AM225: Assignment 4 solutions*

1. See previous solutions.

2. **Fitting a square peg in a round hole**

   Consider two coordinate systems $\mathbf{v} = (v, w)$ and $\mathbf{x} = (x, y)$. In the $\mathbf{v}$ system, introduce the circle of unit radius, $\Omega = \{\mathbf{v} \in \mathbb{R} : ||\mathbf{v}||_2 < 1\}$. In the $\mathbf{x}$ system, introduce the square $S = (-1, 1)^2$. The mapping $\mathbf{v} = f(\mathbf{x})$ defined by

   $$v = x\sqrt{1 - \frac{y^2}{2}}, \qquad w = y\sqrt{1 - \frac{x^2}{2}} \tag{1}$$

   is a differentiable map from S to Ω (Fig. 1). Consider the Poisson problem

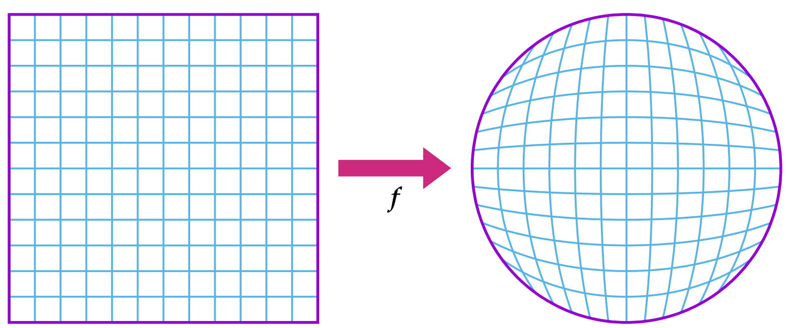

Figure 1: The mapping f from the square S to the circle Ω used in question 2.

   $$\nabla^2 u = f \tag{2}$$

   on the circle Ω with Dirichlet condition $u(\mathbf{v}) = 0$ for $\mathbf{v} \in \partial\Omega$. Introduce an $N \times N$ grid of squares on S, each with side length $h = 2/N$. Let ϕ_i be the set of bilinear elements on S corresponding to the nodal basis at square corners. Since the elements on the boundary can be neglected, this gives $(N-1)^2$ basis functions in total. On the circle, define basis functions ψ_i via the mapping from S, such that

   $$\psi_i(\mathbf{v}) = \phi_i(f^{-1}(\mathbf{v})). \tag{3}$$

   We represent the solution as

   $$u(v, w) = \sum_i u_i \psi_i(v, w) \tag{4}$$

   and solve the PDE problem using the Ritz–Galerkin method. Let us assume that the function f also has the expansion

   $$f(x) = \sum_i f_i \psi_i(v, w). \tag{5}$$

   Then Eq. 2 gives the weak form

   $$-a(u, \psi_i) = (f, \psi_i)_{0,\Omega} \tag{6}$$

*Solutions to problems 2 and 3 written by Dan Fortunato and Nick Derr, respectively.

where

$$a(u,v) = \int_\Omega \nabla_\mathbf{v} u \cdot \nabla_\mathbf{v} v d\mathbf{v}, \qquad (f,v)_{0,\Omega} = \int_\Omega f v d\mathbf{v} \tag{7}$$

For this problem, the stiffness and mass matrix calculations vary from element to element, and we compute them by pulling back the integrals from Ω to S. That is,

$$\begin{aligned} A_{ij} = a(\psi_j, \psi_i) &= \int_\Omega \nabla_\mathbf{v} \psi_j \cdot \nabla_\mathbf{v} \psi_i d\mathbf{v} \\ &= \int_S (D^{-1} \nabla_\mathbf{x} \phi_j) \cdot (D^{-1} \nabla_\mathbf{x} \phi_i)(\det D) d\mathbf{x} \end{aligned} \tag{8}$$

and

$$\begin{aligned} M_{ij} &= \int_\Omega \psi_j \cdot \psi_i d\mathbf{v} \\ &= \int_S \phi_j \cdot \phi_i (\det D) d\mathbf{x} \end{aligned} \tag{9}$$

where $D = \partial \mathbf{v} / \partial \mathbf{x}$ is the Jacobian of the mapping:

$$D = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \dfrac{y^2}{2}} & \dfrac{-xy}{2\sqrt{1 - \dfrac{y^2}{2}}} \\ \dfrac{-xy}{2\sqrt{1 - \dfrac{x^2}{2}}} & \sqrt{1 - \dfrac{x^2}{2}} \end{pmatrix} = \begin{pmatrix} S_y & -\frac{xy}{2S_y} \\ -\frac{xy}{2_x} & S_x \end{pmatrix} \tag{10}$$

where

$$S_x = \sqrt{1 - \frac{x^2}{2}}, \qquad S_y = \sqrt{1 - \frac{y^2}{2}}. \tag{11}$$

