AM225: An example finite-element calculation

We now consider a specific example of the finite-element method. Let $u(x)$ be a function on the interval $\Omega = (1, 2)$ that satisfies

$$-\frac{d}{dx}\left(x\frac{du}{dx}\right) = f(x) \tag{1}$$

with the boundary conditions

$$u(1) = 0, \qquad x\frac{du}{dx}\bigg|_{x=2} = 2g \tag{2}$$

where g is a real constant. Following the discussion in the lectures, a classical solution of Eqs. 1 & 2 will satisfy the variational problem

$$J(v) = \frac{1}{2}a(v, v) - (f, v)_{0,\Omega} - (2g, v)_{0,\Gamma} \to \min \tag{3}$$

where

$$a(u, v) = \int_1^2 xu'v'\,dx, \qquad (f, v)_{0,\Omega} = \int_1^2 fv\,dx, \tag{4}$$

and the minimization is taken over all functions that satisfy the essential (Dirichlet) boundary condition that $v(1) = 0$. Here, the boundary Γ where natural (Neumann) boundary conditions are imposed consists of the single point at $x = 2$. Hence

$$(2g, v)_{0,\Gamma} = 2gv(2). \tag{5}$$

To solve this problem with the finite element method, introduce a grid with $3N + 1$ gridpoints with spacing $h = 1/3N$ between them. The gridpoints are located at $x_i = 1 + ih$ for $i = 0, \ldots, 3N$. Split the domain into N intervals $I_{3q} = [x_{3q}, x_{3q+3}]$ for $q = 0, \ldots, N - 1$ and define the function space

$$S_h = \{v \in C(\bar{\Omega}) \,:\, v \text{ is cubic on each interval } I_{3q} \text{ and } v(1) = 0\}. \tag{6}$$

Within S_h, we introduce a nodal basis ψ_i for $i = 1, \ldots, 3N$ such that

$$\psi_i(x_k) = \delta_{ik}. \tag{7}$$

Note that we can also define a ψ_0 function, but we do not include it in the basis, because the Dirichlet condition eliminates it from contributing to the solution. Figure 1 shows an example of the basis functions for $N = 3$. When i is not a multiple of three, the function ψ_i is non-zero only on a single interval. When i is a multiple of three, the function ψ_i is non-zero on two intervals I_i and I_{i-3}.

