AM205: An explicit calculation of a cubic spline

In the lectures, we discussed the *cubic spline* as a particular example of a piecewise polynomial interpolation of a collection of points $(x_0, y_0), \ldots, (x_n, y_n)$. A cubic spline $s(x)$ is defined as a set of n different cubic polynomials valid over each interval $[x_j, x_{j+1}]$ for $j = 0, 1, \ldots, n-1$. The cubics must satisfy the following conditions:

1. the cubic polynomials must agree with the data points at the end of each interval (giving $2n$ constraints),
2. the first derivatives of the cubics on either side of each point $x_1, x_2, \ldots, x_{n-1}$ must match (giving $n-1$ constraints),
3. the second derivatives of the cubics on either side of each point $x_1, x_2, \ldots, x_{n-1}$ must match (giving $n-1$ constraints).

Since each cubic has four unknown parameters, there are a total of $4n$ degrees of freedom. Since there are only $4n-2$ constraints listed above, it is necessary to add two additional constraints. There are several different methods of doing this, with a common approach being to make the second derivatives vanish at the end points, so that $s''(x_0) = s''(x_n) = 0$.

In practice, finding a cubic spline requires solving a large linear system for the $4n$ cubic polynomial coefficients in order to satisfy the above constraints. However, to understand exactly how the constraints affect the cubics, we consider here an example of constructing a cubic spline through the three points $(0,0), (1,0), (2,1)$, which is small enough to be explicitly calculated. In this case, s is given by two polynomials over the intervals $[0,1]$ and $[1,2]$. To simplify the analysis, we introduce the four cubics

$$\begin{aligned} c_0(x) &= x^2(3-2x), \\ c_1(x) &= -x^2(1-x), \\ c_2(x) &= (x-1)^2 x, \\ c_3(x) &= 2x^3 - 3x^2 + 1. \end{aligned} \tag{1}$$

on the interval $x \in [0,1]$. These cubics are plotted in Fig. 1; they are linearly independent and form a basis to specify any cubic. As shown in Table 1, their function values and derivatives at $x = 0$ and $x = 1$ have useful properties that make it easy to satisfy the cubic spline constraints. The second derivatives of these cubics are given by

$$c_0''(x) = 6 - 12x, \qquad c_1''(x) = 6x - 2, \qquad c_2''(x) = 6x - 4, \qquad c_3''(x) = 12x - 6. \tag{2}$$

Let us now construct the cubic spline using these basis functions. Over the interval $[0,1]$, there must be zero contribution from $c_0(x)$ and $c_3(x)$, since $s(0) = s(1) = 0$. Over the interval from $[1,2]$, we use the same basis as in Eq. 1, but shift them by using the argument $x-1$ instead of x, so that the properties of Table 1 will be true at the endpoints $x = 1$ and $x = 2$. There must be a zero contribution from $c_3(x-1)$ to ensure $s(1) = 0$, and a single

Function	$c_i(1)$	$c_i'(1)$	$c_i'(0)$	$c_i(0)$
$c_0(x) = x^2(3-2x)$	1	0	0	0
$c_1(x) = -x^2(1-x)$	0	1	0	0
$c_2(x) = (x-1)^2 x$	0	0	1	0
$c_3(x) = 2x^3 - 3x^2 + 1$	0	0	0	1

Table 1: Properties of the four cubics used as a basis to compute the cubic spline.

$c_0(x-1)$ term to ensure that $s(2) = 1$. The cubic spline of the three points can therefore be written as

$$s(x) = \begin{cases} \alpha c_1(x) + \beta c_2(x) & \text{for } 0 \leq x < 1, \\ c_0(x-1) + \eta c_2(x-1) + \gamma c_1(x-1) & \text{for } 1 \leq x \leq 2. \end{cases} \tag{3}$$

for some unknown parameters α, β, γ, and η. To ensure that the first derivative matches at $x = 1$, we must have $\eta = \alpha$ and hence η can be eliminated. Ensuring that the second derivatives at $x = 1$ match gives the constraint

$$4\alpha + 2\beta = 6 - 4\alpha - 2\gamma. \tag{4}$$

Setting $s''(0) = 0$ gives

$$0 = -2\alpha - 4\beta \tag{5}$$

and setting $s''(2) = 0$ gives

$$0 = -6 + 4\gamma + 2\alpha. \tag{6}$$

Equation 5 shows that $\beta = -\frac{\alpha}{2}$. Eliminating β from Eq. 5 therefore gives

$$3\alpha = 6 - 4\alpha - 2\gamma \tag{7}$$

and hence

$$7\alpha + 2\gamma = 6. \tag{8}$$

Equations 6 and 8 are two simultaneous equations that can be solved to give $\alpha = \frac{1}{2}$ and $\gamma = \frac{5}{4}$. Hence $\beta = -\frac{1}{4}$ and

$$s(x) = \begin{cases} \frac{1}{2}c_1(x) - \frac{1}{4}c_2(x) & \text{for } 0 \leq x < 1, \\ c_0(x-1) + \frac{1}{2}c_2(x-1) + \frac{5}{4}c_1(x-1) & \text{for } 1 \leq x \leq 2. \end{cases} \tag{9}$$

Figure 2 shows a plot of this cubic spline, and as expected it smoothly passes through the three data points. The spline becomes approximately straight at the two endpoints, due to the enforcement of the conditions $s''(0) = s''(2) = 0$.

