AM205: Assignment 3 (due 5 PM, October 28)

*For this assignment, first complete problems 1, 2, 3, and 4, and then complete **either** problem 5 (on theory) or problem 6 (on an application). If you submit answers for both, then your grade will be calculated using the maximum score from the two.*

1. **Convergence rates of two integrals.** Consider the function

$$f(x) = \frac{1}{\frac{5}{4} - \cos x}. \tag{1}$$

   (a) It can be shown that

$$I_A = \int_0^{\pi/3} f(x)dx = \frac{8\pi}{9}. \tag{2}$$

   Write a program to numerically evaluate I_A using the composite trapezoid rule with n intervals of equal size h, for $n = 1, 2, \ldots, 50$, and make a log–log plot of the absolute error as a function h. On the same axes, overlay the composite trapezoid error bound from the lectures, $E(h) = \frac{h^2\pi}{36}\|f''\|_\infty$, and show that your numerically computed results are smaller than the bound.

   (b) It can be shown that

$$I_B = \int_0^{2\pi} f(x)dx = \frac{8\pi}{3}. \tag{3}$$

   Write a program to numerically evaluate I_B using the composite trapezoid rule with n equally sized intervals, for $n = 1, 2, \ldots, 50$. Make a log–log plot of the absolute error as a function h. Does the absolute error scale like h^m for some m?

   (c) **Optional.** Use residue calculus to prove the integral in Eq. 3.

2. (a) **Adaptive integration.** Given that the cubic Legendre polynomial is $P_3(x) = \frac{1}{2}x(5x^2 - 3)$, derive the 3-point Gauss quadrature rule on the interval $[-1, 1]$ by evaluating the relevant integrals by hand. Demonstrate that this quadrature rule integrates all polynomials up to the expected degree exactly.

   (b) In the lectures we discussed a method of evaluating integrals $\int_a^b f(x)dx$ by adaptively refining the calculation in regions where the function varies rapidly. To begin, a tolerance level T is introduced, and the calculation starts using a single integration interval $[a, b]$. Let $I_{a,b}$ be the integral of f over this interval using the three-point quadrature rule from part (a). In addition, define $c = \frac{a+b}{2}$ and calculate $\hat{I}_{a,b} = I_{a,c} + I_{c,b}$ as a more refined estimate of the integral. Then $E_{a,b} = |I_{a,b} - \hat{I}_{a,b}|$ is an estimate of the error of $I_{a,b}$. If $E_{a,b} < Tl$, where l is the length of the interval, then the error is acceptable, and the method can terminate. Otherwise this interval must be subdivided into $[a, c]$ and $[c, b]$, and the above procedure must be applied to these two intervals. The procedure must be applied recursively until the errors become smaller than the tolerance.

   Write a function to implement this adaptive integration scheme. Using $T = 10^{-6}$, apply it to the integrals

$$\int_{-1}^{9/4} (x^m - x^2 + 1)dx \tag{4}$$

for $m = 4, 5, 6, 7, 8$. For each case, report the value of the integral, the total estimated error (by summing the relevant $E_{\alpha,\beta}$ terms), and the total number of intervals that are used.

(c) Use your adaptive integration routine from part (a) and $T = 10^{-6}$ to evaluate the three integrals

$$\int_{-1}^{1} |x| dx, \qquad \int_{-1}^{2} |x| dx, \qquad \int_{0}^{1} x^{4/5} \sin \frac{1}{x}\, dx.$$

For each case, report the value of the integral, the total estimated error (by summing the relevant $E_{\alpha,\beta}$ terms), and the total number of intervals that are used.

3. **Integration of a family of functions.** For a given parameter $\phi \in (0, 1)$, define a sequence of functions recursively on the interval $x \in [-\frac{1}{2}, \frac{1}{2}]$ by putting $f_0(x;\phi) = |x|$ and $f_k(x;\phi) = |f_{k-1}(x;\phi) - \phi^k|$. Let g be the limiting function, given by

$$g(x;\phi) = \lim_{k\to\infty} f_k(x;\phi). \tag{5}$$

For a given ϕ, the function g can be numerically approximated by f_n, where n is taken to be the smallest value such that $\phi^n < 10^{-16}$.

(a) Plot $g(x; \frac{1}{3})$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

(b) Using your adaptive integration routine from question 2 and a tolerance of $T = 10^{-6}$, make a plot of

$$I(\phi) = \int_{-1/2}^{1/2} g(x;\phi) dx \tag{6}$$

for $\phi \in (0, 1)$. To do this, you should calculate $I(\phi)$ at 99 points $\phi = 0.01, 0.02, \ldots, 0.99$. In addition, make a plot of the number of integration intervals used as a function of ϕ. Approximately what range of values of ϕ require the most intervals?

