Harvard Applied Mathematics 205

Group Activity: Fluid instability in coffee[1]

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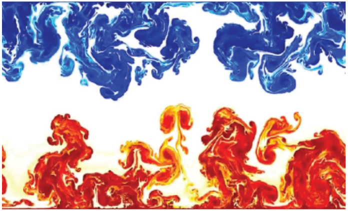
November 16 & 18, 2021

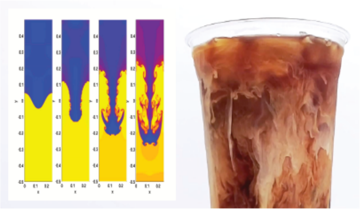
[1] Based off notes from (a) ES220 Spring 2021 (b) MIT 18.357 Spring 2021 (c) AM205 Fall 2020 workshop by Nick Derr.

Outline

Examples of fluid instability



Rayleigh-Bénard instability (convection)



Rayleigh-Taylor instability

Kelvin-Helmholtz instability

Plateau-Rayleigh instability

[1] Rayleigh–Bénard instability: `https://youtu.be/OM0l2YPVMf8`

[2] Rayleigh–Taylor instability: `https://fyfluiddynamics.com/2015/11/pouring-cream-in-coffee-produces-some-of-the-most/`

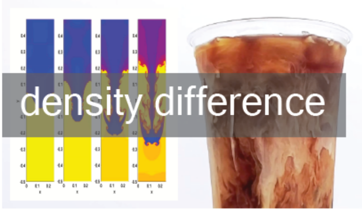
[2] Rayleigh–Taylor instability: `https://en.wikipedia.org/wiki/Rayleigh%E2%80%93Taylor_instability`

[3] Kelvin–Helmholtz instability: `https://en.wikipedia.org/wiki/Kelvin%E2%80%93Helmholtz_instability`

[4] Plateau–Rayleigh instability: `https://youtu.be/wzEiZdcss88`

Examples of fluid instability

Rayleigh-Bénard instability (convection)



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Navier–Stokes equations

Generally, the governing equations for many fluid dynamics problems are the incompressible[2] Navier–Stokes equations:

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}_{\text{body}} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0} \tag{1}$$

where $\rho, \mathbf{u}, \mu$ are fluid density, velocity, and dynamic viscosity.

The material derivative $\dfrac{D}{Dt}$ is defined as a nonlinear operator:

$$\boxed{\frac{D}{Dt} \overset{\text{def}}{=} \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla} \tag{2}$$

which describes the time rate of change of some physical quantity of a material element moving with the flow at velocity $\mathbf{u}$.

[2] Incompressibility means the fluid density is constant.

Navier–Stokes equations: background

Although the equations are named after Claude-Louis Navier and George Gabriel Stokes, they are in essence just mass conservation and momentum conservation equations.

One fundamental hypothesis in studying continuum mechanics[3] is that the material we are studying, at the scale of interest, is a continuum—rather than individual particles or molecules.

[3] *e.g.* fluid dynamics and solid mechanics

Navier–Stokes equations: mass conservation

Mass conservation states that the rate of change in mass is equal to the net flux. In differential form, we have:

$$\boxed{\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0} \tag{3}$$

Expand the divergence operator, we have

$$\begin{aligned} &\implies \quad \underbrace{\rho_t + \mathbf{u} \cdot \nabla \rho}_{\text{material derivative}} + \rho(\nabla \cdot \mathbf{u}) = 0 \\ &\implies \quad \frac{D\rho}{Dt} = -\rho(\nabla \cdot \mathbf{u}) \end{aligned} \tag{4}$$

If ρ is constant everywhere (in space and time), we have

$$\frac{D\rho}{Dt} = 0 \quad \implies \quad \underbrace{\nabla \cdot \mathbf{u} = 0}_{\substack{\text{incompressibility} \\ \text{constraint}}} \tag{5}$$

