Math 221 – Practice problems for the Second midterm (fall 2013)

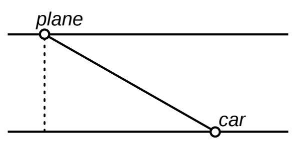
The following problems are examples of the types of problems you can expect for the second midterm. They are not all the problems you need to know, so be sure you also work on other problems from homework and from the notes.

1. Find the equations of the tangent and normal to the curve defined implicitly by $e^{2x+y}+xy = x+1$ at $(0, 0)$.

2. Suppose a function $y = f(x)$ satisfies $y^3 + xy = 12$ for all x

- If $f(x) = 2$ then what is x?
- Find $\frac{dy}{dx}$ at the point $x = 11$, $y = 1$.

3. (20 points) A highway patrol plane flies 3 miles above a straight road at a steady velocity of 120 miles per hour. The pilot sees an oncoming car and with radar determines that at the moment when the line-of-sight distance from the plane to the car is 5 miles, the distance from plane to car is decreasing at a rate of 160 miles per hour. Find the car's speed along the highway at that moment.



4.

(1) $y = e^{2x\cos x} \implies \dfrac{dy}{dx} = ?$

(2) $y = \ln \dfrac{(x+1)^2}{(x^2+1)^3} \implies \dfrac{dy}{dx} = ?$

(3) Use the implicit function theorem to calculate $\theta = \arctan(1 - z)$

(4) $y = x^{\ln x} \implies \dfrac{dy}{dx} = ?$

5. *From these statements two are correct and two are wrong. Explain which one is which and why.* **Explain your answer, more than half the points go to the explanation.**

(1) If f is nonincreasing, then $f'(x) \le 0$ always.
(2) If f is a function and f increases, then $f'(x) > 0$ always.
(3) If f is a function and $f' \ge 0$ always, then f increases.
(4) If $f'(x) < 0$ always, then f is decreasing.

6. Calculate the following limits

(1) $\lim\limits_{x\to+\infty} x^3 e^{-1/x}$.

(2) $\lim\limits_{t\to+\infty} \dfrac{e^t + e^{-t}}{2e^t - te^{-t}}$.

(3) $\lim\limits_{x\to 0+} x(\ln x)^2$. (Hint: either substitute $x = t^2$ or write $x = (x^{1/2})^2$ and combine.)

(4) $\lim\limits_{x\to+\infty} \dfrac{(\ln x)^2}{x}$. (Same hint as above.)

(5) $\lim\limits_{x\to 1} \dfrac{a\ln x + x - 1}{x^3 - x}$, where a is any constant.

7. The sum of two nonnegative numbers is 20. Find the numbers such that one number plus the square root of the other number is the largest possible.

8. If b, c and d are constants, for which value of b will the curve $f(x) = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$?

9. The ACME Boxes & Containers company is going to produce a new type of container in shape of a rectangular box (the base of the box is a square, and the four vertical sides are rectangles; the container will have no top). The material that is used to make the base of the container costs \$2 per square meter. The material from which the sides will be made costs \$3 per square meter. If the container must hold 9 cubic meters, then what is the cheapest container that can be made?

In your solution specify

- which function you are going to minimize, and
- explain what the variables that you use stand for.

10. Sketch the graph of the function $f(x) = \dfrac{2x}{1+x^2}$. In particular

(1) Find the intervals where the function is positive or negative;
(2) find the intervals on which the function is increasing (decreasing);
(3) find the local maxima and minima;
(4) indicate which local maxima and minima are in fact global maxima or minima (explain how you reached your conclusion);
(5) find the inflection points on the graph and where the function is concave and convex;
(6) find asymptotes.

11. Consider the function $f(x)$ defined by $f(x) = \begin{cases} x \ln x & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$ Find the absolute minimum of $f(x)$ over $[0, \infty)$.

12. Sketch the graph of $f(x) = \dfrac{1}{x^2+3}$ using the same steps as in the previous graph.

13. Consider the parametrized curve

$$x(t) = \frac{1-t^3}{1+t^2} \quad \text{and} \quad y(t) = \frac{2t}{1+t^2}.$$

(1) Which points have horizontal tangent lines?
(2) Find one point with a vertical tangent line.
(3) Calculate the tangent line when $t = 2$.

