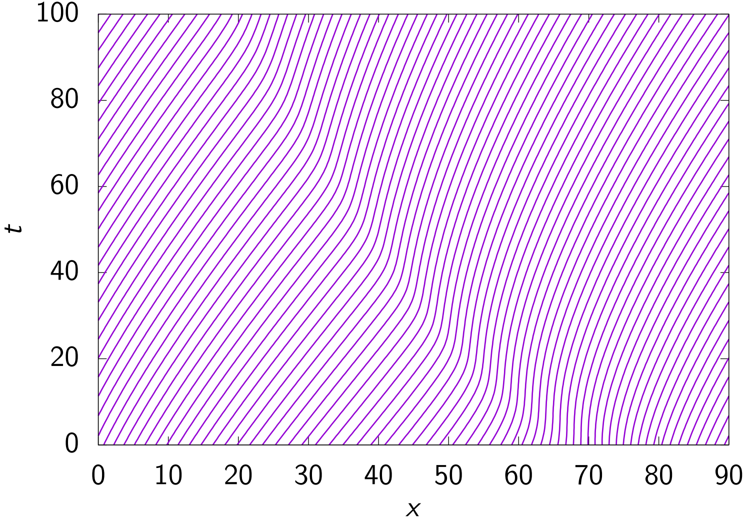
We can obtain a feel for the model by simulating a set of discrete car trajectories $X_k(t)$ on a periodic interval. Assume that the speed of the kth car, is based on a local estimate of density from the car in front, so that

$$X_k'(t) = U\left(\frac{1}{X_{k+1}(t) - X_k(t)}\right)$$

An example code `cm_solve.cc` is provided that simulates this model using sixty cars on a periodic interval $[0, 90)$, with an initially non-uniform density.

Car trajectory simulation

Showing a density wave moving backward while cars move forward.



A general solution method

The traffic equation can be rewritten as

$$q_t + f'(q)q_x = 0.$$

Assume q is smooth. Consider the characteristic curve $X(t)$ that satisfies the ODE

$$X'(t) = f'(q(X(t), t)). \tag{5}$$

Then

$$\frac{d}{dt}q(X(t), t) = X'(t)q_x + q_t = 0$$

and so the solution is constant along characteristics. Consequently Eq. (5) tells us that $X'(t)$ is constant, and so the characteristic is a straight line, so long as the solution remains smooth.

Characteristic speed

For the traffic equation the characteristic speed is

$$f'(q) = u_{\max}(1 - 2q),$$

which is different from the speed of an individual car, $U(q) = u_{\max}(1 - q)$. This is consistent with the discrete car example above.

In particular $f'(q)$ can be negative, corresponding to backward-moving characteristics. Individual cars always move forward.

General solution procedure: smooth case

Suppose we are given some initial data $q(x, 0) = \mathring{q}(x)$. Then the solution will be given by

$$q(x, t) = \mathring{q}(\xi)$$

where ξ solves the equation

$$x = \xi + f'(\mathring{q}(\xi))t$$

There will be a unique solution provided that the characteristics do not cross.

Comparison: Burgers' equation

At this point, it is useful to introduce a different nonlinear equation, the inviscid Burgers equation,

$$u_t + \left(\frac{u^2}{2}\right)_x = 0. \tag{6}$$

Burgers studied[6] the generalization of this known as the viscous Burgers equation,

$$u_t + \left(\frac{u^2}{2}\right)_x = \epsilon u_{xx}. \tag{7}$$

This was originally studied as a reduced equation containing the nonlinear hyperbolic term in gas dynamics. The ϵu_{xx} term represents viscosity.

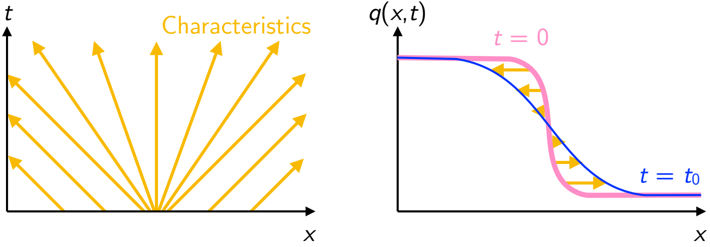
Expanding Eq. (6) gives the simple form $u_t + uu_x = 0$.

---

[6] J. M. Burgers, *A mathematical model illustrating the theory of turbulence*, Adv. Appl. Mech. **1**, 171–179 (1948). doi:10.1016/S0065-2156(08)70100-5

Rarefaction waves

Returning to the traffic equation, let us consider the solution in region where q decreases, so $q_x(x,0) < 0$.