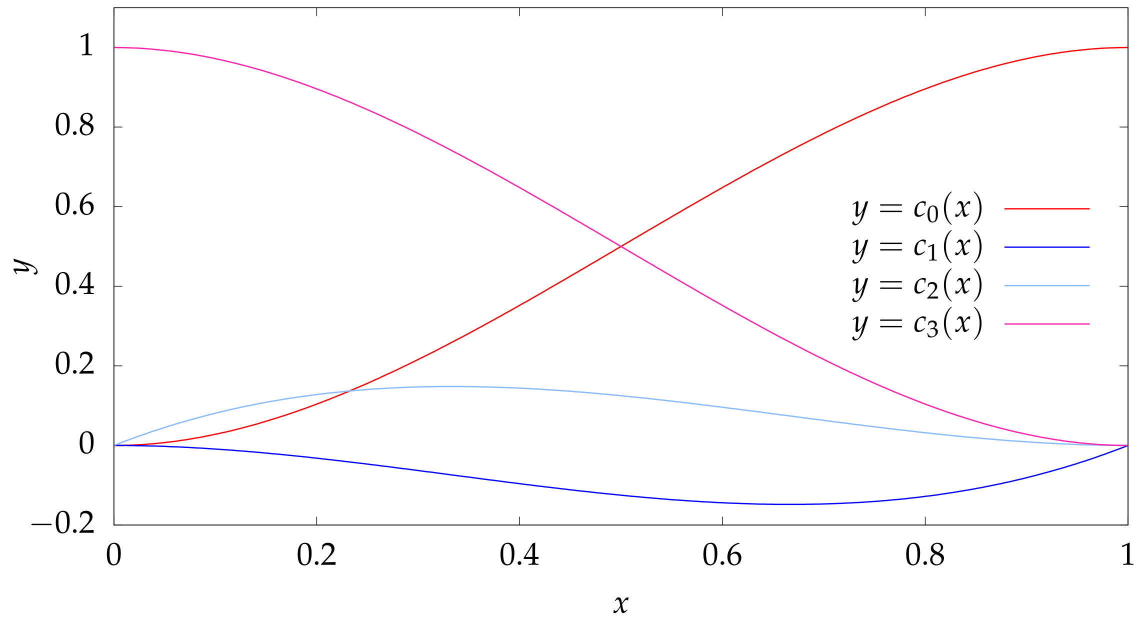

Figure 1: The four cubic basis functions used to construct the cubic spline.

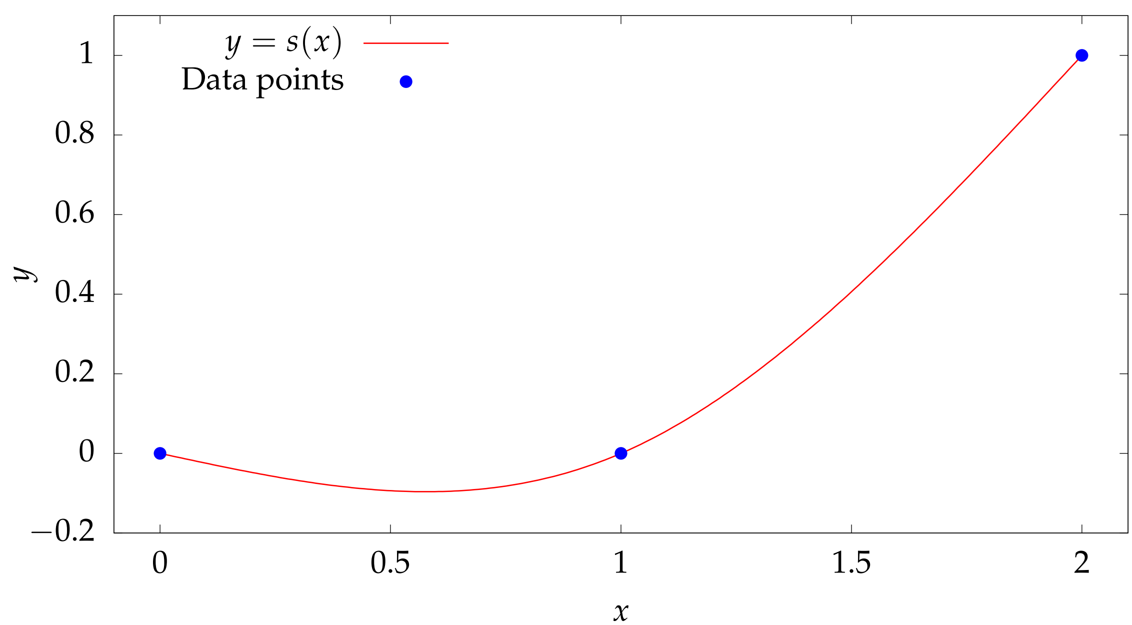

Figure 2: Plot of the cubic spline through the three data points.