14. Let $y(t)$ be the amount of certain radioactive element in a sample at time t (in years) and suppose $y(t)$ is decaying at a rate proportional to itself. If it takes one year for the amount of this radioactive element to reduce to one half of the original amount, how long does it take for the amount to reduce to 1/3 of the original amount? (Your answer may be written in terms of natural logarithms.)

15. *One of the following statements is true. State which one and explain why, including why the others are wrong.* **Explain your answer in detail, more than half the credit goes to the explanation.**

(1) The integral

$$\int_a^b f(x)dx$$

represents the area below the function f and above the interval $[a, b]$.

(2) $\displaystyle\int_0^x F'(t)dt = F(x)$

(3) $\displaystyle\int_0^x F'(t)dt = F(t)$

(4) $\dfrac{d}{dx}\displaystyle\int_0^x f(t)dt = f(x)$

16. Compute the following integrals

(1) $\displaystyle\int_0^\pi \left(\sqrt{x^3} + \sin x\right) dx$

(2) $\dfrac{\cos x}{1+\sin^2 x}dx$

(3) $\displaystyle\int_0^{\pi/2} e^{\cos t}\sin t dt.$

(4) $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}dx.$

(5) $\displaystyle\int \frac{e^{x}}{1+e^{2x}}dx.$

►►► See also problems on page 148 of the text.

17. Find $\dfrac{dy}{dx}$, where $y = \displaystyle\int_{2x}^{3x} \sin(t^3+1)\ dt.$

18. Find the area of the bounded region above the x-axis and below the graph of $f(x) = 4x^2 - x^4$.

19. Consider the "triangular" region T in the first quadrant that is bounded above by the curve $y = \cos x$, below by the curve $y = \sin x$ and on the left by the y-axis. Find the area of T.

20. Find the area of the region bounded by the functions $y = 2 - x^2$ and $y = x^2$.

21. Find the area of the "triangular" region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.

Answers and explanations

(1) This is an implicit function problem, we need to find $\frac{dy}{dx}$ but the function y is given as solution of an equations. Thus, we differentiate both sides of the equation with respect to x

$$e^{2x+y}\frac{d(2x+y)}{dx} + \frac{d(xy)}{dx} = \frac{dx}{dx} = 1$$

$$(2+\frac{dy}{dx})e^{2x+y} + x\frac{dy}{dx} + y = 1$$

which, at $(0,0)$ becomes $2 + \frac{dy}{dx} = 1$ or $\frac{dy}{dx} = -1$. From here, the tangent line will be

$$y = -1x + b$$

where b is found by substituting $x = 0$ and $y = 0$. Thus $b = 0$ and the line is $y = -x$. The normal line is of the form

$$y = x + c$$

here again $c = 0$ since it needs to go through $(0,0)$, The normal is $y = x$.

(2) – If x and y are such that $y = f(x) = 2$ then by definition $y^3 + xy = 12$, i.e.

$$2^3 + x\cdot 2 = 12,\ \text{so that}\ x = \frac{12-2^3}{2} = \frac{12-8}{2} = 2.$$

– We must use implicit differentiation. From the equation that $y = f(x)$ satisfies we get

$$\begin{aligned}
y^3 + xy &= 12 && \text{true for all } x\text{, so we are allowed to differentiate both sides w.r.t. } x \\
\frac{dy^3}{dx} + \frac{d(xy)}{dx} &= \frac{d\,12}{dx} \\
3y^2\frac{dy}{dx} + 1 \cdot y + x\frac{dy}{dx} &= 0 \\
\big(3y^2 + x\big)\frac{dy}{dx} + y &= 0 && \text{now solve for } \frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{-y}{3y^2 + x}.
\end{aligned}$$

This is still true for all x and y provided they satisfy $y^3 + xy = 12$.
At the point that we are given in the problem we have $x = 11$ and $y = 1$, so there we have

$$\frac{dy}{dx} = \frac{-1}{3 \cdot 1^2 + 11} = -\frac{1}{14}.$$

(3) This is a related rates problem. In such problems it is important to distinguish between facts that are true at all times (and which we are therefore allowed to differentiate with respect to time), and facts that are true at only one instant.

Let the road be the x-axis, call the x-coordinates of the plane and the car $P(t)$ and $C(t)$. These are functions of time. We are given

$$P'(t) = \frac{dP}{dt} = 120 \text{ mph.}$$

We are asked to find $C'(t)$ (the velocity of the car).

We are given that the height of the plane above the ground is 3 miles.