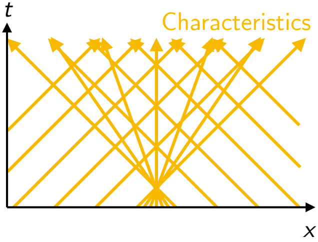


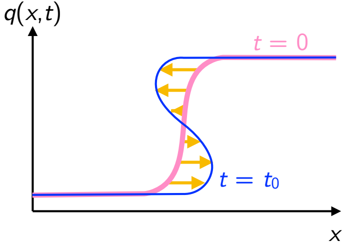
The characteristics do not cross (left image). Since the solution is constant along characteristics, we can use this to sketch the profile at a later time t_0 (right image).

The profile will smooth out—this is known as a rarefaction wave.

Compression waves

If we apply the same approach when $q_x(x, 0) > 0$, we encounter a problem: the characteristics cross and the solution at a later time t_0 becomes multi-valued.





This is called a compression wave and we want to resolve what happens physically in this case.

Vanishing viscosity approach

One method to gain insight into a compression wave is to add a small second derivative term (analogous to the viscous term in the Burgers equation) to obtain

$$q_t + f(q)_x = \epsilon q_{xx}.$$

The ϵq_{xx} term heavily penalizes sharp discontinuites. It can also be shown that by including this term, the solution will remain smooth for all time. Examining the limit of vanishing viscosity where $\epsilon \to 0$ should give us insight into the original equation.

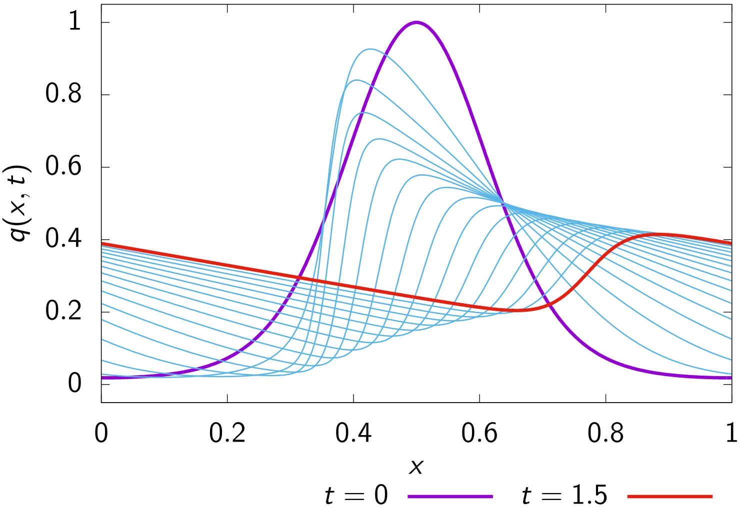
Vanishing viscosity numerical example

The test code `vv_solve.cc` shows an example of simulating the traffic equation using a simple second-order finite-difference method, using the initial condition

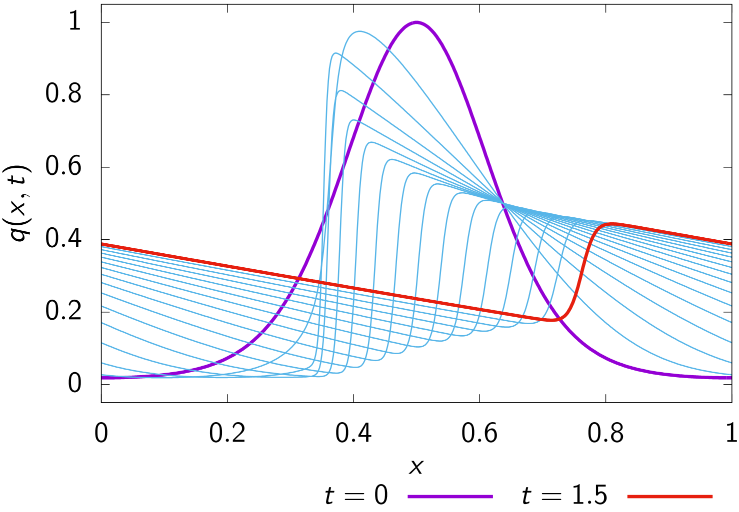
$$q(x,0) = e^{-2(1+\cos 2\pi x)}.$$

on the periodic interval $[0,1)$. It uses 1500 gridpoints in space, and simulates with three different values of the ϵ parameter.

