Math 126: Homework 11

1. Consider the traffic equation $\rho_t + v_m(1 - \frac{2\rho}{\rho_m})\rho_x = 0$ with initial condition

$$\rho(x,0) = \begin{cases} \frac{\rho_m}{4} & \text{for } x \le 0 \\ \frac{\rho_m(L+x)}{4L} & \text{for } 0 < x \le L \\ \frac{\rho_m}{2} & \text{for } x > L \end{cases}$$

for some constant $L > 0$. By considering the characteristics, or otherwise, show that a shock forms after a finite time t_s. Find $\rho(x,t)$ for all $x \in \mathbb{R}$ and all $t \ge 0$.

2. The green light problem discussed in Section 4.3.3 of the textbook has solution

$$\rho(x,t) = \begin{cases} \rho_m & \text{for } x \le -v_m t \\ \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right) & \text{for } |x| < v_m t \\ 0 & \text{for } x \ge v_m t. \end{cases}$$

   (a) Consider a car starting from $x = a$ where $a < 0$, and compute its trajectory $c(t)$ for all $t \ge 0$. *[Hint: the substitution $c(t) = b(t)t^{1/2}$ may help.]*

   (b) Show that the speed of the car approaches v_m as $t \to \infty$. Define $d(t) = v_m t - c(t)$ to be the distance between the car and the front of the rarefaction fan. How does $d(t)$ behave as $t \to \infty$?

3. Now suppose that there is traffic jam at a distance L ahead of the green light, so that the initial condition is

$$\rho(x,0) = \begin{cases} \rho_m & \text{for } x \le 0 \\ 0 & \text{for } 0 < x \le L \\ \rho_m & \text{for } x > L. \end{cases}$$

   (a) The traffic density has a shock starting from $x = L$. Determine the trajectory $s(t)$ of the shock for all $t \ge 0$.

   (b) Find the solution $\rho(x,t)$ for all $x \in \mathbb{R}$ and all $t \ge 0$, and sketch ρ for several values of t.

   (c) Consider a car starting from $x = a$ where $a < 0$, and compute its trajectory $c(t)$ for all $t \ge 0$. What happens to $c(t)$ as $t \to \infty$? Why should this be expected?