Navier–Stokes equations: momentum conservation

Newton's second law states that the net force in a system of interest is equal to the change in momentum w.r.t. time, we have

$$\frac{D(\rho \mathbf{u})}{Dt} = \sum \text{forces} \tag{6}$$

Categorizing forces[4] into body forces (*e.g.* gravity, electrostatic force) and surface force (*e.g.* pressure, shear). We use $\mathbf{f}_{\text{body}}$ for all body forces on the small continuum element, and represent surface forces with Cauchy stress $\boldsymbol{\sigma}$ (derivation omitted here[5], we have

$$\frac{D(\rho \mathbf{u})}{Dt} = \underbrace{\nabla \cdot \boldsymbol{\sigma}}_{\substack{\text{surface} \\ \text{forces}}} + \underbrace{\mathbf{f}_{\text{body}}}_{\substack{\text{body} \\ \text{forces}}} \tag{7}$$

[4] Precisely speaking, here we are working with force density: $\mathbf{F} = m\mathbf{a} \implies \mathbf{F}/\text{area} = p = \rho \mathbf{a}$.

[5] If interested in the derivation, you can read p.p.46-49 in ES241 notes "Finite deformation: general theory".

Navier–Stokes equations: momentum conservation

Assume incompressible material[6], we can simplify the derivative:

$$\begin{aligned} \implies \quad & \rho \frac{D\mathbf{u}}{Dt} + \mathbf{u}\cancel{\frac{D\rho}{Dt}} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_{\text{body}} \\ \implies \quad & \boxed{\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_{\text{body}}} \end{aligned} \tag{8}$$

which gives us the Cauchy momentum equation.

We can further simplify the Cauchy stress $\boldsymbol{\sigma}$ by assuming isotropic material[7]:

$$\boldsymbol{\sigma} = -\underbrace{p\mathbb{1}}_{\text{pressure}} + \underbrace{\boldsymbol{\tau}}_{\text{deviatoric stress}} \tag{9}$$

Therefore, the Cauchy momentum equation becomes

$$\boxed{\rho\left(\partial_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \nabla\cdot\boldsymbol{\tau} + \mathbf{f}_{\text{body}}} \tag{10}$$

[6] For compressible material, $D\rho/Dt$ can be handled by mass conservation, so the same equation holds.

[7] Isotropic material means properties of a material are identical in all directions.

Navier–Stokes equations: momentum conservation

So far, the Cauchy momentum equation is true for all materials (*e.g.* solids or any continuum). We need to specify this equation for fluid, which is realized through a constitutive model between $\boldsymbol{\tau}$ and $\mathbf{u}$.

Assume the fluid is Newtonian[8], the constitutive model is

$$\boldsymbol{\tau} = \underbrace{\mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)}_{\text{viscous stress}} \tag{11}$$

where μ is a constant called the dynamic viscosity. The viscous stress represents the fluid resistance to applied deformation.

In summary, the incompressible Navier–Stokes equations are

$$\boxed{\underbrace{\rho\left(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right)}_{\text{inertial force}} = -\nabla p + \underbrace{\mu\nabla^2\mathbf{u}}_{\text{viscous force}} + \mathbf{f}_{\text{body}}} \tag{12}$$

[8] In the Newtonian fluid model, viscous stresses are linearly proportional to the deformation rate. Examples where this is a good model include air, water, and glycerol. Detailed calculations can be found at Wikipedia: Derivation of the NSE—incompressible Newtonian fluid.

7 steps for solving instability problems

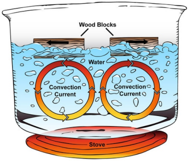
1. Set up the problem and illustrate the system.
2. Write down the governing equations.
3. Nondimensionalize the equations with dimensionless numbers. (Extract dominant terms by scaling nondimensional equations.)
4. Find the steady state solutions.
5. Apply linear stability theory and linearize about steady state.
6. Simplify the PDEs and specify initial and boundary conditions.
7. Solve analytically or numerically for non-trivial solutions.

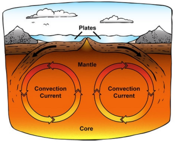
Rayleigh–Bénard convection

A type of thermal convection: a layer of fluid is heated from below, and the fluid develops a regular pattern of convection cells. RBC is often studied for its analytical and experimental accessibility.

Henri Bénard did the experiment in 1900, and Lord Rayleigh did the mathematical analysis[9] in 1916.