Using the Ritz–Galerkin method, the solution is written as

$$u_h(x) = \sum_{k=1}^{3N} z_k\psi_k(x) \tag{8}$$

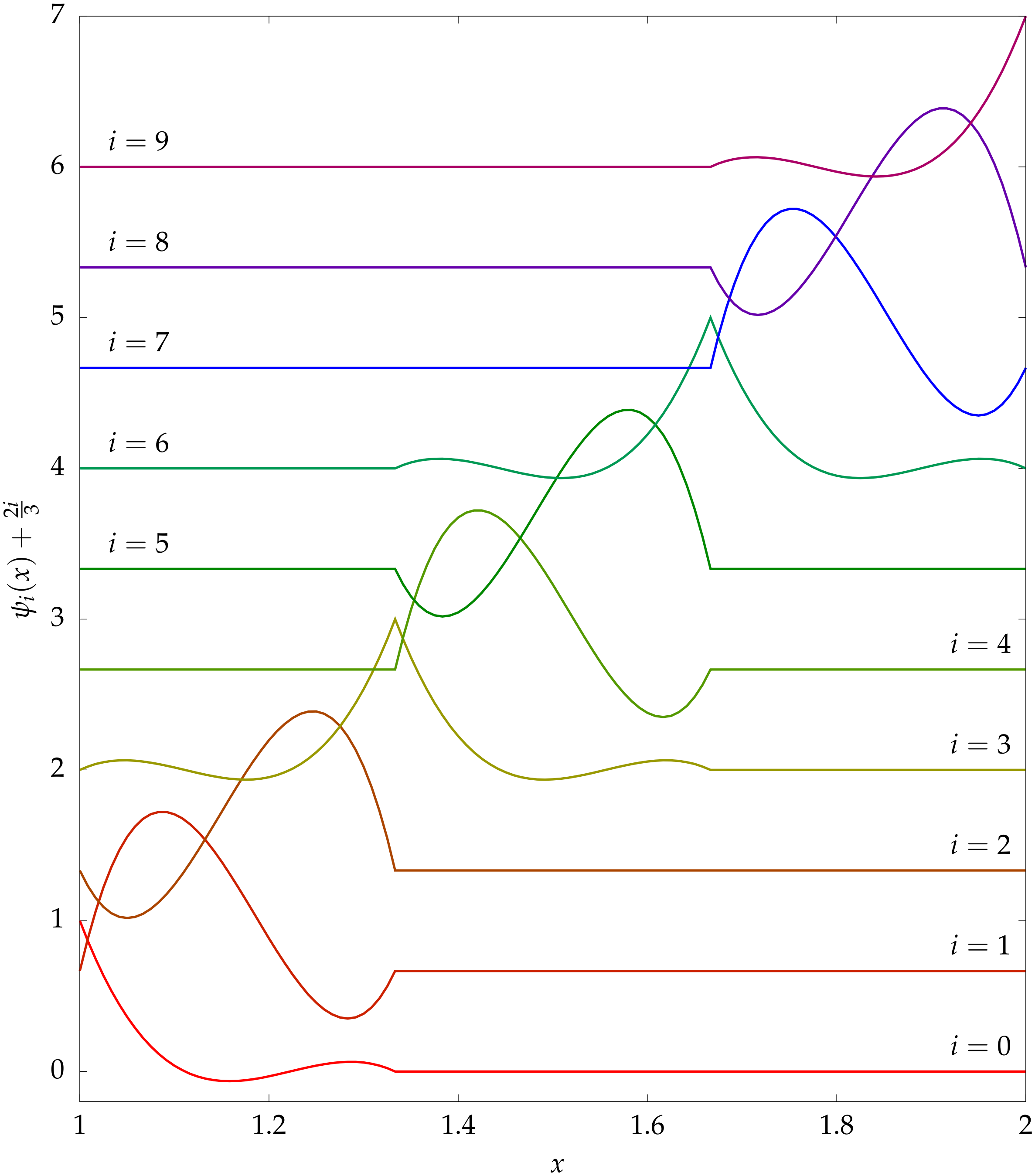

Figure 1: The ten piecewise cubic functions $\psi_i(x)$ for the case of $N = 3$. The functions are plotted with vertical offsets to better visualize them. Note that ψ_0 is shown for completeness, but is not part of the basis due to the imposed Dirichlet condition at $x = 1$.

and must satisfy

$$a(u_h, \psi_i) = \langle l, \psi_i \rangle \tag{9}$$

for all $i = 1, \ldots, 3N$. Here $\langle l, \psi_i \rangle = (f, \psi_i)_{0,\Omega} + (2g, \psi_i)_{0,\Gamma}$. To solve this, we first evaluate the terms $a(\psi_k, \psi_i)$ that form the stiffness matrix. Consider two distinct basis functions that overlap in an interval I_{3q}. Let $k = 3q + \alpha$ and $i = 3q + \beta$ for $\alpha, \beta \in \{0, 1, 2, 3\}$. Define the Lagrange interpolants on the interval for the set of points $\{0, 1, 2, 3\}$ as

$$L_0(z) = -\frac{(z-1)(z-2)(z-3)}{6}, \tag{10}$$

$$L_1(z) = \frac{z(z-2)(z-3)}{2}, \tag{11}$$

$$L_2(z) = -\frac{z(z-1)(z-3)}{2}, \tag{12}$$

$$L_3(z) = \frac{z(z-1)(z-2)}{6}. \tag{13}$$

Then in the interval I_{3q},

$$\psi_k(x) = L_\alpha\left(\frac{x-1}{h} - 3q\right), \qquad \psi_i(x) = L_\beta\left(\frac{x-1}{h} - 3q\right). \tag{14}$$

