Predicting the Madden-Julian Oscillation (MJO) and the Monsoon Intraseasonal Oscillation (MISO)

Instructor: Nan Chen

Department of Mathematics
University of Wisconsin-Madison

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MWF 8:50AM – 9:40AM, B239 Van Vleck Hall
Singular Spectrum Analysis (SSA) is a nonparametric method for time series analysis. It consists of two complementary stages - decomposition and reconstruction.

SSA can be an aid in the decomposition of time series into a sum of components, each having a meaningful interpretation.
Steps of SSA

- **Decomposition**
  - Create a time-lagged matrix for time series.
  - Do the Singular Value Decomposition (SVD) for the time-lagged matrix.

- **Reconstruction**
  - Group the resultant elementary matrices from SVD.
  - Do the diagonal averaging for the matrix grouped.
The first step of decomposition

For an one-dimension time series \([x_1, x_2, \ldots, x_N]\), the time-lagged matrix \(A\) is defined as

\[
A = \begin{bmatrix}
    x_1 & x_2 & \ldots & x_{N-M+1} \\
    x_2 & x_3 & \ldots & x_{N-M+2} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{M-1} & x_M & \ldots & x_{N-1} \\
    x_M & x_{M+1} & \ldots & x_N
\end{bmatrix}
\]

where \(N\) is the length of the time series, and \(M\) is called Window Length.

All of the next steps are regarding this matrix!
### The second step of decomposition

For the time-lagged matrix $A$ with rank-$d$, we do the SVD to decompose it to some rank-1 matrices:

$$ A = \sqrt{\lambda_1} y_1 x_1^T + \cdots + \sqrt{\lambda_d} y_d x_d^T. $$

where $\lambda_i$ is the $i$-th eigenvalue and $x_i$ is the corresponding unit eigenvector of $A^T A$, and $y_i$ satisfies $Ax_i = \sqrt{\lambda_i} y_i$.

In many cases, $d = M$.

Naturally, we define $A_i = \sqrt{\lambda_i} y_i x_i^T$ and call it the elementary matrix so that we finish the decomposition:

$$ A = A_1 + \cdots + A_d. $$
Eigendecomposition v.s. SVD

Consider the eigendecomposition $A = QDQ^{-1}$ and SVD $M = U\Sigma V^*$. 

1. The vectors in the eigendecomposition matrix $Q$ are not necessarily orthogonal, so the change of basis isn’t a simple rotation. On the other hand, the vectors in the matrices $U$ and $V$ in the SVD are **orthonormal**, representing rotations.

2. In the SVD, the non-diagonal matrices $U$ and $V$ are not necessarily the inverse of one another. In the eigendecomposition the non-diagonal matrices $Q$ and $Q^{-1}$ are inverses of each other.
Eigendecomposition v.s. SVD

Consider the eigendecomposition $A = QDQ^{-1}$ and SVD $M = U\Sigma V^*$. 

3. In the SVD the entries in the diagonal matrix $\Sigma$ are all real and nonnegative. In the eigendecomposition, the entries of $D$ can be any complex number - negative, positive, imaginary, whatever.

4. The SVD always exists for any sort of rectangular or square matrix, whereas the eigendecomposition can only exist for square matrices, and even among square matrices sometimes it doesn’t exist.
More details of SVD

From what we learned in linear algebra, for the real symmetric matrix $\mathbf{M}$, we can get the eigendecomposition

$$\mathbf{M} = \mathbf{X} \Lambda \mathbf{X}^T$$

where $\mathbf{X}$ consists of the eigenvectors of $\mathbf{M}$, and $\Lambda$ consists of the eigenvalues of $\mathbf{M}$.

Note that the eigenvectors which correspond to distinct eigenvalues of the real symmetric matrix are orthogonal!
(Why do we need this note? Just wait for a moment!)
Note that the eigenvectors which correspond to distinct eigenvalues of the real symmetric matrix are orthogonal!

