Math 126: Homework 12 solutions

1. For this equation, the flux is given by $q(u) = u^2/2$. Let the trajectory of the shock starting from $x = 1$ be given by $s(t)$. By the Rankine–Hugoniot condition,

$$\dot{s}(t) = \frac{q(1) - q(0)}{1 - 0} = \frac{1}{2}$$

and thus $s(t) = 1 + \frac{t}{2}$ initially. From the regions where $u(x, 0) = 0$, the characteristics will have velocity zero, and from the regions where $u(x, 0) = 1$, the characteristics will have velocity one. This will lead to a rarefaction fan of characteristics in the region $0 < x < t$ where the characteristics are

$$x = q'(u)t$$

and hence $u_{\text{fan}}(x, t) = x/t$. The front of the rarefaction fan is at $x = t$, and will hit the shock when

$$t = s(t) = 1 + \frac{t}{2}$$

corresponding to $t = 2$. After this time, the shock's velocity will be given by

$$\dot{s}(t) = \frac{q(u_{\text{fan}}(s(t), t)) - q(0)}{u_{\text{fan}}(s(t), t) - 0} = \frac{u_{\text{fan}}(s(t), t)}{2} = \frac{s(t)}{2t},$$

and thus the shock's velocity will satisfy the ODE

$$\frac{ds}{dt} = \frac{s}{2t}$$

which can be integrated to

$$s(t) = K\sqrt{t}$$

for some constant K. Since $s(2) = 2$, $K = \sqrt{2}$, and so

$$s(t) = \sqrt{2t}$$

for $t \geq 2$. Hence the general solution is

$$u(x, t) = \begin{cases} 0 & \text{for } x < 0 \\ x/t & \text{for } 0 \leq x < t \text{ and } x < s(t) \\ 1 & \text{for } x \geq t \text{ and } x < s(t) \\ 0 & \text{for } x \geq s(t). \end{cases}$$

The characteristics and shock are shown in Fig. 1(a). Since $q(u) = u^2/2$, it follows that $q''(u) = 1 > 0$. In this situation, the entropy condition implies that $q'(u_+) < \dot{s} < q'(u_-)$ at a shock, which is satisfied for the shock in this solution. Hence the solution is physical.

An unphysical integral solution can be constructed as a function with shocks, such that it is a classical solution away from the shocks, and satisfies the Rankine–Hugoniot condition at the shocks. In a similar manner to the examples considered in the textbook, a possible solution is

$$u(x,t) = \begin{cases} 0 & \text{for } 2x < t \\ 1 & \text{for } t \leq 2x < t+2 \\ 0 & \text{for } 2x \geq t+2. \end{cases}$$

Away from the two shocks, the function is constant and is therefore a classical solution. At the first shock, the Rankine–Hugoniot condition gives

$$\dot{s}_1 = \frac{q(1) - q(0)}{1 - 0} = \frac{1}{2}$$

which is consistent with the solution. Similarly, at the second shock, the Rankine–Hugoniot condition gives

$$\dot{s}_2 = \frac{q(0) - q(1)}{0 - 1} = \frac{1}{2}$$

which is also consistent. Hence the function is an integral solution to the equation. The characteristics are shown in Fig. 1. It can be seen that the solution is unphysical since characteristics emanate from the shock at $2x = t$. In this situation, the entropy condition that $u_- > u_+$ at a shock is violated for $2x = t$.

2. (a) The shock's velocity is given by

$$\dot{s}(t) = \frac{q(\frac{\rho_m}{2}) - q(0)}{\frac{\rho_m}{2}} = v_m \left(1 - \frac{\frac{\rho_m}{2}}{\rho_m}\right) = \frac{v_m}{2}$$

and hence the density is

$$\rho(x,t) = \begin{cases} 0 & \text{for } x < \frac{v_m t}{2} \\ \frac{\rho_m}{2} & \text{for } x \geq \frac{v_m t}{2}. \end{cases}$$

