AM205: Assignment 0

This assignment will not be graded, and consists of several warm-up problems that can be used to test and refresh your mathematical and programming skills. You do not need to submit your answers.

1. The Chebyshev polynomials $T_k(x)$ can be defined using the recursive relation

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$$

   and $T_0(x) = 1$, $T_1(x) = x$. Evaluate and plot the Chebyshev polynomial of degree 5 at 101 evenly spaced points in the interval $x \in [-1, 1]$. Draw a 2D surface plot of the function $T_3(x)T_5(y)$ on a 101×101 grid on the domain $(x, y) \in [-1, 1]^2$.

2. Use the iteration

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$$

   to approximate $\sqrt{a}$. This is known as Heron's formula[1] and it is equivalent to the Newton–Raphson method for the function $f(x) = x^2 - a$. Choose an initial starting value of $x_0 = a$ and iterate until $|x_{k+1} - x_k| < \epsilon$ for some tolerance ϵ. Determine the number of iterations required to compute $\sqrt{5}$ for the cases of $\epsilon = 10^{-3}$ and $\epsilon = 10^{-9}$.

3. (a) Let $f(x) = \tan x$ and consider the finite-difference approximation

$$f_{\text{diff},2}(x; h) = \frac{f(x+h) - f(x-h)}{2h}$$

   Make a log–log plot the relative error in $f_{\text{diff},2}(x; h)$ at $x = 1$ as a function of h for $h = 10^{-k}$, using $k = 1, 1.5, 2, 2.5, \ldots, 15.5, 16$. Use linear regression to fit the relative error y to the straight line

$$\log y = \log(\alpha) + \beta \log h$$

   for some coefficients α and β. Show that $\beta \approx 2$, meaning that the approximation is second-order accurate.

   (b) Repeat the analysis for the stencil

$$f_{\text{diff}}(x; h) = \frac{-11f(x) + 18f(x+h) - 9f(x+2h) + 2f(x+3h)}{6h}$$

   and determine the rate of convergence β.

4. In the first lecture we discussed Archimedes' method of finding an error bound for π by drawing inscribed and superscribed regular polygons inside a circle with radius 1.

[1] Heron of Alexandria, 10–70 AD.

(a) Let a_n and b_n be the areas of inscribed and superscribed regular polygons with 3×2^n sides, respectively. The case of $n = 0$ therefore corresponds to inscribed and superscribed equilateral triangles. Use geometry to show that $a_0 = \frac{3}{4}\sqrt{3}$ and $b_0 = 3\sqrt{3}$.

(b) Show that

$$\frac{2}{b_{n+1}} = \frac{1}{a_{n+1}} + \frac{1}{b_n}, \qquad a_{n+1}^2 = a_n b_n$$

and write a program to evaluate (a_n, b_n) for $n = 0, 1, \ldots, 40$. In addition, calculate $c_n = \frac{1}{2}(a_n + b_n)$.

(c) Make a semi-log plot of the absolute error $|a_n - \pi|$ and $|c_n - \pi|$ as a function of n. How fast do these two sequences converge to π? Is there a difference in the convergence rate between a_n and c_n?