Let us call the distance between the plane and the car $D(t)$.

By Pythagoras' theorem we have

$$\big(C(t) - P(t)\big)^2 + 3^2 = D(t)^2.$$

This is true at all times t so we may differentiate this (using the chain rule):

$$2\big(C(t) - P(t)\big) \cdot \big(C'(t) - P'(t)\big) + 0 = 2D(t)D'(t),$$

i.e., canceling the 2s

$$\big(C(t) - P(t)\big) \cdot \big(C'(t) - P'(t)\big) = D(t)D'(t),$$

In the problem we are also given that at one given moment $D = 5$ miles and $D' = -160$ miles per hour (D' is negative because the distance between car and plane is given to be decreasing.)

From Pythagoras we then also have

$$C - P = \sqrt{5^2 - 3^2} = 4 \text{ miles.}$$

Combining all the facts we have found so far we see that

$$4 \cdot \big(C'(t) - 120\big) = 5 \cdot (-160).$$

Solving this for C' leads to

$$C'(t) = 120 - \frac{5 \cdot 160}{4} = 120 - 200 = -80 \text{ mph.}$$

So at the moment that the pilot measures the speed of the car the car is moving at 80 miles per hour in the direction of the plane (since $C' < 0$.)

(4) (1) $\dfrac{dy}{dx} = e^{2x\cos x}\dfrac{d(2x\cos x)}{dx} = e^{2x\cos x}(2\cos x - 2x\sin x)$

(2) First we simplify
$$\ln\frac{(x+1)^2}{(x^2+1)^3} = 2\ln(x+1) - 3\ln(x^2+1)$$
then we differentiate, using the chain rule
$$\frac{dy}{dx} = \frac{2}{x+1} - 3\frac{2x}{x^2+1}.$$

(3) First we transform the equation to
$$\tan\theta = 1 - z$$
and we apply the implicit function theorem differentiating with respect to z in both sides
$$(1+\tan^2\theta)\frac{d\theta}{dz} = -1 \implies (1+(1-z)^2)\frac{d\theta}{dz} = -1 \implies \frac{d\theta}{dz} = \frac{-1}{1+(1-z)^2}.$$

(4) When we have functions in both the base and exponent of a function, as in this case, the best is to apply logarithms to both sides. Thus
$$\ln y = \ln x^{\ln x} = \ln x \ln x = (\ln x)^2.$$
we then apply implicit differentiation to get
$$\frac{y'}{y} = 2\ln x\frac{1}{x}$$
and from here
$$y' = \frac{2\ln x}{x}y = x^{\ln x}\frac{2\ln x}{x}$$

(5) (1) This is correct, since if $f'(x) > 0$ the function needs to be increasing at x (by a theorem in the book), so it cannot be nonincreasing.

(2) This is false, take for example $f(x) = x^3$. The function is always increasing, but $f'(x) = 3x^2$ which is zero at $x = 0$.

(3) This is false, take the function $f(x) = 0$, $f' \geq 0$ but it is not increasing.

(4) This is correct, this is one of the theorems in the book. (You can make a stronger statement using the mean values theorem, but I would accept something like this). The stronger statement would be: if $a < b$, then
$$\frac{f(b) - f(a)}{b - a} = f'(c)$$
for some $a < c < b$. And since $f'(c) < 0$, we have $f(b) < f(a)$, so f is decreasing.

(6) (1) If we make $u = -\frac{1}{x}$, we have $\lim\limits_{x\to+\infty} x^3 e^{-1/x} = \lim\limits_{u\to 0^-} -\dfrac{e^u}{u^3} = +\infty.$

(2) We factor the exponential with the highest base, that is e^t. We get

$$\lim_{t\to+\infty}\frac{e^t + e^{-t}}{2e^t - te^{-t}} = \lim_{t\to+\infty}\frac{1+e^{-2t}}{2 - te^{-2t}}.$$

And since $\lim_{t\to+\infty} e^{-2t} = \lim_{t\to+\infty} te^{-2t} = 0$ we have

$$\lim_{t\to+\infty} \frac{e^t + e^{-t}}{2e^t - xe^{-t}} = \frac{1}{2}.$$

(3) $\lim\limits_{x\to 0+} x(\ln x)^2 = \lim\limits_{x\to 0+} (x^{1/2}\ln x)^2 = 0$, since $\lim\limits_{x\to 0+} x^{1/2}\ln x = 0$