The determinant is

$$\det D = \frac{2 - x^2 - y^2}{\sqrt{2 - x^2}\sqrt{2 - y^2}} = \frac{2 - x^2 - y^2}{2 S_x S_y}. \tag{12}$$

The inverse Jacobian is

$$D^{-1} = \frac{1}{2 - x^2 - y^2} \begin{pmatrix} (2 - x^2) S_y & xy S_x \\ xy S_y & (2 - y^2) S_x \end{pmatrix}. \tag{13}$$

We numerically compute the integrals in Eqs. 8 & 9 using Gaussian quadrature.

The final linear system looks like

$$-Au = Mf \tag{14}$$

where u and f are coefficient vectors of length $(N-1)^2$.

To test the code, we use a source term of $f(v,w) = -e^{-v}(3 + (v-4)v + w^2)$, which gives the analytical solution $u(v,w) = (1 - v^2 - w^2)e^{-v}$. We run the code for a variety of choices of N and calculate the L_2 error between the numerical solution and the analytical solution; Fig. 2 shows that the rate of convergence is $\mathcal{O}(h^2)$. A solution for $N = 40$ is shown in Fig. 3.

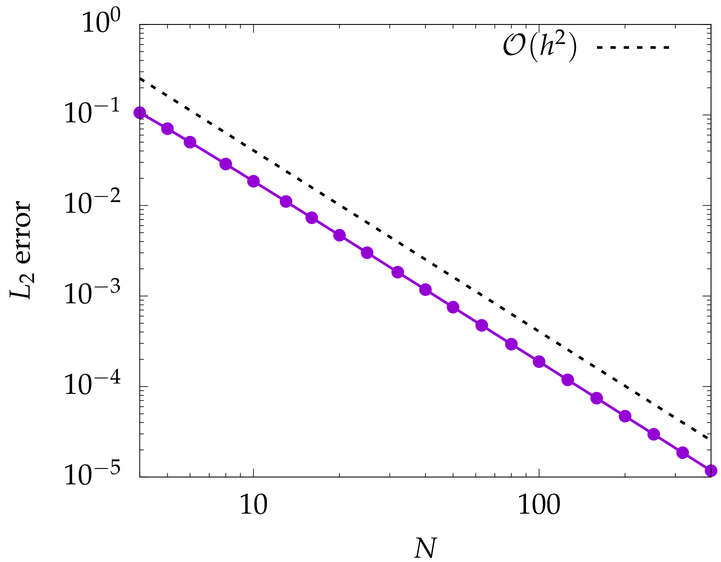

Figure 2: The method demonstrates $\mathcal{O}(h^2)$ convergence.

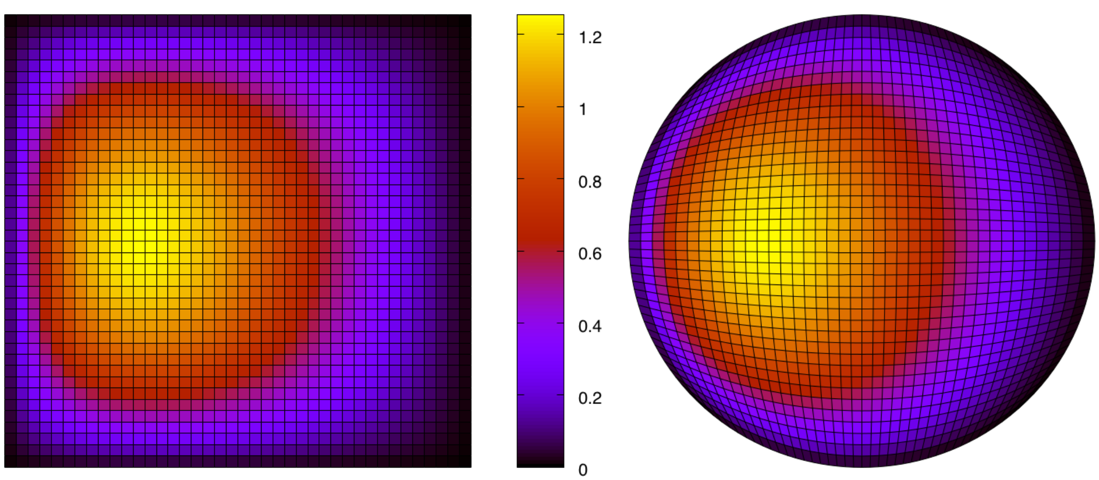

Figure 3: Solution to Eq. 2 with $N = 40$.

