Math 126: Homework 13 solutions

1. (a) The initial data can be parameterized according to

$$x = f(s) = s, \qquad y = g(s) = 0, \qquad u(f(s), g(s)) = h(s) = \sin s.$$

The characteristics are determined by the ODE system

$$\frac{dx}{dt} = \frac{2xy}{1+y^2}, \tag{1}$$

$$\frac{dy}{dt} = 1, \tag{2}$$

$$\frac{dz}{dt} = \frac{-zy}{4}. \tag{3}$$

Equation 2 gives $y = t$, and hence

$$\frac{dx}{dt} = \frac{2xt}{1+t^2}$$

so

$$\frac{dx}{x} = \frac{2t\,dt}{1+t^2}$$

which gives

$$\log x = C + \log(1+t^2)$$

for some constant C. Hence, the characteristic passing through $(s, 0)$ at $t = 0$ is

$$x = s(1+t^2).$$

Equation 3 then gives

$$\frac{dz}{dt} = -\frac{zt}{4}$$

so

$$\frac{dz}{z} = -\frac{t\,dt}{4}$$

which can be integrated to give

$$\log z = -\frac{t^2}{8} + D$$

for some constant D. By using the initial data, it can be seen that

$$z = Z(s,t) = e^{-t^2/8} \sin s$$

The family of characteristics for different values of s is shown is Fig. 1

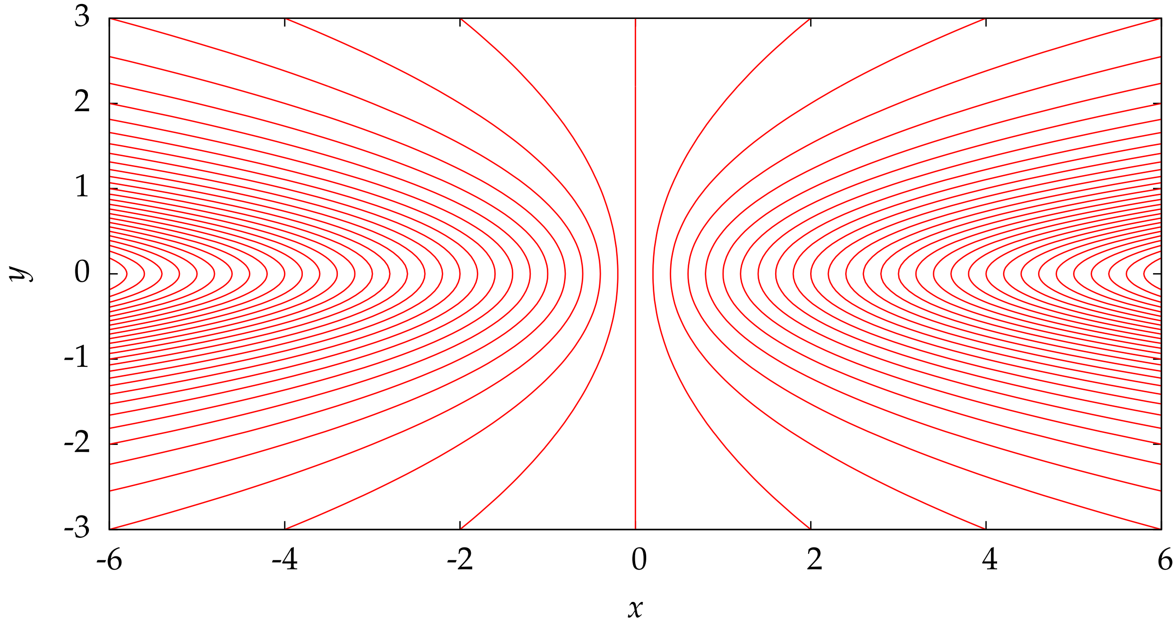

Figure 1: Characteristics of the partial differential equation considered in question 1, projected into the xy plane.

(b) From part (a), it can be seen that

$$s = \frac{x}{1+y^2}, \qquad t = y,$$

and hence

$$u(x,y) = Z(s,t) = e^{-y^2/8} \sin\left(\frac{x}{1+y^2}\right). \tag{4}$$

(c) Taking partial derivatives of Eq. 4 gives

$$\begin{aligned} u_x(x,y) &= \frac{e^{-y^2/8}}{1+y^2} \cos\left(\frac{x}{1+y^2}\right) \\ u_y(x,y) &= e^{-y^2/8}\left(-\frac{y}{4}\sin\left(\frac{x}{1+y^2}\right) - \frac{2xy}{(1+y^2)^2}\cos\left(\frac{x}{1+y^2}\right)\right) \end{aligned}$$

