Math 126: Homework 13

1. Consider the Cauchy problem

$$\frac{2xy}{1+y^2}u_x + u_y = -\frac{uy}{4},$$

for the function $u(x,y)$ with initial data $u(x,0) = h(x) = \sin x$.

   (a) Calculate the characteristics and plot their trajectories in the xy-plane.

   (b) Find u for all $x, y \in \mathbb{R}$.

   (c) Verify that u is a solution by directly substituting it into the partial differential equation.

   (d) Use a computer plotting program to plot u in the domain $-25 < x < 25$ and $-4 < y < 4$.

2. Consider the linear PDE

$$-yu_x + xu_y = 0$$

for the function $u(x,y)$ defined in $\mathbb{R}^2$, with initial data $u(x,0) = h(x)$. Calculate the characteristics, and determine a necessary and sufficient condition on the function $h(x)$ for a solution to exist. In the case when this condition is satisfied, calculate the general solution of $u(x,y)$.

3. Consider the problem

$$u_x + 3x^2u_y = 3ux^2$$

for the function $u(x,y)$ defined in $\mathbb{R}^2$, with initial data $u(x,0) = h(x)$.

   (a) Calculate the characteristics, expressing them in the form $X(s,t)$ and $Y(s,t)$ where s is a coordinate along the initial data and t is a coordinate along each characteristic, with $t = 0$ corresponding to the initial data.

   (b) Calculate

$$J(s,t) = \begin{vmatrix} X_s(s,t) & Y_s(s,t) \\ X_t(s,t) & Y_t(s,t) \end{vmatrix}.$$

   By considering $J(s,0)$, determine a necessary condition on $h'(0)$ in order for C^1 solution to exist in a neighborhood of the x axis.

   (c) Consider the case when $h(x) = x^2$. What is $h'(0)$? Find the solution $u(x,y)$ and determine whether it is C^1.

4. Find two solutions to the equation

$$u_x^2 + u_y^2 = 4u$$

in the disk $\{x, y \in \mathbb{R} : x^2 + y^2 < 4\}$, with initial data $u(\cos s, \sin s) = 1$ for $s \in \mathbb{R}$.