3. **A generalization of the Lax–Wendroff scheme**

(a) Consider characteristics $X(t)$ which move with the spatially dependent velocity $A(x)$. By definition, such characteristics satisfy the ODE

$$\frac{dX}{dt} = A\left(X(t)\right).$$

Now consider the value of $q(x,t)$ along this characteristic - in other words, consider the function $q_c(t)$ defined by

$$q_c(t) = q\left(X(t), t\right).$$

We can write

$$\frac{dq_c}{dt} = \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x}\frac{dX}{dt},$$

and plug in the ODE above and $\partial_t q + \partial_x\left(A(x)q\right) = 0$ to obtain a second ODE

$$\frac{dq_c}{dt} = -\left(\left.\frac{dA}{dx}\right|_{x=X(t)}\right) q_c(t).$$

Along with the initial conditions

$$X(0) = x_0,\ q(x_0, 0) = q_0,$$

this system of coupled ODEs can be solved analytically; see Figure 4 for a method of doing so using Mathematica. The resulting system of equations, after some algebra, reveals that $X(t)$ and $q_c(t)$ are time-periodic with the same period.
To see this, we can introduce the variables

$$\theta = \left(\frac{2\sqrt{5}}{3}\right) t,$$

$$\phi = 2\arctan\left[\frac{1}{\sqrt{5}}\left(-2 - 3\tan\left(\frac{x_0}{2}\right)\right)\right],$$

$$\varphi = \theta - \phi,$$

and write the solutions in terms of φ. If we define

$$f(x) = -\left(3 + 2\sin x\right),$$

$$g(\varphi) = \frac{9 + 4\cos\varphi - 2\sqrt{5}\sin\varphi}{15},$$

then the solution to the coupled ODE in terms of the phase is

$$X(\varphi) = -2\arctan\left[\frac{1}{3}\left(2 - \sqrt{5}\tan\left(\frac{\varphi}{2}\right)\right)\right],$$

$$q_c(\varphi) = q_0\, f(x_0)\, f\left[X(\varphi)\right]^{f[X(\varphi)]g(\varphi)}.$$

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(* spatially dependent velocity *)
A[x_] := 2+(4/3)Sin[x];

(* two ODEs *)
Xeq := X'[t] == A[X[t]];
Qeq := Q'[t] == -A'[X[t]] Q[t];

(* two initial conditions *)
Xic := X[0] == x0;
Qic := Q[0] == q0;

(* the system is solvable *)
DSolve[{Xeq,Qeq,Xic,Qic},{X[t],Q[t]},t]
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Figure 4: An approach for solving the system of coupled ODEs with Mathematica

You can verify that $X(-\phi) = x_0$ and $g(-\phi) = -1/f(x_0)$, so that $q_c(-\phi) = q_0$ as required for the satisfcation of the initial conditions.

Note that $X(\varphi)$ and $q_c(\varphi)$ are both 2π-periodic in φ. This is equivalent to $X(t)$ and $q_c(t)$ being periodic in time t with period

$$\boxed{T = \frac{3\pi}{\sqrt{5}}.}$$

Using these facts, we can write

$$\begin{aligned} q(x, t+T) &= q(X(t), t+T) = q(X(t+T), t+T), && (X(t) \text{ is time-periodic}) \\ &= q_c(t+T) = q_c(t), && (q_c(t) \text{ is time-periodic}) \\ &= q(X(t), t) = q(x, t), \end{aligned}$$

showing that $q(x,t) = q(x, t+T)$. ■

(b) The CFL condition states that we require $\Delta t < {}^{h}/_{c}$, where c is the velocity at which information in the problem propagates; in this case, this is just the velocity A in the advection equation.)