(4) $\lim\limits_{x\to+\infty} \dfrac{(\ln x)^2}{x} = \lim\limits_{x\to+\infty} \left(\dfrac{\ln x}{x^{1/2}}\right)^2 = 0$ since $\lim\limits_{x\to+\infty} \dfrac{\ln x}{x^{1/2}} = 0.$

(5) In this one we use l'Hopital

$$\lim_{x\to 1} \frac{a\ln x + x - 1}{x^3 - x} = \frac{0}{0} = \lim_{x\to 1} \frac{\frac{a}{x} + 1}{3x^2 - 1} = \frac{a+1}{2}.$$

(7) Let x and y be two numbers such that $x + y = 20$. We want to maximize

$f(y) = x + \sqrt{y} = 20 - y + \sqrt{y}.$

We use the usual method

$$f'(y) = -1 + \frac{1}{2\sqrt{y}} = 0 \implies \sqrt{y} = \frac{1}{2}, \implies y = \frac{1}{4}.$$

This value indeed produces a max since using the second derivative test

$$f''(y) = -\frac{1}{4}y^{-3/2} < 0.$$

The value for x is $x = 20 - y = \frac{79}{4}$ and the maximum value is $f(\frac{1}{4}) = \frac{81}{4}$.

(8) We need to calculate the second derivative of the function $f''(x) = 6x + 2b$ and find the values of b for which it vanishes at $x = 1$. This gives us $b = -3$. We then check that $f''(0) < 0$ and $f''(2) > 0$ and so the point is an inflection point.

(9) Let x be the size of the sides of the squared base and y the height of the rectangular box, both in meters. If the container must hold 9 cubic meters, then

$$x^2y = 9, \quad y = \frac{9}{x^2}.$$

Since we want to have the cheapest box, we need to minimize the cost of the box. The cost for the base is \$2 per square meter, and since it has x^2 square meters, the cost is $2x^2$. The material for the sides is \$3 per square meter, and since the area of one side is xy square meters, we have that the cost for one side is $3xy$, and for four sides is $12xy$. Thus, the total cost is

$C(x, y) = 2x^2 + 12xy.$

We need this function **in one variable only** before we start the study, so we use $y = 9/x^2$ and

$C(x) = 2x^2 + (12)9/x.$

We can now find the minimum or maximum.

$C'(x) = 4x - (12)9/x^2 = 0, \implies x^3 = (12)9/4 = 27. \implies x = 3 \implies y = 1.$

We now check that these values are a minimum. We use the second derivative test

$C''(x) = 4 + (12)18/x^3 > 0$

if $x > 0$. $C''(3) = 12 > 0$, thus it is a minimum.

(10) (1) Since there are no vertical asymptotes, we only need to find zeroes of f to know where f is positive or negative. The only x for which f vanishes is $x = 0$, and $f(-1) < 0$ while $f(1) > 0$. Therefore, f is positive for $x > 0$ and negative for $x < 0$.

(2) To answer this question we need to find where the derivative of the function is positive or negative. We calculate

$$f'(x) = \frac{2(1+x^2) - (2x)^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2} = 0 \implies 1 - x^2 = 0 \implies x = \pm 1.$$

Using that $f'(-2) < 0$, $f'(0) > 0$ and $f'(2) < 0$ we conclude that the function is increasing for $-1 < x < 1$ and decreasing elsewhere.

(3) Since f decreases to the left and increases to the right of $x = -1$, this points represents a local minimum achieved at $f(-1) = -1$. Since it increases to the left and decreases to the right of $x = 1$, $x = 1$ represents a maximum, achieved at $f(1) = 1$.

(4) To see if these are global max and min we need to find the limits at $\pm\infty$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x}{x^2} \frac{2}{\frac{1}{x^2}+1} = 0$$

and so both of the extrema are global (you should include a picture of the graph here to bring your point).