Thermal convection is the force that drives fluid motion due to a temperature gradient. Such fluid flows are everywhere, like boiling water, the sun and all of its consequences, and geophysical flows.

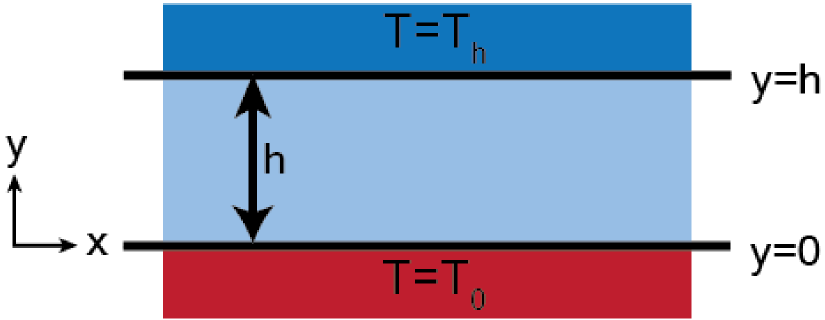




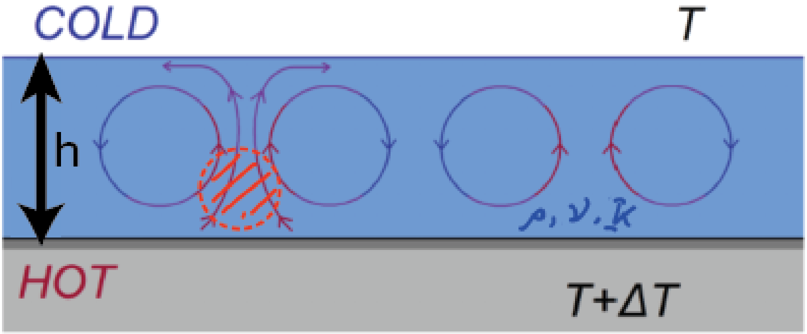
[9] There was a small twist to the story: Lord Rayleigh's original proof did not consider surface tension; however, Bénard's experiment was done in a millimetric dish, where Marangoni stress dominates. Nonetheless, the nomenclature stays.

[9] Convection cells (currents): `https://taylorsciencegeeks.weebly.com/blog/convection-cells-currents`

Rayleigh–Bénard convection: setup



Rayleigh–Bénard convection: setup



Derive the onset of instability

If the temperature difference between the two plates is increased by small steps, the state of rest remains stable until ΔT reaches a certain critical value, where an organized cellular motion starts.

Our goal is to find this critical temperature difference ΔT when the onset of instability happens. We will be using techniques including

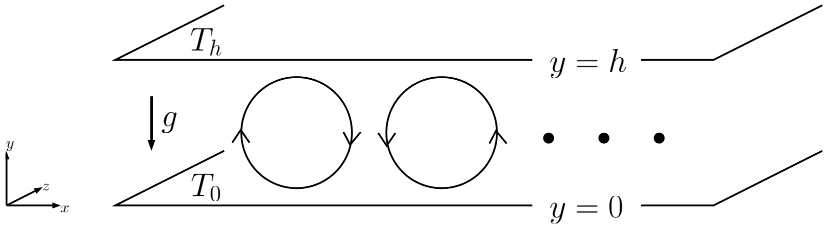
- nondimensionalization
- linear stability theory

and see how this very complicated instability problem/phenomenon can be simplified to an eigenvalue problem.

We will be following the 7 steps in the note "Derive the onset of instability for Rayleigh–Bénard convection".

Derive the onset of instability: Step 1

Set up the problem and illustrate the system.



variables	parameters
fluid density ρ	volume coefficient of thermal expansion α
fluid velocity $\mathbf{u}$	gravitation g
temperature T	dynamic viscosity μ, kinematic viscosity $\nu = \mu/\rho$
pressure p	thermal diffusivity κ

Derive the onset of instability: Step 2

Write down the governing equations.

momentum: $$\rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} - \rho_0 \left(1 + \alpha(T - T_0)\right) g \hat{y}$$

mass: $$\nabla \cdot \mathbf{u} = 0$$

energy: $$\partial_t T + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T$$

equation of state: $$\rho = \rho_0 \left(1 + \alpha(T - T_0)\right)$$

Derive the onset of instability: Step 3

Nondimensionalize the equations with dimensionless numbers.