Since the velocity is spatially varying, we must choose Δt such that the condition is satisfied at all locations in our domain. This corresponds to identifying c as the maximum velocity

$$\boxed{c = \max_{x\in[0,2\pi)} A(x) = \frac{10}{3}.}$$

(c) The initial condition $q(x,t) = \exp\left(\sin x + \frac{1}{2}\sin 4x\right)$ is shown in Figure 5. Snapshots at $T = {}^{T}/_{4}, {}^{T}/_{2}, {}^{3T}/_{4}$, and T for this initial condition are shown in Figure 6.

(d) The calculated L2 error for a range of m is shown in Figure 7. The method converges at second order.

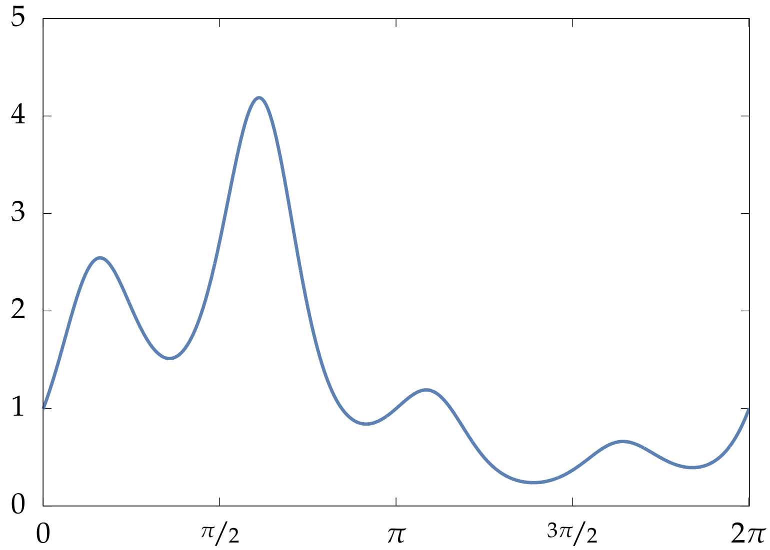

Figure 5: The initial condition $q(x, 0) = \exp(\sin x + \frac{1}{2}\sin 4x)$

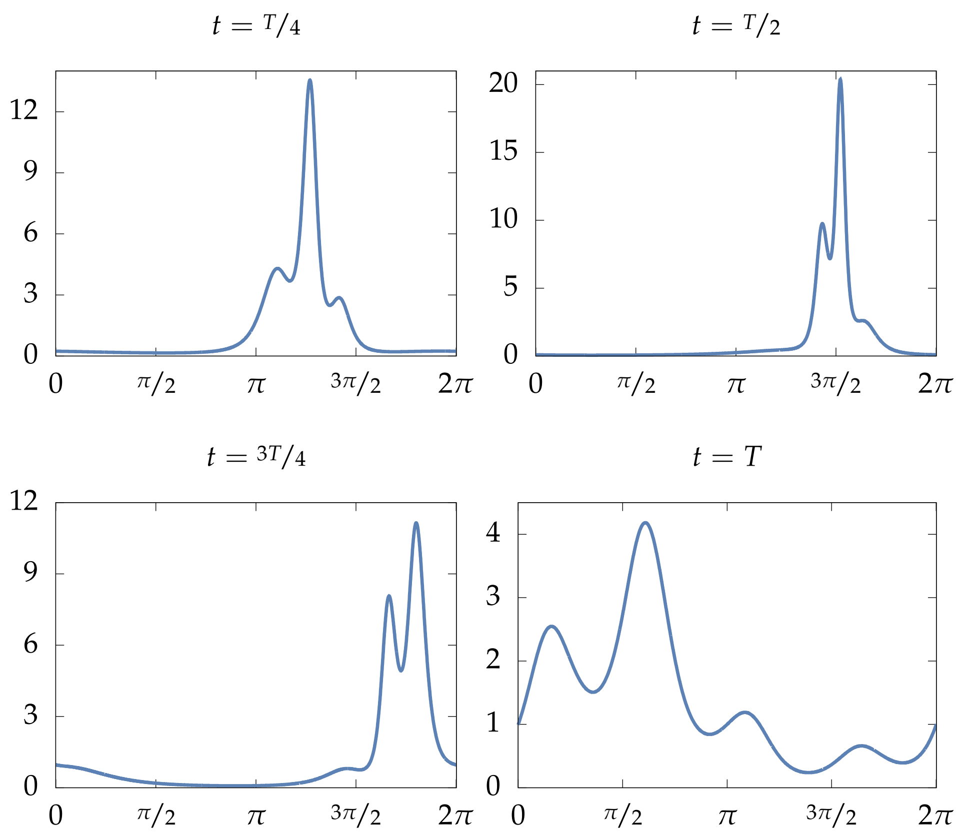