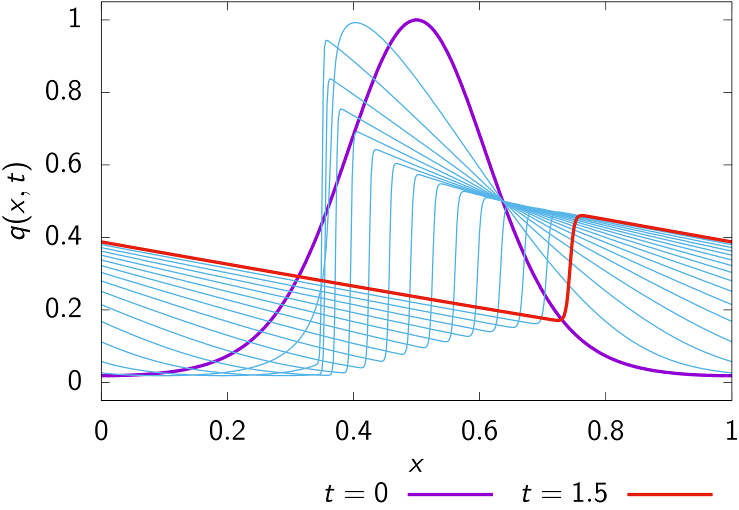
Vanishing viscosity simulation ($\epsilon = 10^{-2}$)



Vanishing viscosity simulation ($\epsilon = 10^{-2.5}$)



Vanishing viscosity simulation ($\epsilon = 10^{-3}$)



Shocks

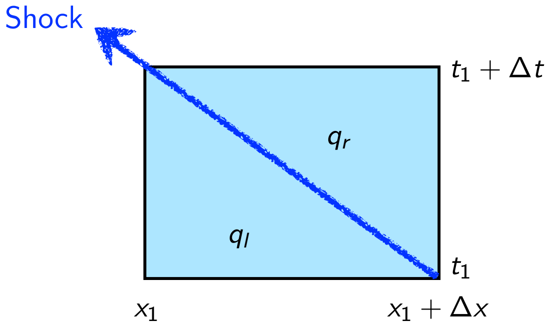
As $\epsilon \to 0$, the solution develops an increasingly sharper jump in $q(x, t)$. In the limit, we could therefore consider replacing this with a moving discontinuity with velocity $s(t)$, known as a shock.

This is physically reasonable—it shows that nonlinear conservation laws of the form $q_t + f(q)_x = 0$ can naturally develop features on an infinitesimaly small length scale. This can model real-world phenomena such as shock waves through a gas.

In reality it is likely that a physical system will have some dissipation on a small enough length scale, which regularizes the discontinuity. For example, this could correspond to a very small but finite ϵ. However, since we may not wish to model such small scales, it is useful to study equations like $q_t + f(q)_x = 0$ and treat the shock as an independent physical object.

Deriving the shock velocity

Consider a shock propagating with velocity s over a small time interval Δt. Let q_l and q_r be the solutions on either side of the shock. This results in the following picture:



We aim to derive s by returning to the integral form of the conservation law. The integral form does not have any difficulty with the discontuity.

Deriving the shock velocity

Integrating over the time interval $(t_1, t_1 + \Delta t)$ and space interval $(x_1, x_1 + \Delta x)$ yields

$$\int_{x_1}^{x_1+\Delta x} q(x, t_1 + \Delta t)dx - \int_{x_1}^{x_1+\Delta x} q(x, t_1)dx$$
$$= \int_{t_1}^{t_1+\Delta t} f(q(x_1, t))dt - \int_{t_1}^{t_1+\Delta t} f(q(x_1 + \Delta x, t))dt$$

Each integral is done over a region where q is constant, and hence

$$\Delta x\, q_r - \Delta x\, q_l = \Delta t f(q_l) - \Delta t f(q_r) + O(\Delta t^2).$$

Since $\Delta x = -s\Delta t$ it follows that $s(q_r - q_l) = f(q_r) - f(q_l)$ and hence

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}.$$

This is the Rankine–Hugoniot jump condition.