4. **Error analysis of a numerical integration rule.** Applying the midpoint quadrature rule (*i.e.* $n = 0$ Newton–Cotes with the quadrature point at the midpoint) on the interval $[t_k, t_{k+1}]$ to $y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(t, y(t)) dt$ leads to the implicit *midpoint method*,

$$y_{k+1} = y_k + hf(t_{k+1/2}, (y_k + y_{k+1})/2), \tag{7}$$

where $t_{k+1/2} = t_k + \frac{h}{2}$.

(a) Use Taylor series expansions to show that the order of accuracy of this method is 2.

(b) What is the stability region of the method for the equation $y' = \lambda y$? In other words, for what values of $\bar{h} = h\lambda \in \mathbb{C}$ is the method stable?

5. (a) **A multi-step method.** Consider solving the differential equation $y' = f(t, y)$ at time-points t_k with corresponding numerical solutions y_k. The multi-step Nyström numerical method is based upon the integral relation

$$y(t_{k+1}) = y(t_{k-1}) + \int_{t_{k-1}}^{t_{k+1}} f(t, y) dt. \tag{8}$$

Derive an implicit multi-step numerical method by approximating the integrand $f(t, y)$ with the polynomial interpolant using the function values at t_{k-2}, t_{k-1}, t_k, and t_{k+1}. Your method should have the form

$$y_{k+1} = y_{k-1} + h(\alpha f_{k-2} + \beta f_{k-1} + \gamma f_k + \eta f_{k+1}) \tag{9}$$

where α, β, γ, and η are constants to be determined, h is the timestep interval size, and $f_l = f(t_l, y_l)$ for an arbitrary l.

(b) Find the exact solution to the second-order differential equation

$$y''(t) + 2y'(t) + 17y(t) = 0. \tag{10}$$

subject to the initial conditions $y(0) = 1$, $y'(0) = 0$.

(c) Write Eq. 10 as a coupled system of two first-order differential equations for $\mathbf{y} = (y, v) = (y, y')$. Solve the system over the interval $0 \le t \le 3$ with a timestep of $h = 0.02$ using your multi-step method from part (a).[1]

Before Eq. 9 can be applied, $\mathbf{y}_1$ and $\mathbf{y}_2$ must be calculated accurately. Use one of the following two approaches:

i. set them based on the exact solution from part (b),
ii. calculate them using the classical fourth-order Runge–Kutta method.

Plot the exact and numerical solutions over the range $0 \le t \le 3$.

Make a log–log plot of the absolute error between the numerical and exact values of y at $t = 3$ as a function of h, over the range from $h = 10^{-3}$ to $h = 10^{-1}$. Show that your method is fourth-order accurate.

(d) Suppose that instead of setting $\mathbf{y}_1$ and $\mathbf{y}_2$ accurately, you instead make use of forward Euler steps. Create a log–log plot of the absolute error of y at $t = 3$ as a function of h. Determine the order of accuracy, and discuss why this is the case.

6. **Asteroid collision.** An asteroid is detected near the Earth and the Moon in the plane of their orbits. Use your ODE-solving skills to determine if there is danger of collision.

This problem can be treated as a circular restricted three-body problem, which considers the motion of an object of negligible mass in the presence of two massive gravitating bodies that circularly orbit each other. The motion of the object is assumed to be restricted to the 2D plane of the circular orbit of the two massive bodies.

The Earth and Moon have circular orbits about each other. If we work in a co-rotating frame, we can consider the Earth and Moon to be at fixed locations and then we only need to determine the trajectory of the asteroid. In dimensionless units, fix the Earth at position $(x, y) = (0, 0)$ and the Moon at position $(x, y) = (1, 0)$. The Earth has radius $R_{\text{Earth}} = 0.02$ and the Moon has radius $R_{\text{Moon}} = 0.005$.

The asteroid (which has negligible mass) will move according to the gravitational forces it feels from the Earth and the Moon. The motion of the asteroid is described by its position

[1] Even though Eq. 9 is an implicit method and would be hard to solve in general, the linearity of the system in Eq. 10 means that for this case, the update equation for $\mathbf{y}_{k+1}$ can be derived analytically.