**Proof:**

By $\mathbf{Mx}_i = \lambda_i \mathbf{x}_i$, we have

$$x_j^T \mathbf{Mx}_i = x_j^T \lambda_i \mathbf{x}_i = \lambda_i x_j^T \mathbf{x}_i.$$  

Meanwhile, since $\mathbf{M}$ is symmetric,

$$x_j^T \mathbf{Mx}_i = x_j^T \mathbf{M}^T \mathbf{x}_i = (\mathbf{Mx}_j)^T \mathbf{x}_i = (\lambda_j \mathbf{x}_j)^T \mathbf{x}_i = \lambda_j x_j^T \mathbf{x}_i.$$  

Hence $\lambda_i x_j^T \mathbf{x}_i = \lambda_j x_j^T \mathbf{x}_i$.

Since $\lambda_i$ and $\lambda_j$ are distinct, we have $x_j^T \mathbf{x}_i = 0$.  

$\square$
More details of SVD

For our time-lagged matrix $A$, we know it’s always non-square, so we couldn’t get its eigendecomposition like the above.

Fortunately, for the matrix with any size, $A^T A$ is a real symmetric square matrix since

$$(A^T A)^T = A^T (A^T)^T = A^T A.$$

Therefore, we can deal with the eigenvectors of $A^T A$ and use the orthogonality.
More details of SVD

Denote $x_i$ and $x_j$ two different eigenvectors of $A^T A$.

Then there is a trick that we consider $Ax_i$.

From the note before, we know the eigenvectors of $A^T A$ are orthogonal. So for $i \neq j$,

$$(Ax_i)^T(Ax_j) = x_i^T A^T A x_j = \lambda_j x_i^T x_j = 0,$$

which means $Ax_i$ and $Ax_j$ are also orthogonal.

In addition,

$$(Ax_i)^T(Ax_i) = x_i^T A^T A x_i = \lambda_i x_i^T x_i = \lambda_i$$

if we scale each eigenvector $x_i$, i.e. $|x_i| = 1$. 


More details of SVD

\[ A = \begin{bmatrix}
  x_1 & x_2 & \cdots & x_{N-M+1} \\
  x_2 & x_3 & \cdots & x_{N-M+2} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{M-1} & x_M & \cdots & x_{N-1} \\
  x_M & x_{M+1} & \cdots & x_N
\end{bmatrix} \]

So we get

\[
\begin{cases}
  (Ax_i)^T (Ax_j) = 0 \\
  (Ax_i)^T (Ax_i) = \lambda_i
\end{cases}
\]

which shows us that with the linear map \( A \), the orthonormal bases \( \{x_1, \ldots, x_d\} \) were transformed to new orthogonal bases \( \{Ax_1, \ldots, Ax_d\} \) with corresponding lengths \( \sqrt{\lambda_1}, \ldots, \sqrt{\lambda_d} \).

Hence we define \( Ax_i = \sqrt{\lambda_i} y_i \) where \( \{y_1, \ldots, y_d\} \) essentially is a set of the orthonormal bases.

Recall that \( d = M \) in many cases.
More details of SVD

By $Ax_i = \sqrt{\lambda_i}y_i$ we can get the matrix form that is

$$AX = Y\Sigma$$

where $X$, $Y$ and $\Sigma$ consist of $x_i$, $y_i$ and $\sqrt{\lambda_i}$ respectively.

Note that $X$ is an orthogonal matrix, so $X^{-1} = X^T$.

Then we have

$$A = Y\Sigma X^{-1} = Y\Sigma X^T.$$
Summary of the first step: decomposition

\[
A = \begin{bmatrix}
  x_1 & x_2 & \cdots & x_{N-M+1} \\
  x_2 & x_3 & \cdots & x_{N-M+2} \\
  \vdots & \vdots & \ddots & \vdots \\
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\end{bmatrix}
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For the time-lagged matrix \( A \) with rank-\( d \), we do the SVD to decompose it to some rank-1 matrices:

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A = \sqrt{\lambda_1} y_1 x_1^T + \cdots + \sqrt{\lambda_d} y_d x_d^T.
\]

where \( \lambda_i \) is the \( i \)-th eigenvalue and \( x_i \) is the corresponding unit eigenvector of \( A^T A \), and \( y_i \) satisfies \( A x_i = \sqrt{\lambda_i} y_i \).