(5) To find inflections points we need to analyze the second derivative of f.

$$f''(x) = \frac{-4x((1+x^2)^2) - 2(1+x^2)2x(2-2x^2)}{(1+x^2)^4} = \frac{-4x(1+x^2) - 4x(2-2x^2)}{(1+x^2)^3} = \frac{4x^3 - 12x}{(1+x^2)^3}.$$

The second derivative vanishes when

$$4x^3 - 12x = 4x(x^2-3) = 0$$

that is, when $x = 0, \pm\sqrt{3}$. Since $f''(-100) =< 0$, $f''(-1) > 0$, $f''(1) < 0$ and $f''(100) > 0$ we know that the function is convex if $\sqrt{3} < x < +\infty$ and $-\sqrt{3} < x < 0$, and concave if $-\infty < x < -\sqrt{3}$ and $0 < x < \sqrt{3}$. The inflection points are $(0,0), (\sqrt{3}, \frac{1}{2}\sqrt{3})$ and $(-\sqrt{3}, -\frac{1}{2}\sqrt{3})$.

(6) There are no vertical asymptotes. Since

$$\lim_{x\to\pm\infty} f(x) = 0$$

there is a horizontal asymptote given by $y = 0$. Since

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x}{x^3} \frac{2}{\frac{1}{x^2}+1} = 0$$

the only slanted asymptote is the horizontal one we already have.

After this evaluation you will need to give a rough description of the graph.

(11) We first find the local minimum for $x > 0$, the usual way

$$f'(x) = \ln x + 1 = 0, \implies \ln x = -1, \implies x = \frac{1}{e}.$$

We can either check the sign of f' to the left and the right of $1/e$ or use the second derivative test, which says

$$f''(x) = \frac{1}{x} > 0$$

for $x > 0$. Therefore, $1/e$ is a local minimum and $f(1/e) = -\dfrac{1}{e}$ is the minimum value. Since $\lim_{x\to 0^+} x\ln x = 0$ and $\lim_{x\to+\infty} x\ln x = +\infty$, we have that $-\dfrac{1}{e}$ is the global minimum.

(12)
- The function is always positive.
-
$$f'(x) = -\frac{2x}{(x^2+3)^2}$$
so $f'(x) = 0$ only if $x = 0$. Since $f'(-1) > 0$ and $f'(1) < 0$ the function is increasing when $x < 0$ and decreasing when $x > 0$. Thus $x = 0$ is a local max achieved at $(0, \frac{1}{3})$.
- Finding the limits we get $\lim_{x\to\pm\infty} f(x) = 0$, and so the function has a global maximum at $x = 0$. No global minimum.
-
$$f''(x) = -\frac{2(x^2+3)^2 - 2(x^2+3)(2x)^2}{(x^2+3)^4} = -\frac{2(x^2+3) - 2(2x)^2}{(x^2+3)^3} = \frac{6x^2-6}{(x^2+3)^3}.$$
From here the inflection points are given by $x^2 - 1 = 0$, that is $x = \pm 1$.
- Since $f''(-100) > 0$, $f''(0) < 0$ and $f''(100) > 0$, they are indeed inflection points since the function is convex when $x < -1$ and $x > 1$ and concave if $-1 < x < 1$.

You then need to sketch the graph.

(13) (1) A point has horizontal tangent line if $y'(t) = 0$. We calculate
$$y'(t) = \frac{2(1+t^2) - (2t)^2}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2}$$
so $y'(t) = 0$ for $t = \pm 1$. These are the points $(0, 1)$ and $(1, -1)$.

(2) To have vertical tangent lines we need $x'(t) = 0$. We calculate
$$x'(t) = \frac{-3t^2(1+t^2) - 2t(1-t^3)}{(1+t^2)^2} = \frac{-t^4 - 3t^2 - 2t}{(1+t^2)^2}$$
This vanishes at $t = 0$, that is at $(1, 0)$.

(3) When $t = 2$ the point is $(-\frac{7}{5}, \frac{4}{5})$ and the slope is
$$\frac{y'(2)}{x'(2)} = \frac{-6/25}{-32/25} = \frac{6}{32} = \frac{3}{16}$$
hence the line is given by
$$y - \frac{4}{5} = \frac{3}{16}(x + \frac{7}{5}).$$

(14) If $y(t)$ is decaying at a rate proportional to itself, it means $y(t)$ follows an exponential decay and
$$y(t) = y(0)e^{kt}$$
for some initial amount $y(0)$ and some constant $k < 0$. Since we are told that the half life is a year, we have that
$$y(1) = \frac{1}{2}y(0) = y(0)e^k, \implies e^k = \frac{1}{2}, \implies k = \ln\frac{1}{2} = -\ln 2.$$

We are asked how long it takes to be 1/3 of the original amount. That translates into

$$y(t) = \frac{1}{3}y(0) = y(0)e^{kt}, \implies e^{kt} = \frac{1}{3}, \implies kt = \ln\frac{1}{3} = -\ln 3, \implies t = \frac{-\ln 3}{k}, \implies t = \frac{\ln 3}{\ln 2}.$$

The time t is given in years.

