AM225: Assignment 4 (due 5 PM, April 16)

Complete ***at least two*** *questions. If you submit answers for more, your grade will be calculated using the best scores.*

1. **A previous question.** Complete a question from HW3 that you have not already done. In particular, question 3 from HW3 on the finite-element method is recommended.

2. **Fitting a square peg in a round hole.** Consider two coordinate systems $\mathbf{v} = (v, w)$ and $\mathbf{x} = (x, y)$. In the $\mathbf{v}$ system, introduce the circle of unit radius, $\Omega = \{\mathbf{v} \in \mathbb{R} : \|\mathbf{v}\|_2 < 1\}$. In the $\mathbf{x}$ system, introduce the square $S = (-1, 1)^2$. The mapping $\mathbf{v} = f(\mathbf{x})$ defined by

$$v = x\sqrt{1 - \frac{y^2}{2}}, \qquad w = y\sqrt{1 - \frac{x^2}{2}} \tag{1}$$

is a differentiable map from S to Ω (Fig. 1). Consider the Poisson problem

$$-\nabla^2 u = f \tag{2}$$

on the circle Ω with Dirichlet condition $u(\mathbf{v}) = 0$ for $\mathbf{v} \in \partial\Omega$. Introduce an $N \times N$ grid of squares on S, each with side length $h = 2/N$. Let ϕ_i be a set of finite element basis functions on S. You can use the bilinear elements corresponding to the nodal basis at square corners.[1] Since the elements on the boundary can be neglected, this gives $(N-1)^2$ basis functions in total. On the circle, define basis functions ψ_i via the mapping from S, such that

$$\psi_i(\mathbf{v}) = \phi_i(f^{-1}(\mathbf{v})). \tag{3}$$

Consider representing your solution as

$$u(v, w) = \sum_i u_i \psi(v, w) \tag{4}$$

and solve the PDE problem using the Ritz–Galerkin method. Use a source term of $f(v, w) = e^{-v}(3 + (v-4)v + w^2)$, which gives the analytical solution $u(v, w) = (1 - v^2 - w^2)e^{-v}$. For this problem, the stiffness and mass matrix calculations will vary from element to element, and you can compute them by pulling back the integrals from Ω to S. For example

$$\begin{aligned} a(\psi_i, \psi_j) &= \int_\Omega \nabla_{\mathbf{v}}\psi_i \cdot \nabla_{\mathbf{v}}\psi_j \, d\mathbf{v} \\ &= \int_S (D^{-1}\nabla_{\mathbf{x}}\phi_i) \cdot (D^{-1}\nabla_{\mathbf{x}}\phi_j)(\det D) d\mathbf{x} \end{aligned} \tag{5}$$

where $D = \partial\mathbf{v}/\partial\mathbf{x}$ is the Jacobian of the mapping. In general these integrals will not analytically solvable, but they can be performed accurately and efficiently using Gaussian quadrature. In the class example files, you will find a demo program `3a_f_element/q2d_test.cc`, which demonstrates integrating a function on a square using Gaussian quadrature.[2]

Use your code for a variety of choices of N to calculate the L_2 error between the numerical solution and analytical solution, and determine the rate of convergence.

[1] You can use an alternative basis if you prefer, such as the C^1 Bogner–Fox–Schmit element.

[2] Note that because each integral like Eq. 5 is only done over a small patch, the integrand should be well-approximated by a low-order Taylor series, and thus not many quadrature points are require to achieve very high accuracy.

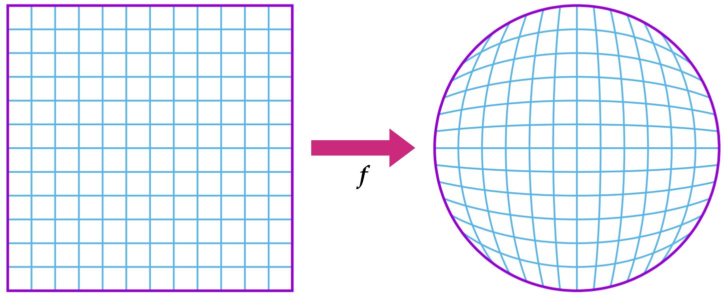

Figure 1: The mapping f from the square S to the circle Ω used in question 2.

3. **A generalization of the Lax–Wendroff scheme.** Consider the hyperbolic conservation equation

$$q_t + [A(x)q]_x = 0 \tag{6}$$

for a function on $q(x,t)$ on the periodic interval $[0, 2\pi)$. Let $A(x) = 2 + \frac{4}{3}\sin x$. Following the finite volume approach, divide the intervals into m domains $\mathcal{C}_i$ of length $h = \frac{2\pi}{m}$, for $i = \{0, 1, \ldots, m-1\}$. Let $Q_i^n \approx q((i + 1/2)h, n\Delta t)$ be the discretized solution at the center of each $\mathcal{C}_i$. The generalized Lax–Wendroff scheme for this equation is given by

$$Q_i^{n+1} = Q_i - \frac{\Delta t}{h}\left[\mathcal{F}^n_{i+1/2} - \mathcal{F}^n_{i-1/2}\right] \tag{7}$$

where the fluxes are

$$\mathcal{F}^n_{i-1/2} = \frac{A_{i-1}Q^n_{i-1} + A_i Q^n_i}{2} - \frac{A_{i-1/2}\Delta t}{2h}\left[A_i Q^n_i - A_{i-1}Q^n_{i-1}\right]. \tag{8}$$

Here, $A_i = A((i + 1/2)h)$ and $A_{i-1/2} = A(ih)$.

(a) By the considering the characteristics, or otherwise, show that the mathematical solution to Eq. 6 is time-periodic, so that $q(x, t + T) = q(x, t)$ for some $T > 0$. Determine T.[3]

(b) The CFL condition requires that $\Delta t \leq \frac{h}{c}$ for stability. What is c in this case?

(c) Implement Eq. 8 and set $\Delta t = \frac{h}{3c}$. Use the initial condition

$$q(x, 0) = \exp\left(\sin x + \tfrac{1}{2}\sin 4x\right). \tag{9}$$

For $m = 512$, plot snapshots of the solution for $t = 0, \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}, T$.[4]

(d) By considering a range of m (*e.g.* 256 and upward) with the initial condition in Eq. 9 calculate the L_2 norm between the numerical solution at $t = T$ and the exact answer. Determine the order of convergence.[5]

(e) Repeat parts (c) and (d) for the initial condition

$$q(x, 0) = \max\{\tfrac{\pi}{2} - |x - \pi|, 0\}. \tag{10}$$

[3] This is not as straightforward as it may seem. You may need to use a symbolic solver such as Mathematica.

[4] Since multiples of Δt do not exactly match these snapshot times, you may need to make a small adjustment to the timestep.

[5] When determining the order of convergence, you are interested in the asymptotic properites of error as m gets large. You can ignore initial transients in error.