The derivatives of the Lagrange interpolants are

$$L_0'(z) = -\frac{11}{6} + 2z - \frac{z^2}{2}, \tag{15}$$

$$L_1'(z) = 3 - 5z + \frac{3z^2}{2}, \tag{16}$$

$$L_2'(z) = -\frac{3}{2} + 4z - \frac{3z^2}{2}, \tag{17}$$

$$L_3'(z) = \frac{1}{3} - z + \frac{z^2}{2}. \tag{18}$$

Hence, using the substitution $x = (3q + z)h + 1$,

$$\begin{aligned} a(\psi_k, \psi_i) &= \int_{I_{3q}} x\psi_k'\psi_i' dx = \frac{1}{h}\int_0^3 (3qh + zh + 1) L_\alpha'(z) L_\beta'(z) dz \\ &= (3q + h^{-1}) B_{\alpha\beta} + C_{\alpha\beta} \end{aligned} \tag{19}$$

where

$$B_{\alpha\beta} = \int_0^3 L_\alpha'(z) L_\beta'(z) dz, \qquad C_{\alpha\beta} = \int_0^3 z L_\alpha'(z) L_\beta'(z) dz. \tag{20}$$

The terms $B_{\alpha\beta}$ and $C_{\alpha\beta}$ are elementary integrals that can be performed using Mathematica. Their values are shown in Table 1.

$B_{\alpha\beta}$	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
$\beta = 0$	$\frac{37}{30}$	$-\frac{63}{40}$	$\frac{9}{20}$	$-\frac{13}{120}$
$\beta = 1$	$-\frac{63}{40}$	$\frac{18}{5}$	$-\frac{99}{40}$	$\frac{9}{20}$
$\beta = 2$	$\frac{9}{20}$	$-\frac{99}{40}$	$\frac{18}{5}$	$-\frac{63}{40}$
$\beta = 3$	$-\frac{13}{120}$	$\frac{9}{20}$	$-\frac{63}{40}$	$\frac{37}{30}$

$C_{\alpha\beta}$	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
$\beta = 0$	$\frac{17}{40}$	$-\frac{51}{80}$	$\frac{3}{8}$	$-\frac{13}{80}$
$\beta = 1$	$-\frac{51}{80}$	$\frac{27}{8}$	$-\frac{297}{80}$	$\frac{39}{40}$
$\beta = 2$	$\frac{3}{8}$	$-\frac{297}{80}$	$\frac{297}{40}$	$-\frac{327}{80}$
$\beta = 3$	$-\frac{13}{80}$	$\frac{39}{40}$	$-\frac{327}{80}$	$\frac{131}{40}$

Table 1: The terms $B_{\alpha\beta}$ (left) and $C_{\alpha\beta}$ that are used to assemble the stiffness matrix in the finite element calculation.

The above calculation also applies to $i = k$, which implies $\alpha = \beta$. If $\alpha = 1$ or $\alpha = 2$ then Eq. 19 applies directly. If $\alpha = 3$ and $i < 3N$, then

$$a(\psi_i, \psi_i) = \int_{I_{3q}} x\psi_i'\psi_i' dx + \int_{I_{3(q+1)}} x\psi_i'\psi_i' dx \tag{21}$$

since the basis functions overlap in two intervals. In this case, two integral contributions of the form of Eq. 19 must be counted.

Let us assume that the function f has the expansion

$$f(x) = \sum_{k=0}^{3N} f_k \psi_k(x). \tag{22}$$

Then to evaluate $(f, \psi_i)_{0,\Omega}$ we must evaluate $(\psi_k, \psi_i)_{0,\Omega}$. Again, choose distinct k and i so that their basis function overlap in an interval I_{3q} and define $k = 3q + \alpha$ and $i = 3q + \beta$ for $\alpha, \beta \in \{0, 1, 2, 3\}$. Then

$$(\psi_k, \psi_i)_{0,\Omega} = \int_{I_{3q}} \psi_k \psi_i dx = h \int_0^3 L_\alpha(z) L_\beta(z) dz = h D_{\alpha\beta} \tag{23}$$

where

$$D_{\alpha\beta} = \int_0^3 L_\alpha(z) L_\beta(z) dz. \tag{24}$$

The values of $D_{\alpha\beta}$ are shown in Table 2. The case of $i = k$ is handled using the same procedure as for the stiffness matrix.

