Math 126: Homework 11 solutions

1. The equation of a characteristic starting from $\xi \in [0, L]$ is

$$\begin{aligned} x &= q'(\rho(\xi))t + \xi \\ &= v_m \left(1 - \frac{L+\xi}{2L}\right) t + \xi \\ &= \frac{v_m (L-\xi)t}{2L} + \xi. \end{aligned} \tag{1}$$

As ξ increases, the velocities of the characteristics decrease, and thus it should be expected that at some point the characteristics will intersect. To verify this, note that at $t = 2L/v_m$,

$$x = (L - \xi) + \xi = L$$

and hence all characteristics for $\xi \in [0, L]$ meet at $(x, t) = (L, 2L/v_m)$. Since all the characterisitcs intersect here, there will be no intersections at an earlier time, and thus the time to the first shock is $t_s = 2L/v_m$.

On the positive side of the shock, the density will be $\rho_m/2$, and on the negative side of the shock, the density will be $\rho_m/4$. Hence the shock will move with constant velocity

$$\dot{s}(t) = \frac{q(\rho_m/2) - q(\rho_m/4)}{\rho_m/4} = v_m \frac{\frac{1}{4} - \frac{3}{16}}{1/4} = \frac{v_m}{4}.$$

Figure 1 shows the characteristics and the shock. From the diagram, the regions that are covered with characteristics starting from $x \leq 0$ or $x > L$ will have constant density. The only area that must be calculated explicitly is for (x, t) within the triangle of converging characteristics that start in the range $0 < x \leq L$. In this region, Eq. 1 can be rearranged to show that the characteristic passing through (x, t) starts from

$$\xi = \frac{2x - v_m t}{2 - \frac{v_m t}{L}}$$

and thus

$$\rho(x,t) = \rho(\xi, 0) = \frac{\rho_m}{4L} \left(L + \frac{2x - v_m t}{2 - \frac{v_m t}{L}} \right) = \frac{\rho_m}{2} \left(\frac{L + x - v_m t}{2L - v_m t} \right).$$

Hence the general solution is given by

$$\rho(x,t) = \begin{cases} \frac{\rho_m}{4} & \text{for } x < \frac{v_m t}{2} \text{ and } t < t_s, \text{ or } x < s(t) \text{ and } t \geq t_s \\ \frac{\rho_m}{2}\left(\frac{L+x-v_m t}{2L - v_m t}\right) & \text{for } \frac{v_m t}{2} \leq x < L \text{ and } t < t_s \\ \frac{\rho_m}{2} & \text{for } x \geq L \text{ and } t < t_s, \text{ or } x \geq s(t) \text{ and } t \geq t_s. \end{cases}$$

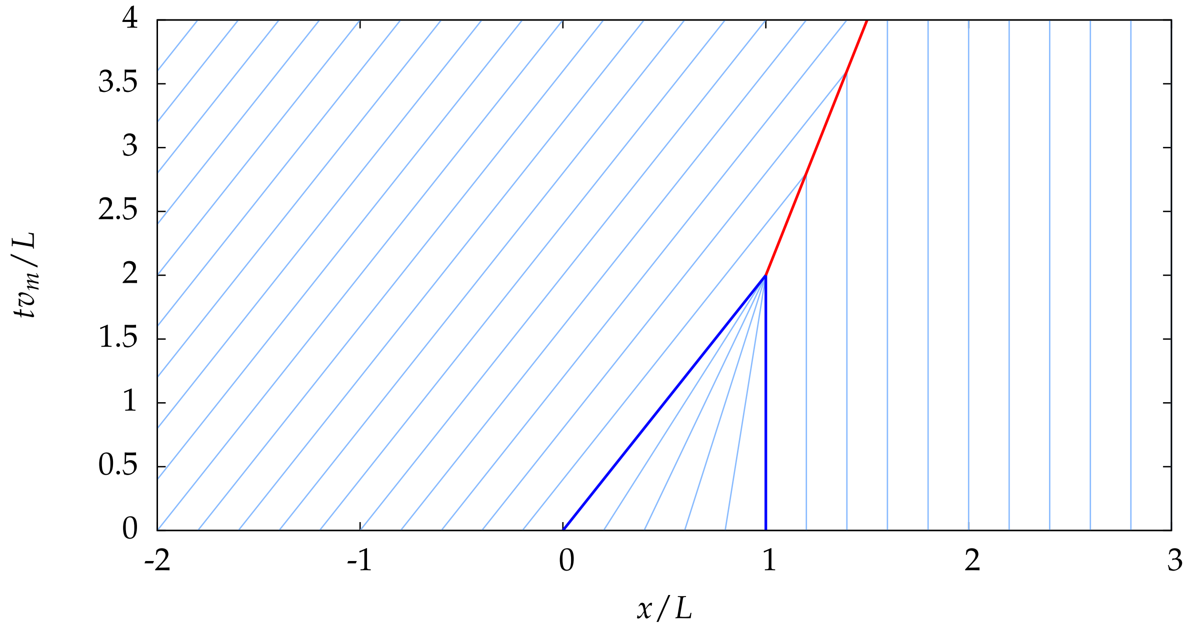

Figure 1: Characteristics for the problem in question 1, plotted in non-dimensionalized units. The characteristics are shown in light blue, with those marking the boundaries between the different regions shown in dark blue. The shock starting at $t_s = 2L/v_m$ is shown in red.