momentum: $$\frac{1}{Pr}\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \nabla^2 \mathbf{u} - Ra\, T\hat{y}$$

mass: $$\nabla\cdot\mathbf{u} = 0$$

energy: $$\partial_t T + (\mathbf{u}\cdot\nabla)T = \nabla^2 T$$

with dimensionless number:

Prandtl number: $$Pr = \frac{\nu}{\kappa} = \frac{\text{momentum diffusivity}}{\text{thermal diffusivity}}$$

Rayleigh number: $$Ra = \frac{\alpha\Delta T g h^3}{\nu\kappa} = \frac{\text{buoyant force}}{\text{viscous force}}$$

Derive the onset of instability: Step 4

Find the steady state solutions.

$$u^{SS} = 0$$
$$v^{SS} = 0$$
$$\partial_y p^{SS}(y) = -Ra\, T^{SS} g$$
$$T^{SS}(y) = y$$

Derive the onset of instability: Step 5

Apply linear stability theory and linearize about steady state.

$$\text{momentum } \hat{x}: \quad \frac{1}{Pr}\partial_t \tilde{u} = -\partial_x \tilde{p} + \nabla^2 \tilde{u}$$

$$\text{momentum } \hat{y}: \quad \frac{1}{Pr}\partial_t \tilde{v} = -\partial_y \tilde{p} + \nabla^2 \tilde{v} - Ra\,\tilde{T} g$$

$$\text{energy} : \quad \partial_t \tilde{T} + \tilde{v}\partial_y T^{SS} = \nabla^2 \tilde{T}$$

$$\text{mass} : \quad \partial_x \tilde{u} + \partial_y \tilde{v} = 0$$

Derive the onset of instability: Step 6

Simplify the governing equations to ODE.

$$(D^2 - k^2)^3 v^* = -Ra\, k^2 v^*$$

with linearized nondimensional boundary conditions:

$$\text{at } y = 0, 1: \quad v^* = Dv^* = (D^2 - k^2)^2 v^* = 0$$

Derive the onset of instability: Step 7

Solve the eigenvalue problem.

Let $D_k = -\frac{(D^2 - k^2)^3}{k^2} = -\frac{(\partial_y^2 - k^2)^3}{k^2}$ and $f = v^*$, then

$$D_k f = Ra\ f \qquad (13)$$

There are multiple ways of solving this matrix equation:

- analytics/algebra (with `Mathematica`)
- Newton's method, shooting method (root-finding methods)
- represent D_k as a matrix and find the eigenvalues

Numerical methods for eigenvalue problem

Assume the ansatz $v^* = A\exp(my)$ to the ODE, we have

$$(m^2 - k^2)^3 + Ra\, k^2 = 0 \tag{14}$$

where m has 6 roots:

$$\begin{cases} m_{1,2} = \pm\sqrt{k^2 - \sqrt[3]{Ra}\, k^{2/3}} \\ m_{3,4} = \pm\sqrt{k^2 + \frac{1}{2}(1 - i\sqrt{3})\sqrt[3]{Ra}\, k^{2/3}} \\ m_{5,6} = \pm\sqrt{k^2 + \frac{1}{2}(1 + i\sqrt{3})\sqrt[3]{Ra}\, k^{2/3}} \end{cases} \tag{15}$$

A general homogeneous solution would be

$$\begin{aligned} v^* = {} & A\exp(m_1 y) + B\exp(m_2 y) + C\exp(m_3 y) \\ & + D\exp(m_4 y) + E\exp(m_5 y) + F\exp(m_6 y) \end{aligned} \tag{16}$$

