MATH 221 PROJECT

Limits

1. Purpose of this project

We will explore how limits can be used to solve interesting problems, some of which were "paradoxes" for thousands of years before calculus came to the rescue. In the next chapter of the notes, we will see how limits can be used to formally define a notion of a "derivative." In this project, we will use the limit for something else: trying to figure out distances traveled and areas under curves (which will later be called an *integral*). This project is theoretical in nature. However, students who put time into trying to fully understand the concepts involved will surely be rewarded for their efforts!

2. Overview

While the concept of the "limit" may seem hard to grasp, it is not an exaggeration to say that it is *the* conceptual breakthrough that allowed for calculus to be developed (and, hence, all the nice things that followed: air travel, mp3 players, etc.). As we have already seen, the limit is the necessary concept required to define a derivative. However, there is another object in calculus, the *integral*, that will be defined in the coming chapters and that relies on the concept of a limit as well. This project will include our first computation of an integral, though we will not formally make this connection until later in the semester. We will also point out how our computation of an integral is related to an ancient paradox.

3. A (seeming?) paradox

The following is a version of a paradox usually attributed to the Greek philosopher Zeno[3]:

Imagine that a person is walking towards a wall that is currently one mile away. It seems that this person should have no trouble walking to the wall. However, we will consider a different way to think about how much distance this person must travel to reach the wall:

(1) The person must walk half to the wall, yielding a total distance of 1/2 mile. This leaves 1/2 mile to go.
(2) The person must walk one half of the remaining 1/2 mile. This leaves 1/4 mile to go.
(3) The person must walk one half of the remaining 1/4 mile. This leaves 1/8 mile to go.

$$\vdots$$

Therefore, we have broken the interval of one mile down into an infinite number of finite pieces of length

$$1/2, \quad 1/4, \quad 1/8, \quad 1/16, \quad etc.,$$

and so the total distance that must be traveled must be infinity! Hence, *the person should never reach the wall!* Another way to put this is that the requirement of walking a mile has been broken into an infinite number of tasks (walking half of the remaining distance over and over again), and this should be impossible.

[3]See, for example, http://en.wikipedia.org/wiki/Zeno's_paradoxes.

3.1. A resolution. If the above argument does not sit well with you, then you have good intuition. Our resolution of the "paradox" rests on limits at infinity (page 29 of the notes). We will let $f(n)$ denote the distance traveled by our wanderer after completion of the first n "tasks" as described in the section above. Thus, we see

$$\begin{aligned} f(1) &= \frac{1}{2} \\ f(2) &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ f(3) &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8} \\ &\vdots \\ f(n) &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}. \end{aligned}$$

Therefore, using the Limit Property (P4) on page 39 of the text, with the conclusion of Example 5.5 on page 37, we have that

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1.$$

Thus, even with the perspective of breaking the interval up into an infinite number of intervals, we arrive at the (reasonable) conclusion that the total distance traveled is one mile.

4. Area under a triangle

We now turn to a seemingly disparate topic: what is the area of a triangle? More specifically, consider the graph of the function $f(x) = x$ on the interval $[0, 1]$. The space between the x-axis and $f(x)$ for $x \in [0, 1]$ makes a triangle and we want to know the area. We will solve this problem in two ways. The first is using simple geometry, and will be familiar to everyone. The second uses limits and is a precursor to the much deeper idea of *integration* that you will be exposed to later in the course.

Solution 1. The triangle has height equal to one, and a base of size one. Hence, the area of the triangle is

$$\frac{1}{2} \cdot \text{base} \times \text{height} = \frac{1}{2}.$$

Solution 2. Limits commonly arise when we wish to make an approximation precise. This is what we will do here. More precisely, we will make a series of approximations, with the nth approximation denote by $A(n)$. We will only assume that we know how to compute the area of rectangle.

For $n \in \{1, 2, 3, \dots\}$, we will break the interval $[0, 1]$ up into n equally spaced intervals. That is,

(1) For $n = 1$, there is only one interval, $[0, 1]$.
(2) For $n = 2$, there are two intervals, $[0, 1/2]$ and $[1/2, 1]$.
(3) For $n = 3$, there are three intervals, $[0, 1/3]$, $[1/3, 2/3]$ and $[2/3, 1]$.
(4) The n intervals for an arbitrary integer n are

$$[0, 1/n], [1/n, 2/n], \dots, [(n-1)/n, 1].$$

Now, we will approximate the area of the space between the graph of $f(x)$ and the x-axis on each subinterval,

$$[i/n, (i+1)/n],$$

by the area of a rectangle, which we know how to compute. Since $f(x) = x$ for all x, we know that

$$f(i/n) = i/n$$

and we can approximate the area of the ith subinterval by

$$f(i/n) \times \frac{1}{n} = \frac{i}{n}\frac{1}{n} = \frac{i}{n^2}.$$

Adding up all these areas then yields the approximation (note that the first interval is always being evaluated at the left endpoint, or zero)

$$A(n) = \sum_{i=0}^{n-1} \frac{i}{n^2} = 0 \times \frac{1}{n^2} + 1 \times \frac{1}{n^2} + 2 \times \frac{1}{n^2} + \cdots + (n-1)\frac{1}{n^2}.$$

Note that n is fixed in the above sum!

5. Problems

1. Draw a detailed picture of the preceding argument. More specifically, for $n = 3, 4$, and 5,

(a) Plot $f(x) = x$ on the interval $[0, 1]$.

(b) Break the interval up into n evenly spaced subintervals.

(c) Show the area of the rectangles being computed.

(d) Compute the approximate area for each $n \in \{3, 4, 5\}$.

2. Let m be an arbitrary integer. We will compute

$$1 + 2 + \cdots + m.$$

To do so, let $S(m)$ be the (unknown) sum and add the following vertically:

$$\begin{array}{rl} 1 \quad + \quad 2 \quad + \cdots + (m-1) + \quad m \quad & = S(m) \\ m \quad + (m-1) + \cdots + \quad 2 \quad + \quad 1 \quad & = S(m) \\ \hline \implies (m+1) + (m+1) + \cdots + (m+1) + (m+1) & = 2S(m) \end{array}$$

Conclude that

$$S(m) = \frac{m(m+1)}{2}.$$

3. Show that

$$A(n) = \frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n}.$$

where $A(n)$ is the approximate to the area of the triangle after breaking $[0, 1]$ into n subintervals. Conclude that

$$\lim_{n \to \infty} A(n) = \frac{1}{2}.$$

Report Instructions

Write a few paragraphs describing Zeno's paradox and its resolution. In particular, note that the results from the class notes (page 29) allowed us to conclude that

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

whereas we used that

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Why is this okay to do? Explain.

Write a few paragraphs describing how to compute the area of a triangle. Be sure to include solutions to all of the exercises. Note especially that the second method of computing the area is much harder than the first. However, what if the function we were approximating, f, where not a straight line, but had a "curve" to it (like a sine function)? Say a few words on how one may hope to find the area between its plot and the x-axis. Note that you *could not* simply resort to using a straight geometric argument like we could for the triangle. We will formalize this method (and call it the *definite integral*) in later chapters.