With these calculations in place, we can formulate a linear system

$$Az = b \tag{25}$$

where Eq. 19 is used to assemble A, and Eq. 23 is used to assemble b. In addition, for the line corresponding to ψ_{3N} a contribution from Eq. 5 is included for $(2g, v)_{0,\Gamma}$.

The program `fe_1d_test.cc` in the AM225 examples Git repository solves this finite element problem. Since the matrix A is symmetric positive definite, the conjugate gradient

	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
$\beta = 0$	$\frac{8}{35}$	$\frac{99}{560}$	$-\frac{9}{140}$	$\frac{19}{560}$
$\beta = 1$	$\frac{99}{560}$	$\frac{81}{70}$	$-\frac{81}{560}$	$-\frac{9}{140}$
$\beta = 2$	$-\frac{9}{140}$	$-\frac{81}{560}$	$\frac{81}{70}$	$\frac{99}{560}$
$\beta = 3$	$\frac{19}{560}$	$-\frac{9}{140}$	$\frac{99}{560}$	$\frac{8}{35}$

Table 2: The terms $D_{\alpha\beta}$ that are used to assemble the source term in the finite element calculation.

algorithm is used. Figure 2 shows the solution for the source term of $f(x) = x - 3/2$ and $g = 1$.

To test the convergence of the method, we use the method of manufactured solutions and propose that

$$u(x) = e^{1-x} \sin 5\pi x. \tag{26}$$

Then

$$u'(x) = e^{1-x} (5\pi \cos 5\pi x - \sin 5\pi x) \tag{27}$$

and therefore $g = u'(2) = e^{-1} 5\pi$. In addition

$$f(x) = -\frac{d}{dx} (xu'(x)) = -e^{1-x} \Big(5\pi(1-2x) \cos 5\pi x + \big((1 - 25\pi^2)x - 1 \big) \sin 5\pi x \Big). \tag{28}$$

The program `fe_1d_conv.cc` tests the convergence of the finite element method using this manufactured solution. It tests using 31 grids from size $N = 10$ to $N = 1000$, and calculates the L_2 error between the exact and numerical solutions using the trapezoid rule. The results, shown in Fig. 3, demonstrate fourth-order convergence.