Numerical methods for eigenvalue problem

A, B, C, D, E, F are constants to be determined based on B.C.s:

$$
\begin{cases}
v^*(0) = 0 & A + B + C + D + E + F = 0 \\
v^*(1) = 0 & A\exp(m_1) + B\exp(m_2) + C\exp(m_3) \\
 & +D\exp(m_4) + E\exp(m_5) + F\exp(m_6) = 0 \\
Dv^*(0) = 0 & Am_1 + Bm_2 + Cm_3 + Dm_4 + Em_5 + Fm_6 = 0 \\
Dv^*(1) = 0 & Am_1\exp(m_1) + Bm_2\exp(m_2) + Cm_3\exp(m_3) \\
 & +Dm_4\exp(m_4) + Em_5\exp(m_5) + Fm_6\exp(m_6) = 0 \\
(D^2 - k^2)^2 v^*(0) = 0 & A(m_1^2 - k^2)^2 + B(m_2^2 - k^2)^2 + C(m_3^2 - k^2)^2 \\
 & +D(m_4^2 - k^2)^2 + E(m_5^2 - k^2)^2 + F(m_6^2 - k^2)^2 = 0 \\
(D^2 - k^2)^2 v^*(0) = 0 & A(m_1^2 - k^2)^2\exp(m_1) + B(m_2^2 - k^2)^2\exp(m_2) \\
 & +C(m_3^2 - k^2)^2\exp(m_3) + D(m_4^2 - k^2)^2\exp(m_4) \\
 & +E(m_5^2 - k^2)^2\exp(m_5) + F(m_6^2 - k^2)^2\exp(m_6) = 0
\end{cases}
$$

Numerical methods for eigenvalue problem

We now have a matrix equation $\mathbf{Mb} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \exp(m_1) & \exp(m_2) & \exp(m_3) & \exp(m_4) & \exp(m_5) & \exp(m_6) \\ m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ m_1\exp(m_1) & m_2\exp(m_2) & m_3\exp(m_3) & m_4\exp(m_4) & m_5\exp(m_5) & m_6\exp(m_6) \\ (m_1^2-k^2)^2 & (m_2^2-k^2)^2 & (m_3^2-k^2)^2 & (m_4^2-k^2)^2 & (m_5^2-k^2)^2 & (m_6^2-k^2)^2 \\ (m_1^2-k^2)^2\exp(m_1) & (m_2^2-k^2)^2\exp(m_2) & (m_3^2-k^2)^2\exp(m_3) & (m_4^2-k^2)^2\exp(m_4) & (m_5^2-k^2)^2\exp(m_5) & (m_6^2-k^2)^2\exp(m_6) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

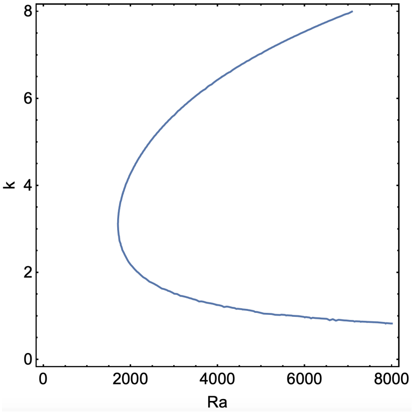
where we numerically solve $\det(\mathbf{M}) = 0$.

When k is fixed, $\mathbf{M}$ only depends on the Rayleigh number. Thus, the matrix equation simplifies to a 1D equation of Ra, and we can solve for different Ra for different fixed k values.

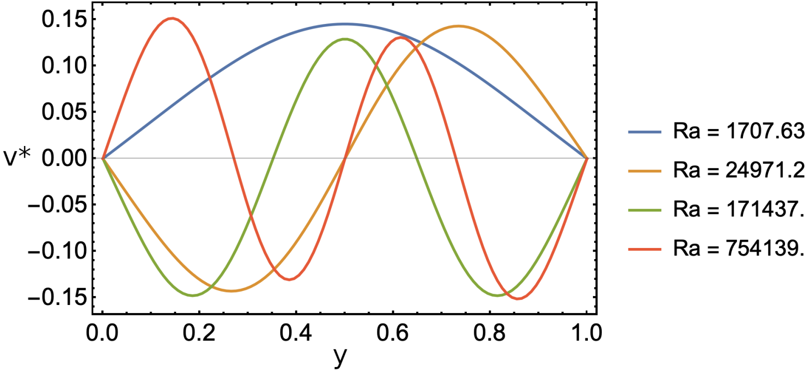
Let us see the Mathematica notebook for three numerical approaches: analytical, shooting method, and matrix equation.