Figure 6: Snapshots of the solution given the initial condition $q(x, 0) = \exp(\sin x + \frac{1}{2}\sin 4x)$ at the times $t = {}^{T}/_{4}$, ${}^{T}/_{2}$, ${}^{3T}/_{4}$, and T for $m = 2048$

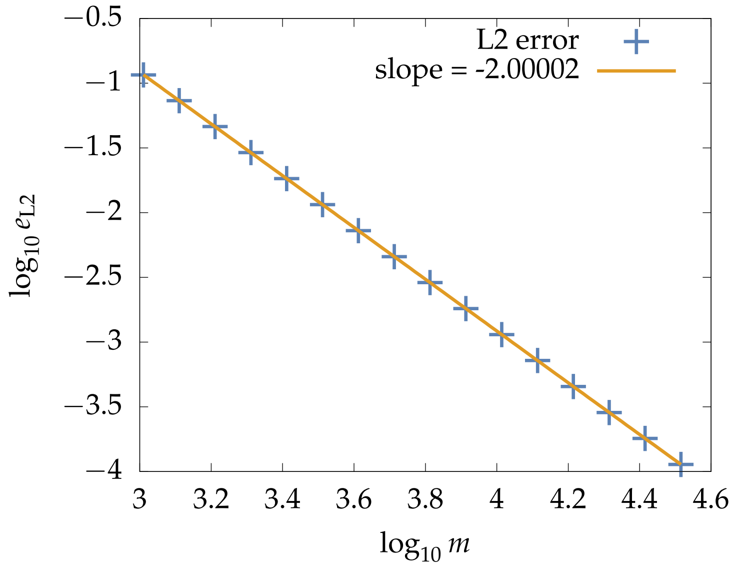

Figure 7: A log-log plot of the L2 error and number of grid points m for the initial condition $q(x, 0) = \exp(\sin x + \frac{1}{2}\sin 4x)$.

(e) The initial condition $q(x, t) = \max\{\frac{\pi}{2} - |x - \pi|,\ 0\}$ is shown in Figure 8. Snapshots at $T = {}^{T}/_{4}$, ${}^{T}/_{2}$, ${}^{3T}/_{4}$, and T for this initial condition are shown in Figure 9. The calculated L2 error for a range of m is shown in Figure 10. The method converges at first rather than second order for the case with the kinked initial condition.