Deriving the shock velocity

For the traffic flow flux of $f(q) = u_{\max} q(1-q)$, the shock speed is

$$s = \frac{u_{\max}\left(q_r(1-q_r) - q_l(1-q_l)\right)}{q_r - q_l}$$
$$= \frac{u_{\max}(q_r - q_l)}{q_r - q_l} - \frac{u_{\max}(q_r^2 - q_l^2)}{q_r - q_l} = u_{\max}(1 - (q_l + q_r)).$$

By using $f'(q) = u_{\max}(1-2q)$, we see that

$$s = \frac{f'(q_l) + f'(q_r)}{2}$$

so that the shock moves at the average of the characteristic velocities on either side. This is true in general for any quadratic flux function (*e.g.* Burgers).

Rarefaction similarity solution

Now that we are willing to consider non-smooth solutions, we can examine the Riemann problem corresponding to a rarefaction wave,

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0, \\ q_r & \text{if } x > 0, \end{cases} \tag{8}$$

where $q_l > q_r$. This problem has no characteristic length scales. The only dimensionless group is $\frac{x}{u_{\max} t}$. We therefore seek for a similarity solution[7]

$$q(x,t) = \tilde{q}(x/t)$$

[7] The factor of $u_{\max}$ is absorbed into $\tilde{q}$.

Rarefaction similarity solution

Then

$$q_t(x,t) = -\frac{x}{t^2}\tilde{q}'(x/t), \qquad f(q)_x = \frac{1}{t}f'(\tilde{q}(x/t))\tilde{q}'(x,t).$$

Substituting into $q_t + f'(q)q_x = 0$ yields

$$f'(\tilde{q}(x/t))\tilde{q}'(x/t) = \frac{x}{t}\tilde{q}'(x/t)$$

and hence either

$$\tilde{q}'(x/t) = 0$$

or

$$f'(\tilde{q}(x/t)) = x/t. \tag{9}$$

Rarefaction similarity solution

We expect a rarefaction fan. The left and right edges of the fan move at $f'(q_l)$ and $f'(q_r)$ respectively. For $f'(q_l) < x/t < f'(q_r)$, $\tilde{q}$ varies and hence Eq. (9) holds, so

$$u_{\max}(1 - 2\tilde{q}(x/t)) = x/t.$$

Rearranging gives

$$\tilde{q}(x/t) = \frac{1}{2}\left(1 - \frac{x}{u_{\max}t}\right)$$

and hence the complete solution is

$$\tilde{q}(x/t) = \begin{cases} q_l & \text{if } x/t \le f'(q_l), \\ 1/2(1 - \frac{x}{u_{\max}t}) & \text{if } f'(q_l) < x/t < f'(q_r), \\ q_r & \text{if } x/t \ge f'(q_r). \end{cases} \quad (10)$$

This is called a rarefaction fan.

Weak solutions

We have now seen several examples—shocks and rarefaction fans—that have a non-differentiable solution and therefore do not satisfy the PDE $q_t + f(q)_x = 0$. We would like to more precisely define a class of solutions to the conservation law.

When deriving the Rankine–Hugoniot condition for a shock, we made use of the integral form of the conservation law. This had no problem with the discontinuity. Hence we try to use integration here.

Weak solution

Suppose we start with a smooth solution that satisfies $q_t + f(q)_x = 0$. Let $\phi \in C_0^1$ be a differentiable function with compact support (defined as in the finite-element section). Then

$$\int_0^\infty dt \int_{-\infty}^\infty dx (q_t + f(q)_x)\phi = 0.$$

Integrating by parts yields

$$\int_0^\infty dt \int_{-\infty}^\infty dx (q\phi_t + f(q)\phi_x) = -\int_{-\infty}^\infty q(x,0)\phi(x,0)dx, \quad (11)$$

where due to the compact support of ϕ we only pick up a single boundary term.

Eq. (11) is valid even if q is discontinuous. Hence we make the following definition: the function $q(x,t)$ is a weak solution of the conservation law given initial data $q(x,0)$ if Eq. (11) holds for all $\phi \in C_0^1$.

Entropy condition

Our definition a weak solution encompasses our previous solutions involving shocks and rarefaction fans.

However, things are a little more complicated. Let us revisit the rarefaction fan Riemann problem[8] with initial condition q_l for $x < 0$ and q_r for $x > 0$ where $q_l > q_r$. One can show that

$$q(x,0) = \begin{cases} q_l & \text{if } x/t < s, \\ q_r & \text{if } x/t > s, \end{cases} \tag{12}$$

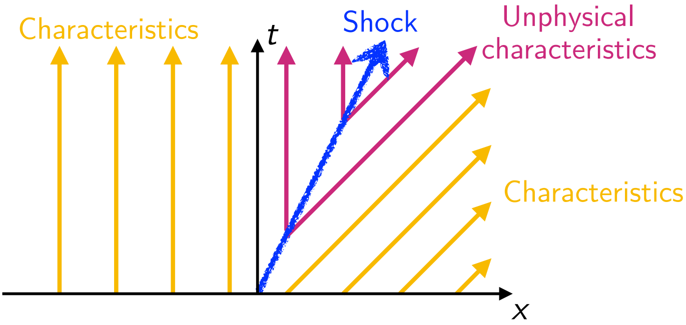
is also a weak solution, where $s = (f(q_r) - f(q_l))/(q_r - q_l)$.