Naturally, we define \( A_i = \sqrt{\lambda_i} y_i x_i^T \) and call it the elementary matrix so that we finish the decomposition:

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A = A_1 + \cdots + A_d.
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Steps of SSA

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The first step of reconstruction

Recall that

\[ A = \sqrt{\lambda_1} y_1 x_1^T + \cdots + \sqrt{\lambda_d} y_d x_d^T. \]

In PCA/EOF method, we select the first \( r \) components whose coefficients weigh sufficiently, that is

\[ A_r = \sum_{i=1}^{r} A_i. \]

In SSA method, we do the same thing at this step!

(Note: Empirical orthogonal functions (EOF), Principal component analysis (PCA), empirical component analysis and many others are simply other names of eigenvalue decomposition.)
The second step of reconstruction

Definition:

A Hankel matrix is a matrix in which all the elements on the anti-diagonal are equal and constant, whose form is like

\[
\begin{bmatrix}
    a_1 & a_2 & a_3 & \cdots & \cdots & a_m & a_{m+1} & \cdots & a_n \\
    a_2 & a_3 & \cdots & \cdots & \cdots & a_m & a_{m+1} & \cdots & a_{n+1} \\
    a_3 & \cdots & \cdots & a_m & a_{m+1} & \cdots & a_{m+n-3} \\
    \vdots & \cdots & a_m & a_{m+1} & \cdots & a_{m+n-3} & a_{m+n-2} \\
    a_m & a_{m+1} & \cdots & a_n & a_{n+1} & \cdots & a_{m+n-3} & a_{m+n-2} & a_{m+n-1}
\end{bmatrix}
\]
The second step of reconstruction

\[ A = \begin{bmatrix}
  x_1 & x_2 & \ldots & x_{N-M+1} \\
  x_2 & x_3 & \ldots & x_{N-M+2} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{M-1} & x_M & \ldots & x_{N-1} \\
  x_M & x_{M+1} & \ldots & x_N 
\end{bmatrix} \]

\( A \) is a Hankel matrix!

Recall that

\[ A_r = \sum_{i=1}^{r} A_i. \]

In terms of the practical meaning, if we transform \( A_r \) into a Hankel matrix, then we will get a new time series!
The second step of reconstruction

For a $m \times n$ matrix

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\
a_{21} & a_{22} & & & & a_{2m} & a_{2,m+1} & \cdots & a_{2n} \\
a_{31} & & & & & & & & \\
\vdots & \ddots & & & & & & & \vdots \\
\vdots & & \ddots & & & & & & \ddots \\
a_{m1} & a_{m2} & \cdots & a_{mm} & a_{m,m+1} & \cdots & a_{m-2,n} & a_{m-1,n} & a_{mn}
\end{bmatrix},
\]

the straightforward idea is to do an averaging on the anti-diagonal so that all the elements on the anti-diagonal are equal!
The second step of reconstruction

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\
  a_{21} & a_{22} & & & & & & & \\
  a_{31} & & & & & & & & \\
  & & & & & & & & \\
  & & & & & & & & \\
  a_{m1} & a_{m2} & \cdots & a_{mm} & a_{m,m+1} & \cdots & a_{m-2,n} & a_{m-1,n} & a_{mn}
\end{bmatrix}
\]

\[\mathbf{A}_r = \]

For \( \mathbf{A}_r = (a_{ij})_{m \times n} \) and \( i + j = s \), we have \( \tilde{\mathbf{A}}_r = (\tilde{a}_{ij})_{m \times n} \) where

\[
\tilde{a}_{ij} = \begin{cases}
  \frac{1}{s - 1} \sum_{l=1}^{s-1} a_{l,s-l} & \text{for } 2 \leq s \leq m \\
  \frac{1}{m} \sum_{l=1}^{m} a_{l,s-l} & \text{for } m + 1 \leq s \leq n + 1 \\
  \frac{1}{m+n-s+1} \sum_{l=s-n}^{m} a_{l,s-l} & \text{for } n + 2 \leq s \leq m + n
\end{cases}
\]

So \( \tilde{\mathbf{A}}_r = (\tilde{a}_{ij})_{m \times n} \) is exactly our result of the reconstruction.
Examples

**Figure:** The real-time multivariate MJO (RMM) indices
Examples

**Figure:** Eigenvalues
Examples

Figure: Principal components 1-4
Examples

Figure: Principal component 30
Examples

Figure: Original data and reconstruction
Predicting the Large-Scale Madden-Julian Oscillation

The Madden-Julian Oscillation (MJO):
(Lau & Waliser 2011)