Hence

$$\frac{2xy}{1+y^2}u_x = \frac{2xye^{-y^2/8}}{(1+y^2)^2}\cos\left(\frac{x}{1+y^2}\right)$$

and so

$$\frac{2xy}{1+y^2}u_x + u_y = -\frac{ye^{-y^2/8}}{4}\sin\left(\frac{x}{1+y^2}\right) = -\frac{yu}{4}$$

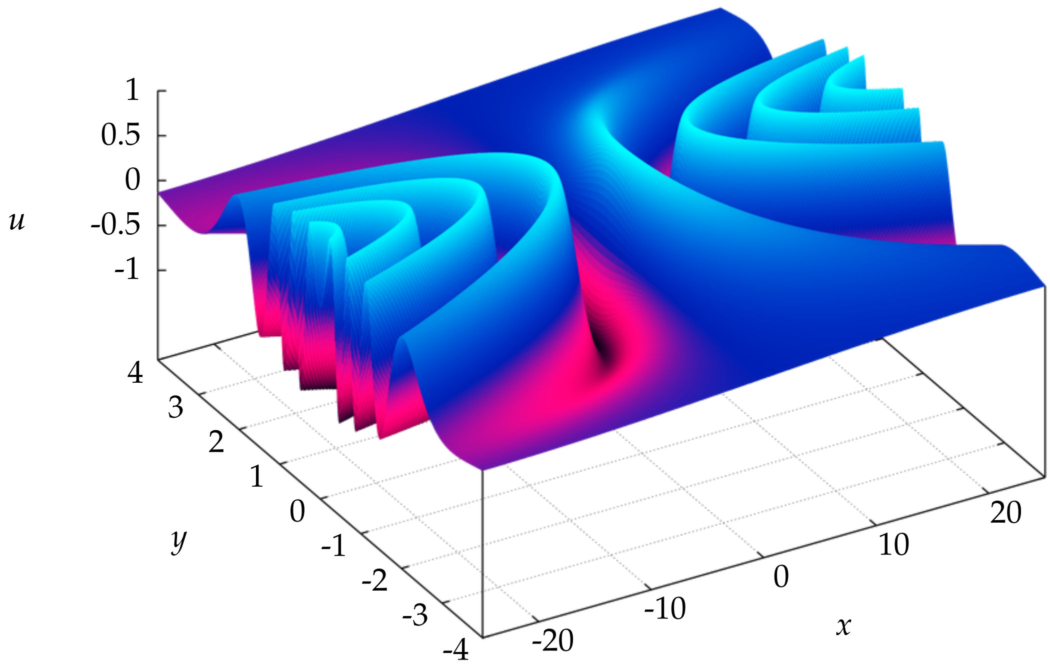

Figure 2: Three-dimensional plot of the solution $u(x, y)$ of question 1.

since the terms involving cosine will cancel each other. Thus $u(x, y)$ satisfies the PDE. In addition

$$u(x,0) = e^0 \sin\left(\frac{x}{1+y^2}\right) = \sin x$$

so the solution satisfies the initial data.

(d) Figure 2 shows a plot of the solution. It can be seen how the sinusoidal initial data is transported along the parabolic characteristics. A small amount of decay is also visible in the y direction, due to the presence of the $\exp(-y^2/8)$ term in the solution.

2. The characteristics will be given by

$$\frac{dx}{dt} = -y, \qquad \frac{dy}{dt} = x$$

which is equivalent to

$$0 = x\,dx - (-y)dy = x\,dx + y\,dy.$$

This has a first integral

$$\psi(x,y) = x^2 + y^2$$

and thus the characteristics are given by the curves where $\psi(x,y)$ is constant. Hence a typical characteristic is

$$x^2 + y^2 = R^2$$

for some constant R^2, which corresponds to a circle of radius R. The characteristics are a family of concentric circles centered on the origin. Since the solution is constant along each characteristic a general solution can then be written as

$$u(x,y) = G(x^2 + y^2)$$

for some function $G(\lambda)$. The initial condition states that

$$u(x,0) = h(x) = G(x^2)$$

and similarly

$$u(-x,0) = h(-x) = G((-x)^2) = G(x^2).$$

For a solution to exist it is therefore necessary that $h(x) = h(-x)$, so h is even. This is reasonable, since any characteristic circle of radius R will intersect the initial data at $\pm R$, so the values must be consistent. For any even choice of h, a general solution can be written as

$$G(\lambda) = h(\sqrt{\lambda})$$

defined for $\lambda \geq 0$, so that

$$u(x,y) = h(\sqrt{x^2 + y^2}).$$

Hence h being even is both a necessary and sufficient condition for a general solution to exist.