2. (a) Initially, the car is a region of maximum density, and is therefore stationary. It will start to move when the rarefaction fan passes it at time $-a/v_m$. The position of the car will then be given by differential equation

$$\begin{aligned} \frac{dc}{dt} &= v(\rho(c(t),t)) \\ &= v_m\left(1-\frac{1}{2}\left(1-\frac{c}{v_m t}\right)\right) \\ &= \frac{v_m}{2}\left(1+\frac{c}{v_m t}\right) \end{aligned}$$

which can be rearranged to give

$$2\frac{dc}{dt}-\frac{c}{t}=v_m. \tag{2}$$

Consider the substitution $c(t)=b(t)t^{1/2}$; this gives

$$\frac{db}{dt}=\frac{dc}{dt}t^{-1/2}-\frac{c}{2}t^{-3/2}=\frac{1}{2\sqrt{t}}\left(2\frac{dc}{dt}-\frac{c}{t}\right)$$

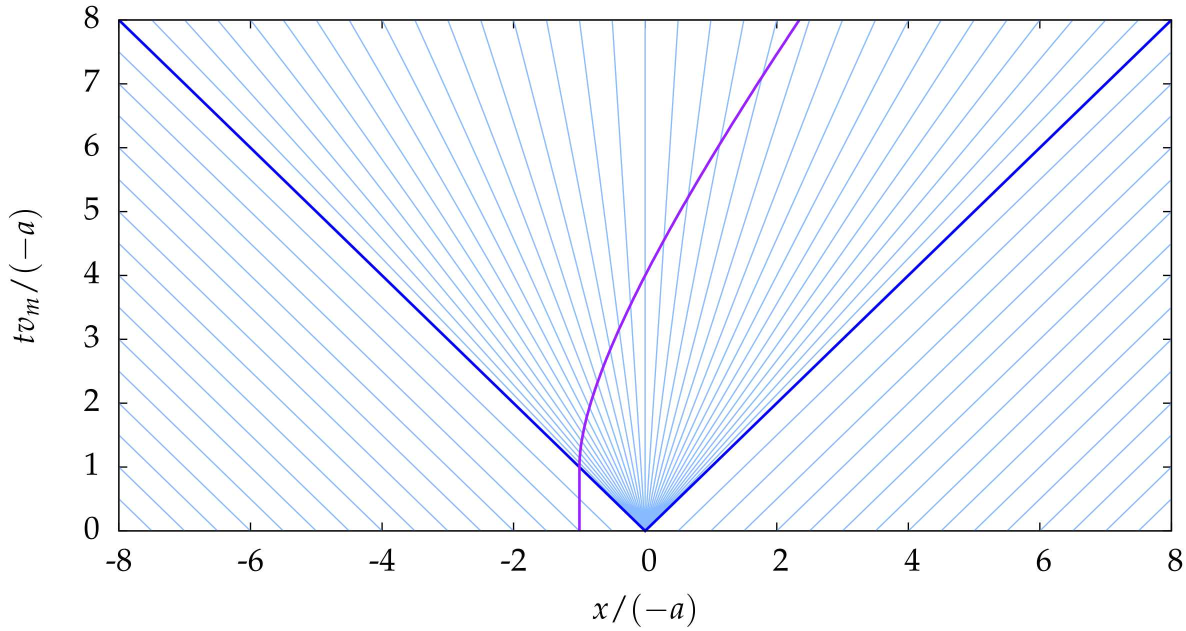

Figure 2: Characteristics for the problem in question 2, plotted in non-dimensionalized units. The characteristics are shown in light blue, with those marking the boundaries between the different regions shown in dark blue. The car's trajectory starting at $x = -a$ is shown in purple.

which can substituted into the equation for $c(t)$ to give

$$\frac{db}{dt} = \frac{v_m}{2\sqrt{t}}.$$

This can be integrated to give

$$b(t) = v_m\sqrt{t} + C$$

for some constant C, and hence

$$c(t) = v_m t + C\sqrt{t}.$$

Since $c(-a/v_m) = a$, it follows that

$$c(t) = v_m t - 2\sqrt{-av_m t}.$$

The characteristics and the trajectory of the car and are shown in Fig. 2.