Numerical methods for eigenvalue problem

The critical Rayleigh number is found at $Ra_c = 1707.63$ numerically, which is consistent with the literature $Ra > 1708$ instability onset.



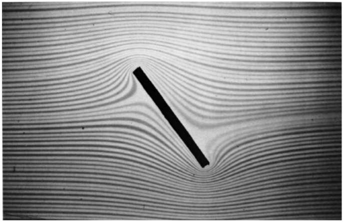
Numerical methods for eigenvalue problem

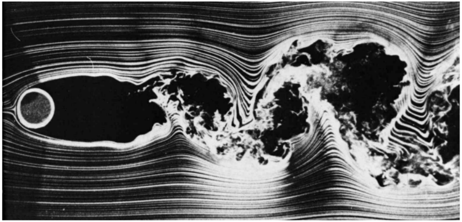


Streamlines, pathlines, and streaklines

We can visualize the velocity field of fluid with three types of lines[10]:

- streamlines: curves that are tangent to the velocity field
- pathlines: trajectories of individual fluid particles follow
- streaklines: dye steadily injecting to fluid at fixed location extends along streaklines



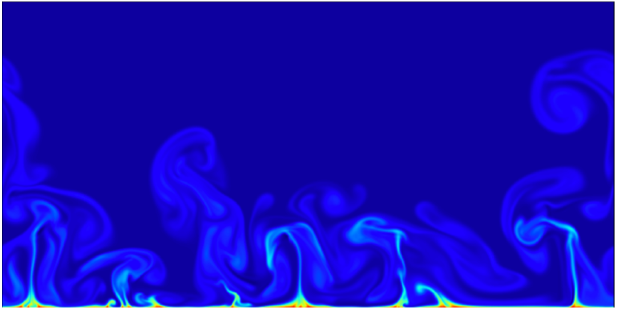


[10]Wikipedia: Streamlines, streaklines and pathlines
[11]Images taken from "An alblum of fluid motion".

Schlieren imaging

We can use Schlieren visualization to examine the flow structure in the temperature field,

$$Sch = \exp\left(-k\frac{|\nabla T|}{\max|\nabla T|}\right) \tag{18}$$

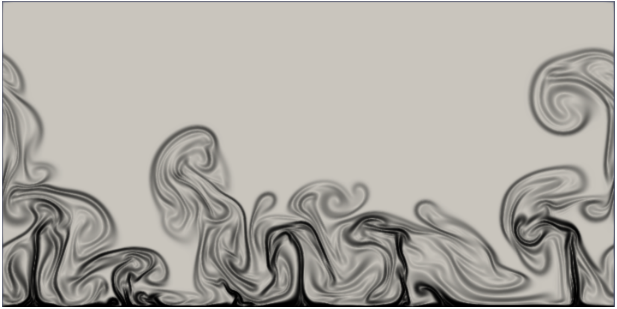


(temperature field)

Schlieren imaging

We can use Schlieren visualization to examine the flow structure in the temperature field,

$$Sch = \exp\left(-k\frac{|\nabla T|}{\max|\nabla T|}\right) \quad (19)$$

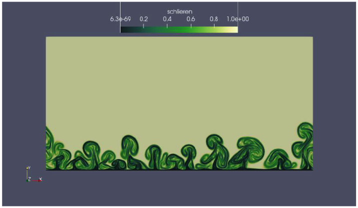


(Schlieren imaging)

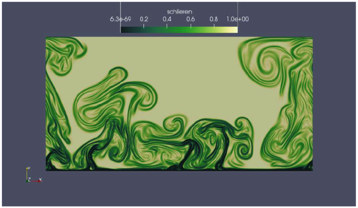
Turbulent Rayleigh–Bénard convection

Poster: Numerical simulation of Rayleigh–Bénard convection

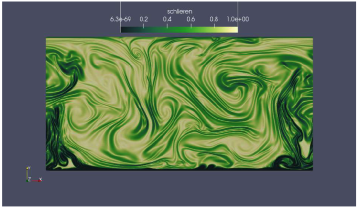
Schlieren Image at Frame 044



Schlieren Image at Frame 062



Schlieren Image at Frame 102



Schlieren Image at Frame 206