- the dominant mode of tropical intraseasonal (30-60 days) variability in boreal winter
- a slow eastward moving large-scale envelope of convection
- affecting tropical and global weather patterns
Predicting the Large-Scale Madden-Julian Oscillation

The Madden-Julian Oscillation (MJO):
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▶ a slow eastward moving large-scale envelope of convection
▶ affecting tropical and global weather patterns

Extracting the large-scale MJO from the noisy and turbulent raw data:

▶ Traditional linear data analysis methods (e.g. Principal Component Analysis) result in large biases.
▶ A new advanced nonlinear techniques, Nonlinear Laplacian Spectral Analysis (NLSA), is applied to the cloudiness data of dimensions $O(10^5)$ (Giannakis & Majda, PNAS, 2012; Chen, Majda & Giannakis, GRL, 2014).
▶ NLSA takes into account the nonlinear manifold structure of complex data sets and it captures nonlinear dynamical features such as intermittency and extreme events.

red: weak convection (clear sky). blue: strong convection (heavy rainfall).

Consistent with the MJOs observed during the field campaign (TOGA-COARE) of 1992-1993 (Webster & Lukas).

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NLSA Large-Scale MJO Patterns

= Spatial Basis (1) \times Time Series (1) + Spatial Basis (2) \times Time Series (2)
NLSA Time-Series Techniques ⇒ 2 components of MJO Cloud Patterns

Intermittent bursts of MJO activity
NLSA Time-Series Techniques → 2 components of MJO Cloud Patterns

Intermittent bursts of MJO activity

Physics-Constrained Low-Order Nonlinear Stochastic Model for Predicting MJO Cloud Patterns (MJO1, MJO2)

Physics-Constrained Low-Order Stochastic Model

\[ d\mathbf{u} = \begin{pmatrix} d\mathbf{u}_1 \\ d\mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} -d_u(t) \mathbf{u}_1 - \hat{\omega} \mathbf{u}_2 \\ -d_u(t) \mathbf{u}_2 + \hat{\omega} \mathbf{u}_1 \end{pmatrix} dt + \sigma_u \begin{pmatrix} \mathbf{dW}_u \end{pmatrix}, \]

with

\[ d_u(t) = d_{u0} + d_{u1} \sin(\omega_f t + \phi). \]

- Observed variables \( \mathbf{u}_1, \mathbf{u}_2 \): MJO 1 and MJO 2 indices from NLSA.
- Standard regression model, insufficient in capturing the key features.
Physics-Constrained Low-Order Stochastic Model

\[ du_1 = (-du(t) \, u_1 + \gamma v \, u_1 - \omega \, u_2) \, dt + \sigma_u \, dW_{u_1}, \]
\[ du_2 = (-du(t) \, u_2 + \gamma v \, u_2 + \omega \, u_1) \, dt + \sigma_u \, dW_{u_2}, \]
\[ dv = (-dv \, v) \, dt + \sigma_v \, dW_v, \]
\[ d\omega = (-d\omega \, \omega + \hat{\omega}) \, dt + \sigma_\omega \, dW_\omega, \]

with
\[ du(t) = du_0 + du_1 \sin(\omega_f t + \phi). \]

- Observed variables \( u_1, u_2 \): MJO 1 and MJO 2 indices from NLSA.
- Hidden variables \( v, \omega \): stochastic damping and stochastic phase.
Physics-Constrained Low-Order Stochastic Model

\[
d u_1 = (-d_u(t) u_1 + \gamma v u_1 - \omega u_2) \, dt + \sigma_u \, dW_{u_1},
\]
\[
d u_2 = (-d_u(t) u_2 + \gamma v u_2 + \omega u_1) \, dt + \sigma_u \, dW_{u_2},
\]
\[
d v = (-d_v v - \gamma (u_1^2 + u_2^2)) \, dt + \sigma_v \, dW_v,
\]
\[
d \omega = (-d_\omega \omega + \hat{\omega}) \, dt + \sigma_\omega \, dW_\omega,
\]

with

\[
d_u(t) = d_{u0} + d_{u1} \sin(\omega_f t + \phi).
\]

▷ Observed variables \(u_1, u_2\): MJO 1 and MJO 2 indices from NLSA.