3. (a) The characteristics are given by the ODE system

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = 3x^2, \qquad \frac{dz}{dt} = 3zx^2$$

and the initial data can be can be parameterized according to

$$x = f(s) = s, \qquad y = g(s) = 0, \qquad u(f(s), g(s)) = h(s).$$

Hence

$$x = X(s,t) = s + t \tag{5}$$

so

$$\frac{dy}{dt} = 3(s+t)^2$$

and therefore

$$y = Y(s,t) = (s+t)^3 - s^3 \tag{6}$$

where the constants of integration have been chosen so that the characteristics pass through the initial data when $t = 0$.

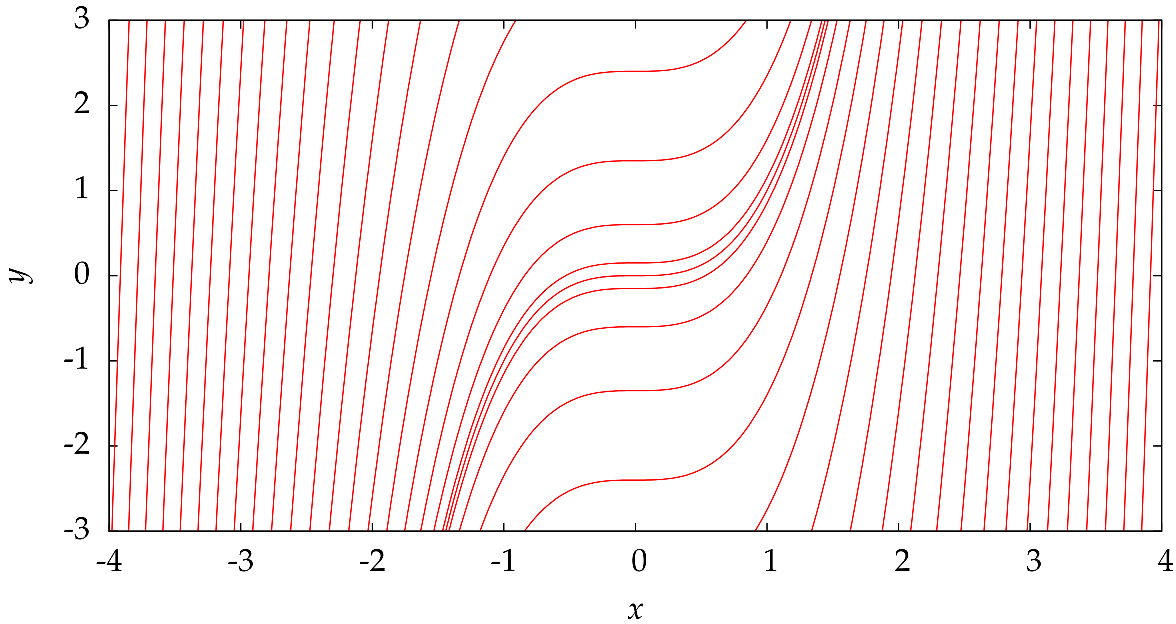

Figure 3: Characteristics for the partial differential equation considered in question 3.

(b) The characteristics are shown in Fig. 3. The Jacobian is given by

$$\begin{aligned} J(s,t) &= \begin{vmatrix} X_s(s,t) & Y_s(s,t) \\ X_t(s,t) & Y_t(s,t) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 3(s+t)^2 - 3s^2 \\ 1 & 3(s+t)^2 \end{vmatrix} \\ &= 3(s+t)^2 - 3(s+t)^2 + 3s^2 = 3s^2 \end{aligned}$$

and thus $J(s,0) = 3s^2$. At $s = 0$, the Jacobian vanishes, implying that the characteristics and the initial data are tangent, which is in agreement with shapes of characteristics shown in Fig. 3. As described in the textbook, for a C^1 solution to exist,

$$\text{rank}\begin{pmatrix} a(0,0,z_0) & b(0,0,z_0) & c(0,0,z_0) \\ f'(0) & g'(0) & h'(0) \end{pmatrix} = 1.$$

This can be written as

$$\text{rank}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & h'(0) \end{pmatrix} = 1$$

and hence $h'(0) = 0$.