We therefore have two weak solutions for the same initial data! Only one should be physical.

[8] Given by Eq. (8)

Entropy condition

Let us consider the previous solution (Eq. (12)) for $q_l = 1/2$ and $q_r = 0$. Then $s = 0$ and we obtain the following picture:



We have a shock at $x = 0$, but unlike the previous cases, characteristics emanate from it. This is unphysical, since it requires information be generated by the shock.

Entropy condition

To resolve this, we impose constraints on the weak solution, referred to as entropy conditions or admissibility conditions, which allow us to select the single physical solution.

There are several examples of entropy conditions, but here we consider one due to Lax.

For a convex scalar conservation law, a discontinuity propagating with velocity s satisfies the Lax entropy condition if

$$f'(q_l) > s > f'(q_r).$$

This condition therefore requires that characteristics do not emanate from shocks.

Numerical solution of nonlinear equations

The reconstruct–evolve–average (REA) method that we considered previously can also be used for nonlinear problems. Here we consider the simplest case of a piecewise constant reconstruction, where we must consider Riemann problems at each finite interval boundary.

We have already solved the Riemann problem for the traffic equation. There are two cases:

- If $q_l > q_r$, we obtain a rarefaction fan, with a complete solution given by Eq. (10).
- If $q_l < q_r$, we obtain a shock propagating with velocity $s = (f(q_r) - f(q_l))/(q_r - q_l)$.

Therefore, to apply the REA method to the traffic equation, we must compute numerical fluxes by switching between the two cases depending on the values of q_l and q_r.

Numerical example

The program `t_solve.cc` implements the REA method for the traffic equation.[9] It uses the periodic interval $[0, 1)$ with initial condition

$$q(x, 0) = \begin{cases} 4/5 & \text{if } 1/4 < x < 3/4, \\ 0 & \text{otherwise.} \end{cases}$$

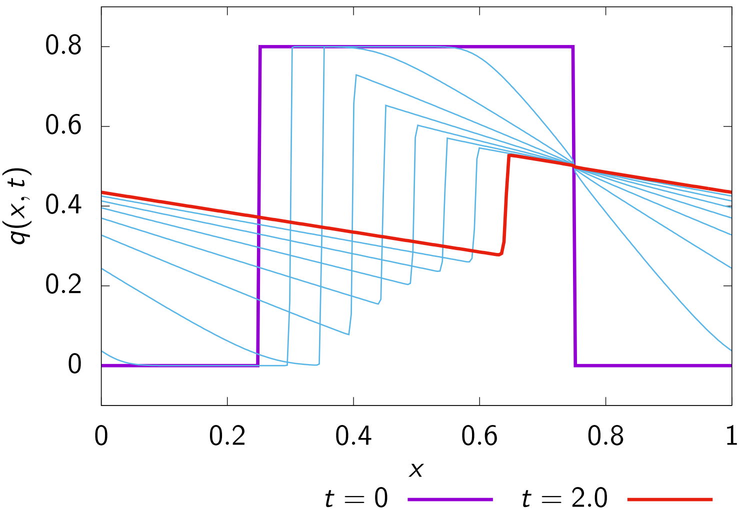
This has a shock with initial velocity

$$s = \frac{f'(4/5) + f'(0)}{2} = 1 - 4/5 = 1/5$$

that starts at $x = 1/4$. It also has a rarefaction fan at $x = 3/4$.

[9] We use $u_{\max} = 1$.

REA method applied to the traffic equation



Notes on the traffic simulation

The graph shows that the simulation is able to model both shocks and rarefactions. Since it is based on the finite-volume approach the numerical method is conservative.

Note that there is a small discrepancy in the rarefaction fan at $q \approx 0.5$. This is a well-known feature and can be removed with more accurate methods—see Chapter 12 of the textbook[10] for more details.

[10] R. J. LeVeque, *Finite Volume Methods for Hyperbolic Problems*. Cambridge, 2002.