▷ Hidden variables \(v, \omega\): stochastic damping and stochastic phase.

▷ Energy-conserving nonlinear interactions between \((u_1, u_2)\) and \((v, \omega)\).

(Majda, Harlim, 2012)
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\[ du_1 = (-d_u(t) u_1 + \gamma v u_1 - \omega u_2) \, dt + \sigma_u \, dW_{u_1}, \]
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\[ dv = (-d_v v - \gamma (u_1^2 + u_2^2)) \, dt + \sigma_v \, dW_v, \]
\[ d\omega = (-d_{\omega} \omega + \dot{\omega}) \, dt + \sigma_{\omega} \, dW_{\omega}, \]

with

\[ d_u(t) = d_{u0} + d_{u1} \sin(\omega_f t + \phi). \]

- Observed variables \( u_1, u_2 \): MJO 1 and MJO 2 indices from NLSA.
- Hidden variables \( v, \omega \): stochastic damping and stochastic phase.
- Energy-conserving nonlinear interactions between \( (u_1, u_2) \) and \( (v, \omega) \).
  (Majda, Harlim, 2012)

**Prediction.** Given the initial values of \( (u_1, u_2) \) and \( (v, \omega) \), run an ensemble forecast.
Physics-Constrained Low-Order Stochastic Model

\[
\begin{align*}
    du_1 &= (-d_u(t) u_1 + \gamma v u_1 - \omega u_2) \, dt + \sigma_u \, dW_{u_1}, \\
    du_2 &= (-d_u(t) u_2 + \gamma v u_2 + \omega u_1) \, dt + \sigma_u \, dW_{u_2}, \\
    dv &= (-d_v v - \gamma (u_1^2 + u_2^2)) \, dt + \sigma_v \, dW_v, \\
    d\omega &= (-d_\omega \omega + \hat{\omega}) \, dt + \sigma_\omega \, dW_\omega,
\end{align*}
\]

with

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d_u(t) = d_{u0} + d_{u1} \sin(\omega_f t + \phi).
\]

▸ Observed variables \(u_1, u_2\): MJO 1 and MJO 2 indices from NLSA.

▸ Hidden variables \(v, \omega\): stochastic damping and stochastic phase.

▸ Energy-conserving nonlinear interactions between \((u_1, u_2)\) and \((v, \omega)\).
  (Majda, Harlim, 2012)

**Prediction.** Given the initial values of \((u_1, u_2)\) and \((v, \omega)\), run an ensemble forecast.

How to determine the initial values of the hidden variables \(v, \omega\)?

![Diagram showing ensemble members, ensemble mean, and truth](image)
Physics-Constrained Low-Order Stochastic Model

\[ du_1 = (-d_u(t) u_1 + \gamma v u_1 - \omega u_2) \, dt + \sigma_u \, dW_{u_1}, \]
\[ du_2 = (-d_u(t) u_2 + \gamma v u_2 + \omega u_1) \, dt + \sigma_u \, dW_{u_2}, \]
\[ dv = (-d_v v - \gamma (u_1^2 + u_2^2)) \, dt + \sigma_v \, dW_v, \]
\[ d\omega = (-d_\omega \omega + \hat{\omega}) \, dt + \sigma_\omega \, dW_\omega, \]

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- Observed variables \( u_1, u_2 \): MJO 1 and MJO 2 indices from NLSA.
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**Prediction.** Given the initial values of \((u_1, u_2)\) and \((v, \omega)\), run an ensemble forecast.

How to determine the initial values of the hidden variables \( v, \omega \)?

**Effective data assimilation of** \( p(u_{II}(t)|u_I(s \leq t)) \)

**based on the conditional Gaussian framework!**
Calibration of parameters using \textbf{Information Theory} (Robust parameters)

almost perfectly capturing the observational statistics (correlation functions, highly non-Gaussian PDFs, and power spectrums)
Skillful prediction at 15- and 25-days lead times
(Ensemble mean prediction)

the most skillful prediction for MJO indices
Long-range forecast starting from different dates (year 2002)

- Ensemble spread captures the long-range forecast uncertainty.
- Ensemble spread predicts the onset and demise of the MJO events.
- Small uncertainty spread at short terms indicates the accuracy of the ensemble mean prediction.
Predicting Monsoon Intraseasonal Precipitation using a Low-Order Nonlinear Stochastic Model
**Monsoon Intraseasonal Oscillation (MISO)**

- the prominent mode of tropical intraseasonal variability in boreal summer
- propagating northeastward
- strongly associated with the boreal summer monsoon rainfall over south Asia
- interactions with El Niño ...