(c) To find the general solution, first consider the differential equation for z given above:

$$\frac{dz}{dt} = 3zx^2.$$

Thus

$$\frac{dz}{z} = 3(s+t)^2 dt$$

and hence

$$\log z = (s+t)^3 + C$$

for some integration constant C. Using the initial condition shows that

$$z = Z(s,t) = h(s)e^{(s+t)^3 - s^3}.$$

Equations 5 and 6 show that

$$s = S(x,y) = \sqrt[3]{x^3 - y}$$

and

$$t = T(x,y) = x - \sqrt[3]{x^3 - y}$$

so the general solution is

$$u(x,y) = Z(S(x,y), T(x,y)) = h(\sqrt[3]{x^3 - y})e^y$$

If $h(x) = x^2$, then $h'(0) = h(0) = 0$, so the condition from part (b) is satisfied. However,

$$u(x,y) = (x^3 - y)^{2/3} e^y$$

and hence

$$u_y(x,y) = \left(\frac{-2}{3\sqrt[3]{x^3 - y}} + (x^3 - y)^{2/3} \right) e^y$$

so

$$u_y(0,y) = \left(\frac{2}{3\sqrt[3]{y}} + (-y)^{2/3} \right) e^y$$

which is unbounded as $y \to 0$ and thus u is not C^1 in a neighborhood of the x axis. This implies that while the condition from part (b) is necessary for a C^1 solution to exist, it is not sufficient. Note that other choices of $h(s)$ would lead to a C^1 solution: if $h(s) = s^3$, then

$$u(x,y) = (x^3 - y)e^y$$

which is C^1 everywhere.

4. This problem can be expressed as

$$F(x,y,u,u_x,u_y) = u_x^2 + u_y^2 - 4u = 0.$$

Following the notation of the textbook, the last two variables of F can be denoted as $p = u_x$ and $q = u_y$, so

$$F(x, y, u, p, q) = p^2 + q^2 - 4u. \tag{7}$$

The characteristic system is then given by

$$\frac{dx}{dt} = F_p = 2p \tag{8}$$

$$\frac{dy}{dt} = F_q = 2q \tag{9}$$

$$\frac{dz}{dt} = pF_p + qF_q = 2p^2 + 2q^2 \tag{10}$$

$$\frac{dp}{dt} = -F_x - pF_u = 4p \tag{11}$$

$$\frac{dq}{dt} = -F_y - qF_u = 4q. \tag{12}$$

The initial data is

$$f(s) = \cos s, \qquad g(s) = \sin s, \qquad h(s) = 1.$$

Let $\varphi(s)$ and $\psi(s)$ be the initial conditions on p and q respectively. Equation 7 requires that

$$\varphi^2 + \psi^2 = 4$$

and the condition $h'(s) = \varphi(s)f'(s) + \psi(s)g'(s)$ requires that

$$-\varphi \sin s + \psi \cos s = 0.$$

This gives one solution as

$$\varphi = 2\cos s, \qquad \psi = 2\sin s$$

and a second solution as

$$\varphi = -2\cos s, \qquad \psi = -2\sin s.$$

Consider the first solution for the initial data. Equation 12 shows that $q = Ce^{4t}$ for some constant C, and to be consistent with the initial data, this leads to

$$q = 2e^{4t}\sin s.$$

Equation 9 then becomes

$$\frac{dy}{dt} = 4e^{4t}\sin s$$

and hence

$$y = e^{4t}\sin s + C_2$$

for some constant C_2. Using the initial data $g(s) = \sin s$ gives

$$y = e^{4t} \sin s.$$

A similar consideration of Eqs. 11 and 8 gives

$$p = 2e^{4t} \cos s, \qquad x = e^{4t} \cos s.$$

Thus

$$\frac{dz}{dt} = 8e^{8t}(\cos^2 s + \sin^2 s) = 8e^{8t}$$

and hence

$$z = e^{8t}.$$

To find a general solution $u(x, y)$, note that

$$x^2 + y^2 = e^{8t}(\cos^2 s + \sin^2 s) = e^{8t}$$

and thus

$$u(x, y) = z = x^2 + y^2.$$

Now consider the second case for the initial data. Then

$$q = -2e^{4t} \sin s$$

and hence Eq. 9 becomes

$$\frac{dy}{dt} = -4e^{4t} \sin s,$$

and by considering the initial data $g(s) = \sin s$,

$$y = (2 - e^{4t}) \sin s.$$

Similarly

$$p = -2e^{4t} \cos s, \qquad x = (2 - e^{4t}) \cos s.$$

The equation for dz/dt is the same with the same initial data and hence $z = e^{8t}$ still holds. To find a general solution $u(x, y)$, note that

$$x^2 + y^2 = (2 - e^{4t})^2$$

so

$$e^{4t} = 2 - \sqrt{x^2 + y^2}$$

and thus the general solution is

$$u(x, y) = \left(2 - \sqrt{x^2 + y^2}\right)^2.$$

Note that extending this solution outside of the disk $x^2 + y^2 = 4$ would require further analysis, since $t \to -\infty$ as $x^2 + y^2 \to 4$.