(b) Substituting $\rho(x,t) = U(x - vt) = U(\xi)$ into the differential equation gives

$$-v\frac{dU}{d\xi} + v_m \left(1 - \frac{2U}{\rho_m}\right) \frac{dU}{d\xi} = \epsilon \frac{d^2U}{d\xi^2}$$

which can be integrated to give

$$-vU + q(U) = \epsilon \frac{dU}{d\xi} + C$$

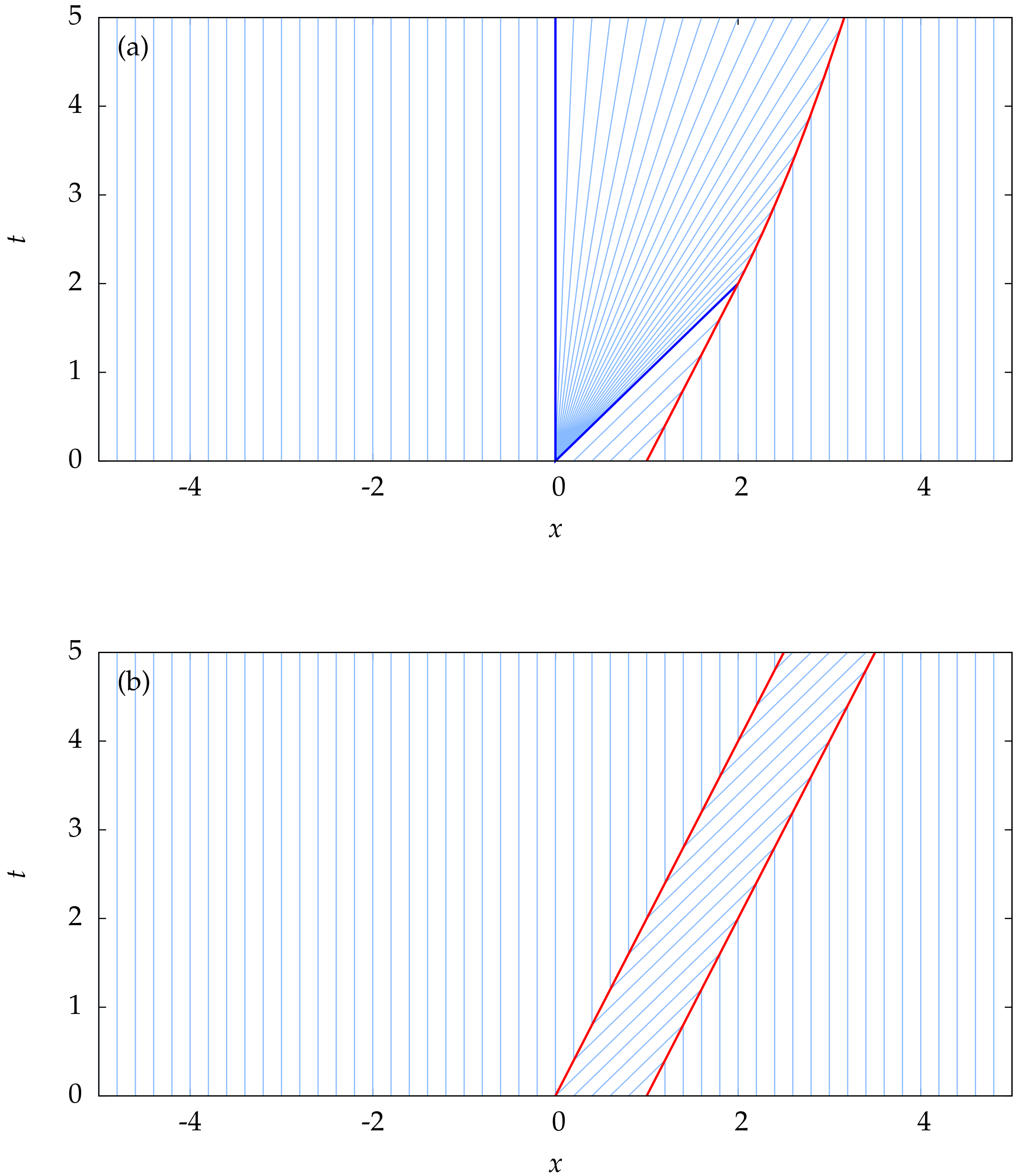

Figure 1: Characteristics for the problem in question 1, for (a) the physical solution and (b) an example of a unphysical solution where the entropy condition is not satisfied. The characteristics are shown in light blue, with those marking the boundaries between the different regions shown in dark blue. The shocks are shown in red.

for some constant C. By assuming that $dU/d\xi \to 0$ as $\xi \to \pm\infty$, it follows that

$$\begin{aligned} -\frac{v\rho_m}{2} + q(\rho_m/2) &= C \\ -0 + q(0) &= C \end{aligned}$$

so $C = 0$ and $v = v_m/2$. The differential equation for U therefore becomes

$$\epsilon \frac{dU}{d\xi} - v_m U \left(1 - \frac{U}{\rho_m}\right) + \frac{v_m U}{2} = 0$$

so

$$\epsilon \frac{dU}{d\xi} = \frac{v_m U}{2} \left(1 - \frac{2U}{\rho_m}\right).$$