**Procedure of prediction with low-order models**

1. developing effective MISO indices (via suitable data decomposition methods)
2. designing low-order models to predict the MISO indices
3. spatiotemporal reconstruction (indices + the associated spatial bases)

Advantages: capturing large-scale features + computationally efficient
Most of the indices are obtained by applying linear decomposition methods to the high-dimensional raw observational data.

Common features of these covariance-based approaches:

✓ capturing the spatiotemporal MISO patterns reasonably well
✓ recovering the northeastward propagation
✗ ad hoc seasonal extraction and longitudinal averaging leading to loss of predictive information
✗ sometimes mixing with other modes due to the nonlinear nature
✗ potential inadequacy in capturing the rare/extreme events

Novel time series technique: **Nonlinear Laplacian Spectrum Analysis (NLSA).**

(Giannakis & Majda, *PNAS* 2012)

- Lagged embedding
- Machine learning
- Adaptive weights
- Spectral entropy criteria

Advantages of NLSA over classical covariance-based techniques

- objective – by design no ad hoc detrending or spatiotemporal filtering of the full data set
- capturing both intermittency and low frequency variability
- higher memory and predictability in the NLSA MISO modes
Apply NLSA to the daily Global Precipitation Climatology Project (GPCP) rainfall data over the Asian summer monsoon region (Sabeerali et al, *Climate Dynamics*, 2016).

Fractional variance of rainfall anomalies:

Capturing the variability over Indo-West Pacific is extremely important in determining the propagation features of MISO (Pillai & Sahai 2015).
Predicting the NLSA MISO indices via a low-order stochastic model

\[
\begin{align*}
\text{MISO 1} \\
\text{MISO 2}
\end{align*}
\]

Observed variables \( u_1, u_2 \): MISO 1 and MISO 2 indices from NLSA.
Predicting the NLSA MISO indices via a low-order stochastic model

Low-Order Stochastic Model

\[
    \begin{align*}
    du_1 &= (-d_u(t) u_1 - \hat{\omega} u_2) \, dt + \sigma_u \, dW_{u_1}, \\
    du_2 &= (-d_u(t) u_2 + \hat{\omega} u_1) \, dt + \sigma_u \, dW_{u_2},
    \end{align*}
\]

with

\[
    d_u(t) = d_{u0} + d_{u1} \sin(\omega_f t + \phi).
\]

- Observed variables \( u_1, u_2 \): MISO 1 and MISO 2 indices from NLSA.
Predicting the NLSA MISO indices via a low-order stochastic model

Low-Order Stochastic Model

\[
\begin{align*}
    du_1 &= (-du(t) u_1 + \gamma v u_1 - \omega u_2) \, dt + \sigma_u \, dW_{u_1}, \\
    du_2 &= (-du(t) u_2 + \gamma v u_2 + \omega u_1) \, dt + \sigma_u \, dW_{u_2}, \\
    dv &= (-d_v v) \, dt + \sigma_v \, dW_v, \\
    d\omega &= (-d_\omega \omega + \hat{\omega}) \, dt + \sigma_\omega \, dW_\omega,
\end{align*}
\]

with

\[
du(t) = du_0 + du_1 \sin(\omega_f t + \phi).
\]

- Observed variables \(u_1, u_2\): MISO 1 and MISO 2 indices from NLSA.
- Hidden variables \(v, \omega\): stochastic damping and stochastic phase.
Predicting the NLSA MISO indices via a low-order stochastic model

**Physics-Constrained Low-Order Stochastic Model**

\[
\begin{align*}
    du_1 &= (-d_u(t) u_1 + \gamma v u_1 - \omega u_2) \, dt + \sigma_u \, dW_{u_1}, \\
    du_2 &= (-d_u(t) u_2 + \gamma v u_2 + \omega u_1) \, dt + \sigma_u \, dW_{u_2}, \\
    dv &= (-d_v v - \gamma (u_1^2 + u_2^2)) \, dt + \sigma_v \, dW_v, \\
    d\omega &= (-d_\omega \omega + \hat{\omega}) \, dt + \sigma_\omega \, dW_\omega,
\end{align*}
\]

with

\[
d_u(t) = d_u0 + d_{u1} \sin(\omega_f t + \phi).
\]