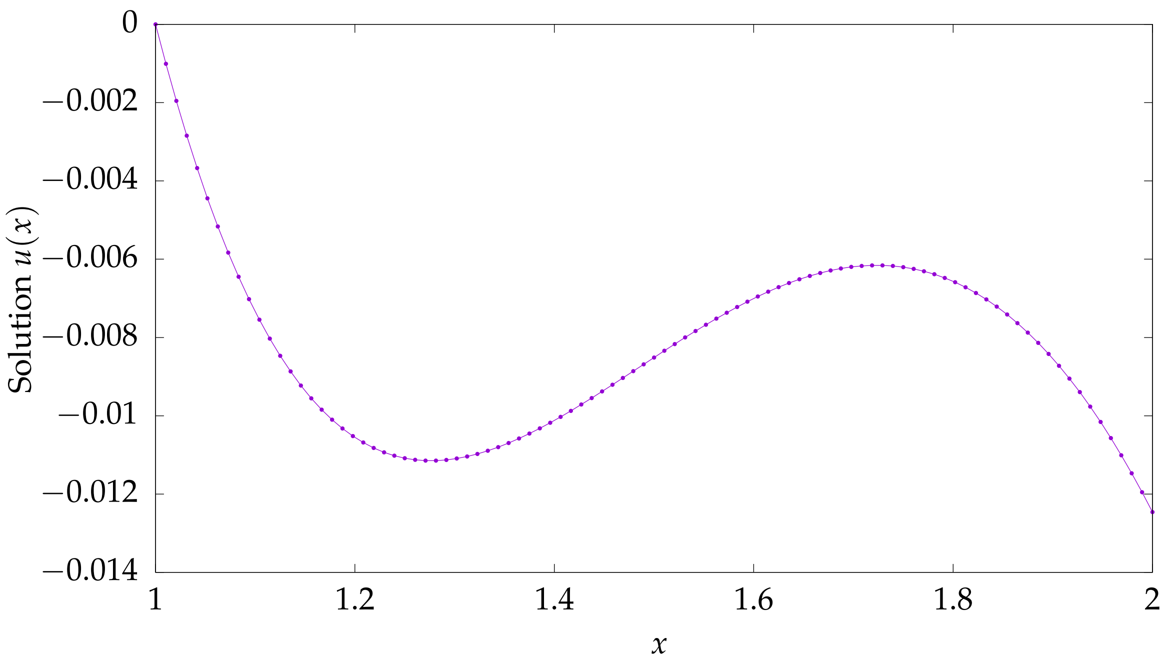

Figure 2: Example finite element solution using the source term $f(x) = x - 3/2$ and boundary conditions $u(1) = 0, u(2) = -1/20$.

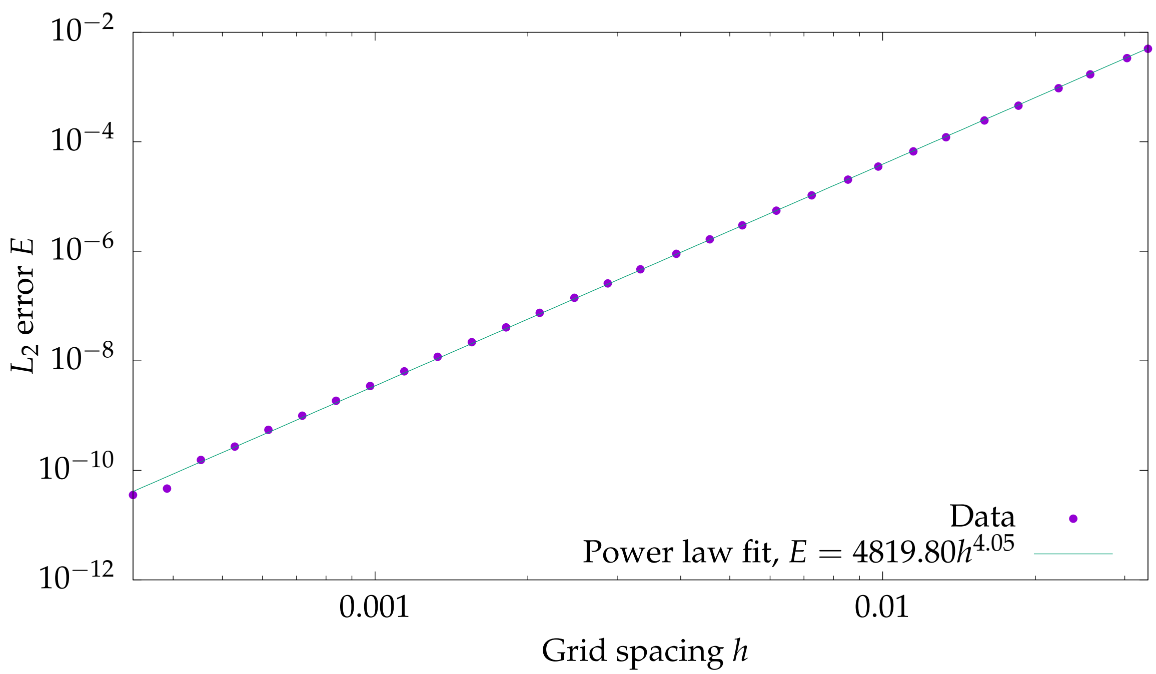

Figure 3: L_2 error of the finite element solution as a function of the grid spacing h, demonstrating fourth-order convergence.