$\mathbf{x}(t) = (x(t), y(t))$, and velocity $\mathbf{v}(t) = (u(t), v(t))$ as functions of time. Throughout the asteroid's movement, the *Jacobi integral*, J, is a constant of motion, and is given by

$$J(x, y, u, v) = (x - \mu)^2 + y^2 + \frac{2(1 - \mu)}{\sqrt{x^2 + y^2}} + \frac{2\mu}{\sqrt{(x - 1)^2 + y^2}} - u^2 - v^2 \tag{11}$$

where $\mu = 0.01$ is the ratio of the Moon's mass to the total mass of the Earth and Moon. The asteroid's equations of motion are given by the system of ODEs,

$$\begin{aligned} x' &= -\frac{1}{2}\frac{\partial J}{\partial u}, & y' &= -\frac{1}{2}\frac{\partial J}{\partial v}, \\ u' &= v + \frac{1}{2}\frac{\partial J}{\partial x}, & v' &= -u + \frac{1}{2}\frac{\partial J}{\partial y}. \end{aligned} \tag{12}$$

(a) Write the system of ODEs to be solved in terms of x, y, u, and v.

(b) In the subsequent analysis, we will need to determine if the computed asteroid trajectory intersects the Earth or Moon. To aid with this, write a program that determines whether a line segment from $\mathbf{x}_0 = (x_0, y_0)$ to $\mathbf{x}_1 = (x_1, y_1)$ intersects a circle of radius R centered at $(0, 0)$.

(c) Two observations of the asteroid's position have been made:

$$\mathbf{x}_{\text{obs}}(0) = (1.0798, 0), \qquad \mathbf{x}_{\text{obs}}(0.02) = (1.0802, -0.0189). \tag{13}$$

However, there is some uncertainty in these observations. The true position $\mathbf{x}(t)$ is given by $\mathbf{x}(t) = \mathbf{x}_{\text{obs}}(t) + (E_x, E_y)$ where for each separate observation, E_x and E_y are independent normal random variables with mean zero and standard deviation 0.002. Assuming that the asteroid trajectory is linear between $t = 0$ and $t = 0.02$, calculate $\mathbf{v}(0)$ in terms of $\mathbf{x}(0)$ and $\mathbf{x}(0.02)$.

Using your favorite ODE solver,[2] integrate the system from part (a) over the range $0 \leq t \leq 10$, for ten initial conditions for $\mathbf{x}(0)$ and $\mathbf{v}(0)$, sampled according to the procedure described above. Make a figure showing the Earth and Moon as circles with the given radii, and the ten computed trajectories overlaid. Use axis ranges of $-1.4 \leq x \leq 2$ and $-1.2 \leq y \leq 1.2$.

(d) Now simulate 2,500 or more trajectories over the range $0 \leq t \leq 10$ using initial conditions from part (c). For each trajectory, determine whether

i. it first collides with the Moon,

ii. it first collides with the Earth,

iii. it collides with neither.

To determine collisions for a given trajectory, take the numerically computed asteroid positions $\mathbf{x}_0, \mathbf{x}_1, \ldots$, and use your program from part (b) to find whether each line segment from $\mathbf{x}_k$ to $\mathbf{x}_{k+1}$ intersects with either the Moon or Earth.

Based on the trajectories, calculate the probability that asteroid will collide with the Moon, and the probability that the asteroid will collide with the Earth. Plot one or more trajectories that collide with the Earth, and one or more that collide with the Moon.

[2] In Python, a good choice would be the `odeint` routine. In MATLAB, the `ode45` routine could be used, using the additional command `odeset('RelTol',1e-10,'AbsTol',1e-10)` to improve accuracy.

(e) **Optional.** If this ODE system were solved exactly, the Jacobi integral would be exactly constant as a function of time, since it is a conserved quantity of the system. We can therefore use the value of the Jacobi integral to check the robustness of our numerical approximation. For the initial conditions $\mathbf{x}(0) = (1.08, 0)$ and $\mathbf{v}(0) = (0, -0.49)$, plot the relative error in the Jacobi integral, $|J(t) - J(0)|/J(0)$, for $t \in [0, 10]$ and explain what you observe.

(f) **Optional.** Find trajectories over the interval $0 \leq t \leq 200$ that neither undergo a collision, nor escape the Earth–Moon system. You should find several different types of trajectory. Estimate the probability of each type occurring.

(g) **Optional.** In part (c), we assumed that the trajectory was linear from $t = 0$ to $t = 0.02$, but it will actually follow a curved path due to the gravitational interaction. Improve your analysis in part (d) to handle this, and determine the alteration in the collision probabilities.