- **Observed variables** \( u_1, u_2 \): MISO 1 and MISO 2 indices from NLSA.
- **Hidden variables** \( v, \omega \): stochastic damping and stochastic phase.
- **Energy-conserving nonlinear interactions** between \((u_1, u_2)\) and \((v, \omega)\).
Predicting the NLSA MISO indices via a low-order stochastic model

**Physics-Constrained Low-Order Stochastic Model**

\[
\begin{align*}
    d u_1 &= (-d_u(t) u_1 + \gamma v u_1 - \omega u_2) \, dt + \sigma_u \, dW_{u_1}, \\
    d u_2 &= (-d_u(t) u_2 + \gamma v u_2 + \omega u_1) \, dt + \sigma_u \, dW_{u_2}, \\
    d v &= (-d_v v - \gamma (u_1^2 + u_2^2)) \, dt + \sigma_v \, dW_{v}, \\
    d \omega &= (-d_\omega \omega + \hat{\omega}) \, dt + \sigma_\omega \, dW_{\omega},
\end{align*}
\]

with

\[
d_u(t) = d_u^0 + d_u^1 \sin(\omega_f t + \phi).
\]

- Observed variables \( u_1, u_2 \): MISO 1 and MISO 2 indices from NLSA.
- Hidden variables \( v, \omega \): stochastic damping and stochastic phase.
- **Energy-conserving nonlinear interactions** between \((u_1, u_2)\) and \((v, \omega)\).
- Conditional Gaussian system: allowing effective data assimilation algorithm to determine the initial values of \((v, \omega)\) that facilitate the ensemble prediction scheme.
Predicting the NLSA MISO indices via a low-order stochastic model

**Physics-Constrained Low-Order Stochastic Model**

\[ du_1 = (-d_u(t) u_1 + \gamma v u_1 - \omega u_2) \, dt + \sigma_u \, dW_{u_1}, \]
\[ du_2 = (-d_u(t) u_2 + \gamma v u_2 + \omega u_1) \, dt + \sigma_u \, dW_{u_2}, \]
\[ dv = (-d_v v - \gamma (u_1^2 + u_2^2)) \, dt + \sigma_v \, dW_v, \]
\[ d\omega = (-d_\omega \omega + \hat{\omega}) \, dt + \sigma_\omega \, dW_\omega, \]

with

\[ d_u(t) = d_{u0} + d_{u1} \sin(\omega_f t + \phi). \]

- **Observed variables** \( u_1, u_2 \): MISO 1 and MISO 2 indices from NLSA.
- **Hidden variables** \( v, \omega \): stochastic damping and stochastic phase.
- **Energy-conserving nonlinear interactions** between \( (u_1, u_2) \) and \( (v, \omega) \).
- **Conditional Gaussian system**: allowing effective data assimilation algorithm to determine the initial values of \( (v, \omega) \) that facilitate the ensemble prediction scheme.
- Only a short training period is needed.
Calibration of parameters using *Information Theory* (Robust parameters)

Model vs. Observations: Non-Gaussian statistics match

(Linear models fail to accurately match the statistics.)
Ensemble mean captures the extreme MISO events for short- and medium-range forecast.

Ensemble spread captures the long-range forecast uncertainty and the onset and demise of the MISO events.
Prediction of the spatiotemporal evolution based on the predicted NLSA MISO indices

- Very skillful in predicting the extreme events in the large-scale MISO.
- Higher prediction skill than the operational models (GCMs).
- May not be quite skillful for predicting weak events.
MISO modes with different lagged embedding window sizes

(a) MISO indices

(b) PDF and spectrum of MISO 1

(c) Autocorrelations and crosscorrelations

Obs Model
64−day window
34−day window
48−day window

20040101 20040110 20040120 20040130

64 day window
48 day window
34 day window
Prediction of the MISO indices with different lagged embedding window sizes

Time-series from a 64-day lagged embedding window

Time-series from a 48-day lagged embedding window

Time-series from a 34-day lagged embedding window

15-day lead prediction of MISO 1