(15) (1) This is false, it is only true if the function is not negative, if negative the integral is the negative of the area.

(2) This is false, the correct answer, by the fundamental theorem of Calculus, would be $\int_0^x F'(t)dt = F(x) - F(0)$ since $F(x)$ is an antiderivative of $F'(x)$.

(3) This is false since t is a dummy variable, not a real variable (also, see the previous one).

(4) This is correct, it is the statement of the second fundamental theorem of Calculus.

(16) (1) $\int_0^\pi \left(\sqrt{x^3} + \sin x\right) dx = \int_0^\pi \left(x^{3/2} + \sin x\right) dx = \left(\frac{x^{5/2}}{5/2} - \cos x\right) |_0^\pi = \frac{2\pi^{5/2}}{5} + 1 + 1 = 2 + \frac{2\pi^{5/2}}{5}$

(2) Let us choose $u = \sin x$ so that $du = \cos x dx$. Substituting in the integral we get $\int \frac{\cos x}{1+\sin^2 x} dx = \int \frac{1}{1+u^2} du = \arctan u + C = \arctan(\sin x) + C$.

(3) Call $u = \cos t$ so that $du = -\sin t dt$. Then $\int_0^{\pi/2} e^{\cos t} \sin t dt = -\int_1^0 e^u du = -e^u|_1^0 = e - 1$.

(4) Call $u = e^{2x}$.

(5) Call $u = e^x$.

(17) $\frac{dy}{dx} = -2\sin(8x^3+1) + 3\sin(27x^3+1)$.

(18) The function f intersects the x-axis at $x = 0$ and $x = \pm 2$. Since $f(100) < 0$, $f(-1) > 0$, $f(1) > 0$ and $f(100) < 0$, the region where the function is above the axis is the interval $[-2, 2]$. Therefore, the area the problem asks for is

$$\int_{-2}^2 (4x^2 - x^4)dx = \frac{4}{3}x^3 - \frac{1}{5}x^5|_{-2}^2 = \frac{4}{3}2^3 - \frac{1}{5}2^5 + \frac{4}{3}2^3 - \frac{1}{5}2^5 = \frac{4}{3}2^4 - \frac{1}{5}2^6 = 2^4(\frac{4}{3} - \frac{4}{5}).$$

(19) We first need to know where the functions $\cos x$ and $\sin x$ intersect in the first quadrant. We know that $\sin x = \cos x$ for $x = \frac{\pi}{4}$, therefore the area will be given by the integral (you will need a picture here)

$$\int_0^{\pi/4} (\cos x - \sin x)dx = \sin x + \cos x|_0^{\pi/4} = 2\frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1.$$

(20) We find intersection in the functions: $2 - x^2 = x^2 \implies x = \pm 1$. Thus, the region we are looking for is the integral between -1 and 1. Since at zero $2 - x^2$ is larger than x^2, the area will be given by

$$\int_{-1}^{1} (2 - x^2 - x^2)dx = 2\int_{-1}^{1} (1 - x^2)dx = 2(x - \frac{1}{3}x^3)|_{-1}^{1} = 2(2/3 - (-2/3)) = 8/3.$$

(21) We first find the intersections:

$6 - x = \sqrt{x}, \implies 36 - 12x + x^2 = x, \implies x^2 - 13x + 36 = 0, \implies x = 4, x = 9.$

For these two values only one ($x = 4$) is valid since $x = 9$ gives $y = \sqrt{9} \neq 6 - 9$.

We also have that $y = \sqrt{x}$ intersects $y = 1$ at $x = 1$ and $y = 6 - x$ intersects $y = 1$ at $x = 5$. You now need to draw a picture here. Since both functions are above $y = 1$, the area we are looking for is then given by the integral

$$\int_{1}^{4} (\sqrt{x} - 1)dx + \int_{4}^{5} (6 - x - 1)dx$$

Finally

$$\int_{1}^{4} (\sqrt{x} - 1)dx + \int_{4}^{5} (6 - x - 1)dx = (\frac{2}{3}x^{3/2} - x)|_{1}^{4} + (5x - \frac{1}{2}x^2)|_{4}^{5} = \text{whatever...}$$