(b) The car's velocity is

$$\frac{dc}{dt} = v_m - \frac{\sqrt{-av_m}}{\sqrt{t}}$$

and thus as $t \to \infty$, $dc/dt \to v_m$. The distance between the car and the front of the rarefaction fan is

$$d(t) = 2\sqrt{-av_m t}$$

and thus $d(t) \to \infty$ as $t \to \infty$. Note that even though the velocity of the car and rarefaction fan approach the same limit, the car becomes further and further away from the front of the fan.

3. (a) Initially, the density on the negative side of the shock is zero, and the density on the positive side of the shock is ρ_m, so the shock's velocity will be

$$\dot{s}(t) = \frac{q(\rho_m) - q(0)}{\rho_m} = 0$$

and hence the shock will initially remain at $x = L$. However, the shock will begin to move when the rarefaction fan hits it at $t = L/v_m$. Within the rarefaction fan, the density will be given by

$$\rho_{\text{fan}}(x,t) = \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right)$$

and hence the shock velocity will be

$$\begin{aligned}
\dot{s}(t) &= \frac{q(\rho_m) - q(\rho_{\text{fan}}(s(t),t))}{\rho_m - \rho_{\text{fan}}(s(t),t)} \\
&= \frac{0 - \frac{\rho_m v_m}{2}\left(1 - \frac{s}{v_m t}\right)\left(1 - \frac{1}{2}\left(1 - \frac{s}{v_m t}\right)\right)}{\rho_m - \frac{\rho_m}{2}\left(1 - \frac{s}{v_m t}\right)} \\
&= \frac{-\frac{\rho_m v_m}{2}\left(1 - \frac{s}{v_m t}\right)\left(\frac{1}{2} + \frac{s}{2v_m t}\right)}{\rho_m\left(\frac{1}{2} + \frac{s}{2v_m t}\right)} \\
&= -\frac{v_m}{2}\left(1 - \frac{s}{v_m}t\right)
\end{aligned}$$

which can be rearranged to give

$$2\frac{ds}{dt} - \frac{s}{t} = -v_m.$$

This is the same as Eq. 2 but with the sign of v_m reversed. Hence the solution is

$$s(t) = -v_m t + C\sqrt{t}$$

for some constant C. Since $s(L/v_m) = L$, it follows that

$$s(t) = -v_m t + 2\sqrt{Lv_m t}.$$

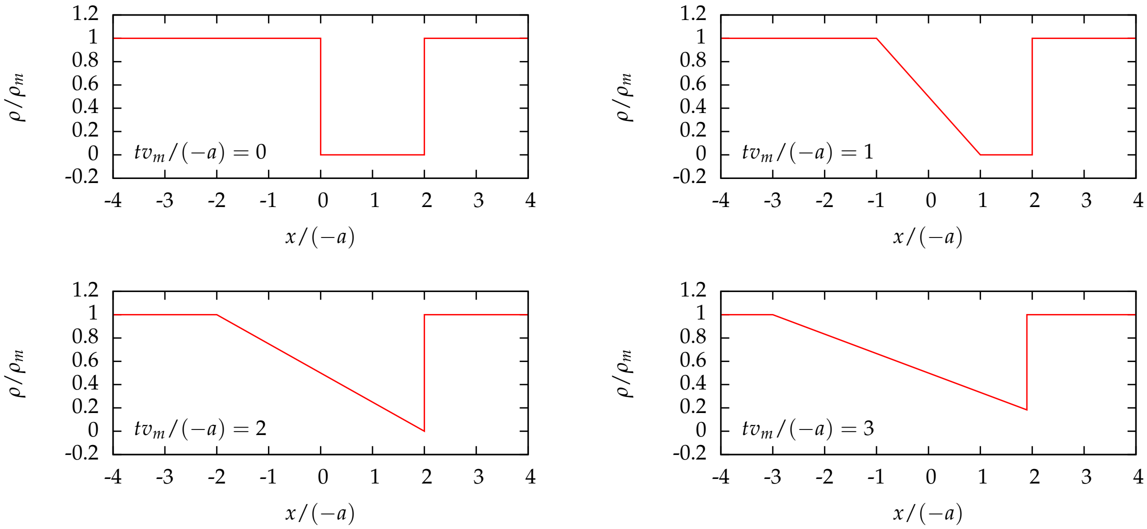

Figure 3: Plots of the traffic density for four different times, plotted in non-dimensionalized units. After the rarefaction fan hits the shock, the shock begins to move in the negative direction.