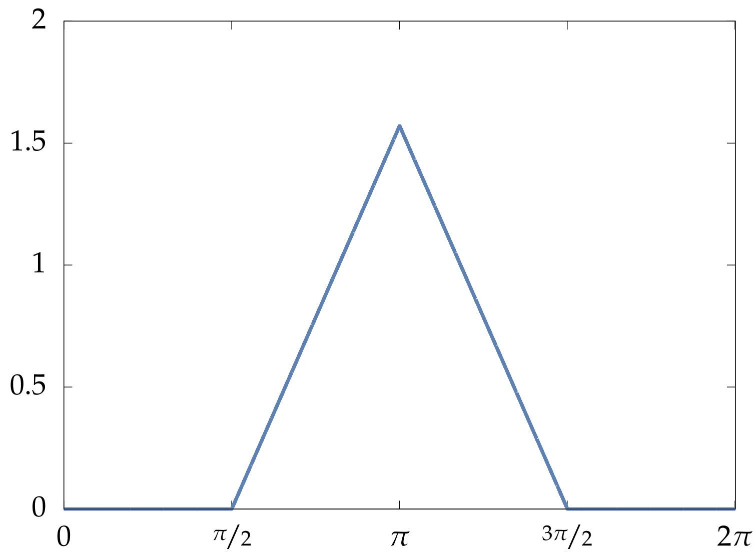

Figure 8: The initial condition $q(x,0) = \max\{\frac{\pi}{2} - |x - \pi|, 0\}$

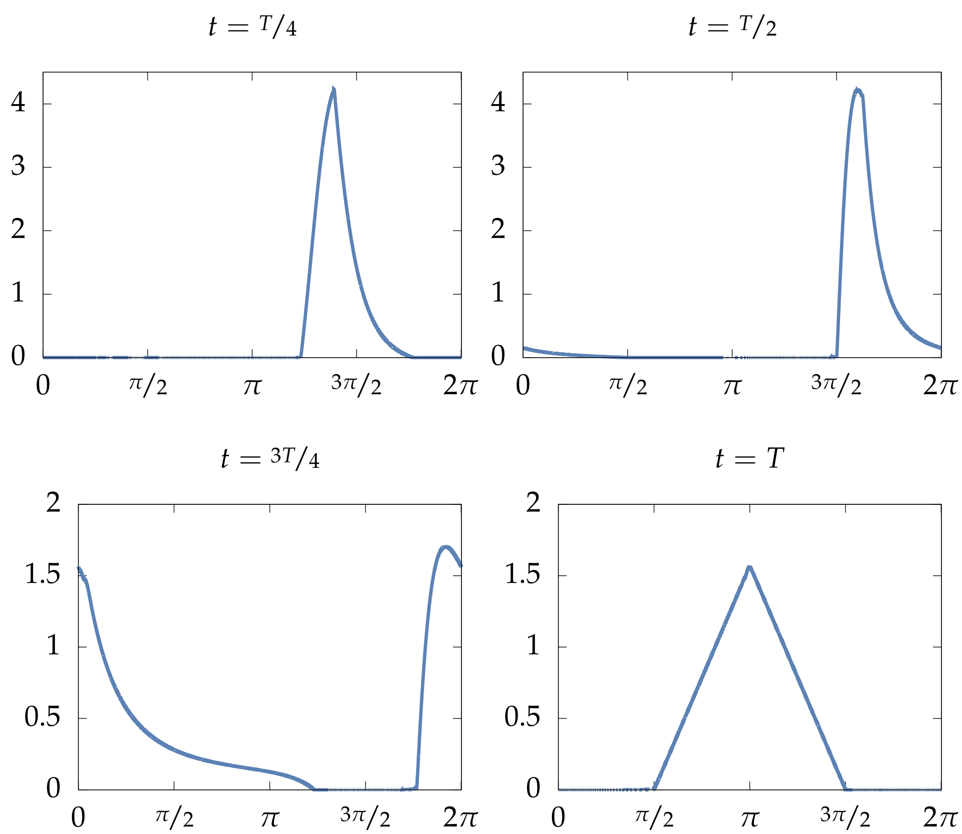

Figure 9: Snapshots of the solution given the initial condition $q(x,0) = \max\{\frac{\pi}{2} - |x - \pi|, 0\}$ at the times $t = {}^{T}/_{4}$, ${}^{T}/_{2}$, ${}^{3T}/_{4}$, and T for $m = 8192$

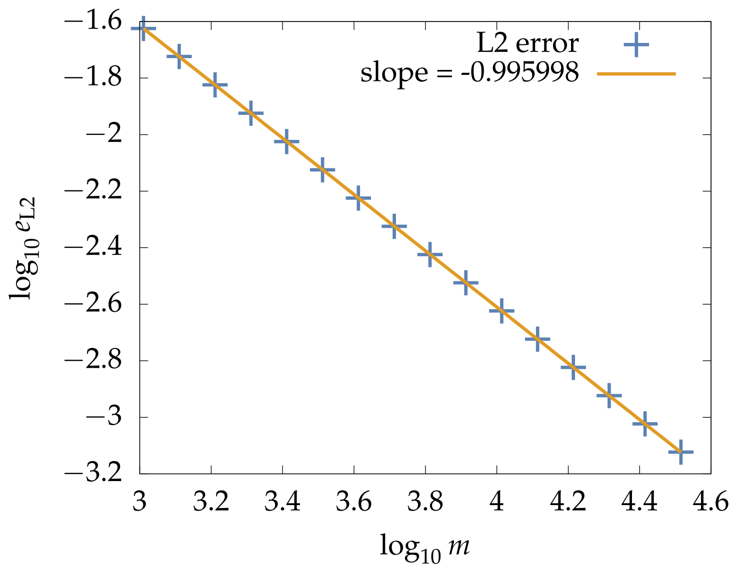

Figure 10: A log-log plot of the L2 error and number of grid points m for the initial condition $q(x, 0) = \max\{\frac{\pi}{2} - |x - \pi|, 0\}$.