By separating variables, this can be written as

$$\frac{dU}{U(\rho_m - 2U)} = \frac{v_m}{2\rho_m \epsilon}\, d\xi.$$

which can be expanded as

$$\frac{dU}{\rho_m} \left(\frac{1}{U} + \frac{2}{\rho_m - 2U}\right) = \frac{v_m}{2\epsilon}\, d\xi.$$

This can be integrated to give

$$\log U - \log(\rho_m - 2U) = \frac{v_m \xi}{2\epsilon} + B$$

for some constant B. Hence

$$\frac{U}{\rho_m - 2U} = D \exp\left(\frac{v_m \xi}{2\epsilon}\right)$$

for some constant D, and thus

$$U = \frac{\rho_m D \exp\left(\frac{v_m \xi}{2\epsilon}\right)}{1 + 2D \exp\left(\frac{v_m \xi}{2\epsilon}\right)}.$$

The constant D just corresponds to a translation of the solution. Choosing $D = 1/2$ ensures symmetry so that $U(0) = \rho_m/4$, which is halfway between the function limits. Hence

$$\rho(x,t) = \frac{\rho_m}{2} \frac{\exp\left(\frac{v_m(x - vt)}{2\epsilon}\right)}{1 + \exp\left(\frac{v_m(x - vt)}{2\epsilon}\right)}$$

which can be also be written as

$$\rho(x,t) = \frac{\rho_m}{4}\left(1 + \tanh\left(\frac{v_m(x - vt)}{4\epsilon}\right)\right).$$

The solution is plotted in Fig. 2 for several different values of ϵ. It can be seen that as $\epsilon \to 0$, the function solutions approaches the shock solution from part (a).

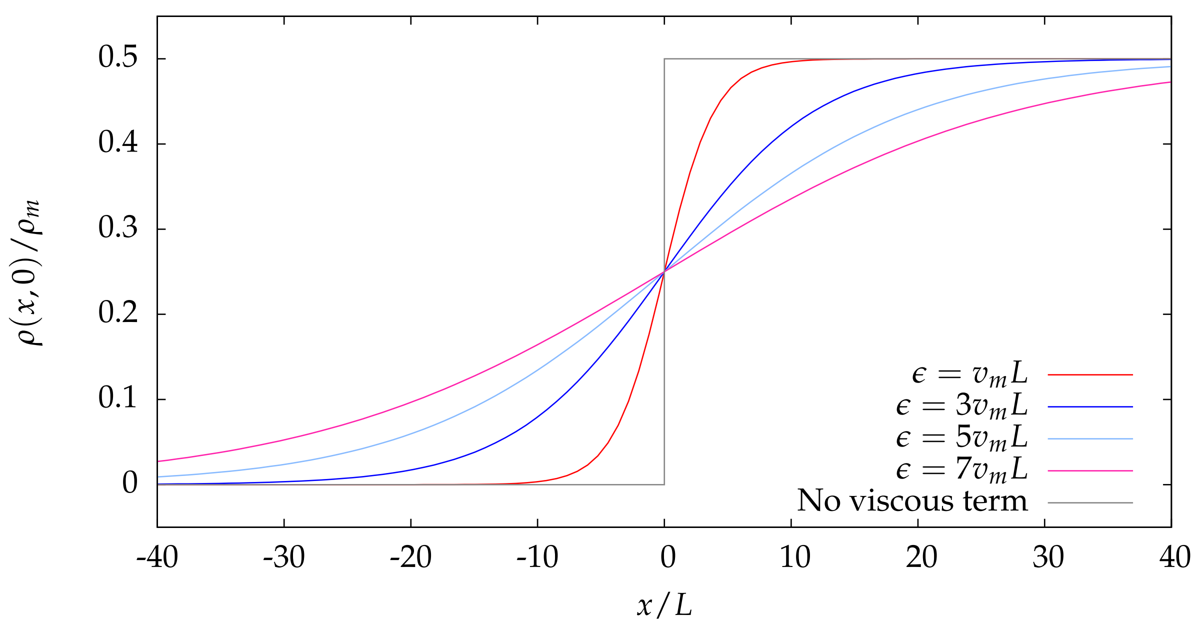

Figure 2: Plots of the travelling wave solution to the traffic equation for several different sizes of the viscous term. The shock solution corresponding to no viscous term is also plotted. Here, L is an arbitrary length scale.