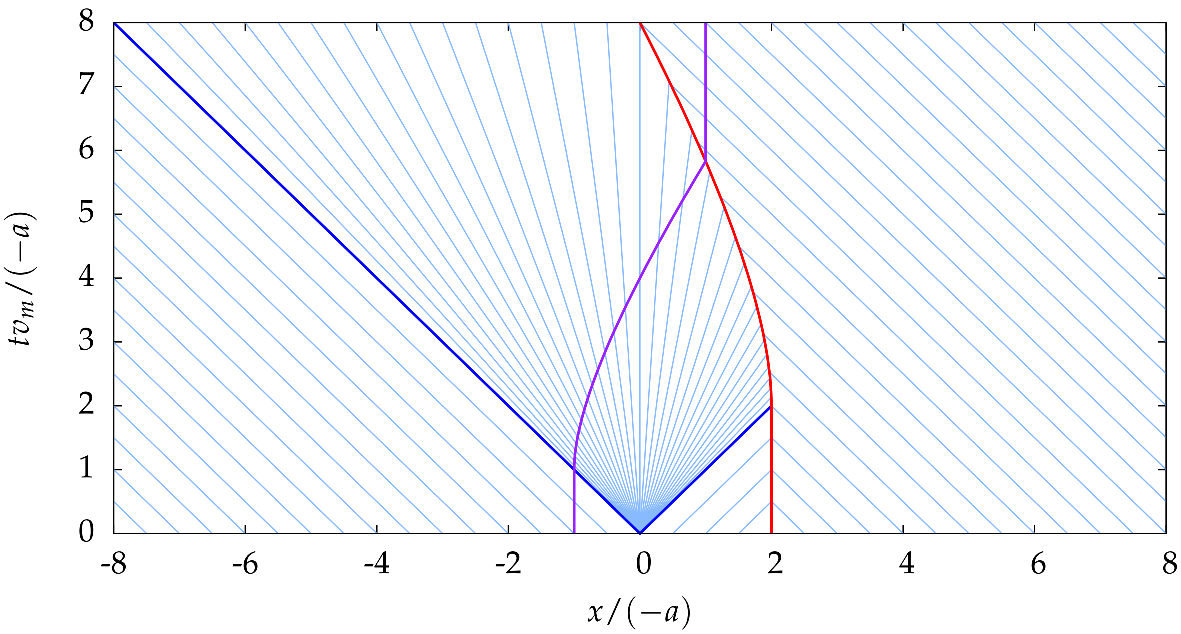

Figure 4: Characteristics for the problem in question 3, plotted in non-dimensionalized units. The characteristics are shown in light blue, with those marking the boundaries between the different regions shown in dark blue. The car's trajectory starting at $x = -a$ is shown in purple, and the shock starting from $x = L$ is shown in red. For this plot, $L = -2a$.

Since $s(t) > -v_m t$ for all t, it follows that as $t \to \infty$, the shock will remain between the rarefaction fan and the jammed traffic, so no further change in behavior will occur.

(b) The traffic density will be the same as the green light problem, but with cars encountering jammed traffic after the shock. Hence the density is given by

$$\rho(x,t) = \begin{cases} \rho_m & \text{for } x \leq -v_m t \text{ or } x > s(t) \\ \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right) & \text{for } |x| < v_m t \text{ and } x \leq s(t) \\ 0 & \text{for } x \geq v_m t \text{ and } x \leq s(t). \end{cases}$$

Plots of the density for four different times are shown in Fig. 3.

(c) Until the car reaches the shock, the traffic density it encounters is exactly the same as the green light problem considered previously. Hence the car will be stationary for $t < -a/v_m$, and then follow

$$c(t) = v_m t - 2\sqrt{-a v_m t}.$$

The car will meet the shock when

$$c(t) = s(t)$$

corresponding to

$$v_m t - 2\sqrt{-a v_m t} = -v_m t + 2\sqrt{L v_m t}$$

from which it follows that

$$2 v_m \sqrt{t} = 2(\sqrt{-a v_m} + \sqrt{L v_m}) = 2\sqrt{v_m}(\sqrt{-a} + \sqrt{L})$$

and hence

$$t = \frac{(\sqrt{-a} + \sqrt{L})^2}{v_m}.$$

The car's position will then be

$$\begin{aligned} c\left(\frac{(\sqrt{-a} + \sqrt{L})^2}{v_m}\right) &= (\sqrt{-a} + \sqrt{L})^2 - 2\sqrt{-a}(\sqrt{-a} + \sqrt{L}) \\ &= -a + L + 2\sqrt{-aL} + 2a - 2\sqrt{-aL} \\ &= a + L. \end{aligned}$$

Once the car has passed the shock, it enters the jammed traffic ahead of the shock and must therefore be stationary, so that

$$c(t) = a + L.$$

This should be expected. Initially, the car at $x = a$ has a region of jammed traffic of width $-a$ ahead of it. These cars will move forward until they hit the jammed traffic at L. Since the car from $x = a$ cannot overtake the cars in front of it, it follows that its final position will be a distance $-a$ behind the location of the initial jam at $x = L$. The characteristics, shock, and the car's trajectory are plotted in Fig. 4.