MATH 221 PROJECT

Derivatives

(and the important topic of ants walking on ropes)

Purpose of this project

In this project we will explore the following problem, which we will solve using the rules of differentiation (and a little integration, which will come later in the class).

Problem statement: [4] Suppose there is a rope that is very pliable stretched to be of size 1 meter. Suppose that one end of the rope is pinned down and an ant is placed at that end. At that moment, the ant starts walking at a rate of 1 meter per hour towards the other end. However, at the same time we begin stretching the rope (away from the pinned down side) so that the end of the rope is moving at speed 10 meters per second. Will the ant ever reach the end of the rope?

While this version of the problem may seem strange, it actually has relevance to a very reasonable question coming from cosmology: can the light from a distant galaxy ever reach us if the universe is expanding? At issue is when a galaxy is so far away that the relative speed between our galaxy and the other is greater than the speed of light due to the expansion of the universe. In this scenario, it seems that the light from that far flung galaxy should never reach us. However, in an analogous manner as the ant on the rubber rope, we can be sure it does without having to resort to a breakdown in the laws of physics.

1. Solution

We will solve the problem with the numbers given in the statement of the problem. The answer is "yes, the ant does reach the end." Surprisingly, the fact that the ant eventually reaches the other end does not depend upon the specific numbers given. Thus, even if the ant were walking at 0.0001 meters per second, and you were stretching this rope near the speed of light, the ant will still eventually reach the end. Of course, the *time* at which the ant reaches the end depends upon the rates chosen, so it would be more accurate to say that an immortal ant would eventually reach the end.

Any good solution to a hard word problem requires good notation! So, we begin with the following.

(1) Let $D(t)$ denote the distance of the ant from its starting point at time t. Thus, for example, $D(0) = 0$.

(2) Note that because the end of the rope is moving at a constant speed of 10 meters per second, and because the end of the rope as located one meter from the pinned down edge at time zero, the position of the end of the rope at time t is

$$1 + 10t.$$

(3) Let $P(t)$ denote the proportion of the rope that the ant has already traversed. Note that until the ant reaches the end, $P(t)$ is always between 0 and 1. For example, if $P(t) = 0$, then the ant is at the beginning of the rope (which is true when $t = 0$), and if $P(t) = 1$, then the ant has reached

[4] See also the wikipedia page on this problem at http://en.wikipedia.org/wiki/Ant_on_a_rubber_rope

the end. Also note that we can combine our knowledge of $D(t)$ and the position of the end of the rope to conclude that

$$P(t) = \frac{D(t)}{1+10t}.$$

With the above notation in hand, we can start thinking about how to solve the problem. It should be clear that we hope to show that

$$P(t^*) = 1,$$

for some finite t^*. However, to get there, we need to first consider $D(t)$.

2. The function $D(t)$

We need to understand how the position of the ant, $D(t)$, is changing with respect to time. The difficult issue is that the ant is moving due to two things

(1) Its own movement relative to the rope.
(2) The movement due to the pulling of the rope.

Whenever we have more than one action forcing something else to change we should try to break up the overall behavior into its components. Therefore, we should expect to have

$$D(t+h) = D(t) + \text{“movement due to ant only”} + \text{“movement due to rope”}.$$

2.1. Movement due to ant.

Since the ant is moving at a speed of 1 m/h, and the amount of time that passes is h hours (which we think of as very small, of course), we have

$$D(t+h) = D(t) + 1 \cdot h + \text{“movement due to rope”}.$$

2.2. Movement due to rope.

The movement due to the rope seems tricker. However, thinking along the following lines will save the day:

(1) We need to understand how fast a particular point on the rope is moving.
(2) We should recognize that for each point on a rope, it's relative position on the rope (i.e. the proportion of the rope behind (or in front of) it) should not change with time.

Thus, if $x \in [0,1]$ was a point on the rope at time zero, and $X(t)$ denotes its position at time t, then

$$\frac{x}{1} = \frac{X(t)}{1+10t}, \tag{6}$$

for all t. Since it holds for all t, we can conclude that

$$\frac{X(t+h)}{1+10(t+h)} = \frac{X(t)}{1+10t}, \tag{7}$$

since both sides equal the same thing (namely, $x/1$).

3. Problem

1. Explain why equation (6) should hold for *all* $t \geq 0$. Show that equation (7) implies that

$$X(t+h) - X(t) = X(t)\frac{10h}{1+10t}. \tag{8}$$

3.1. Returning to $D(t)$. We return to our analysis of $D(t)$ armed with more knowledge. Since the position of the ant at time t is $D(t)$, equation (8) tells us that the total movement of the ant due to the stretching rope over a time period of size h is approximately

$$D(t)\frac{10h}{1+10t}.$$

Combining all of the above, we have

$$\begin{aligned} D(t+h) &= D(t) + \text{“movement due to ant only”} + \text{“movement due to rope”} \\ &= D(t) + h + D(t)\frac{10h}{1+10t}. \end{aligned}$$

Rearranging terms and dividing by h gives

$$\frac{D(t+h)-D(t)}{h} = 1 + D(t)\frac{10}{1+10t}.$$

Of course, we know that as $h \to 0$, the right hand side becomes $D'(t)$. Thus, we can conclude that

$$D'(t) = 1 + D(t)\frac{10}{1+10t}, \tag{9}$$

where we also know that $D(0) = 0$. The above equation tells us that the function D has the following properties:

(1) It has a value of zero at time zero (we already knew that, of course!).
(2) Its derivative is equal to itself multiplied by

$$\frac{10}{1+10t},$$

plus one.

4. So what?!?! – Change perspective.

Unfortunately, equation (9) is a differential equation, and is beyond the scope of this class to solve. That is, at this point in your mathematical education it is not clear at all that there should even be a function satisfying such an odd equation.[5] Therefore, this seems like a dead-end.

However, at this point, we can remember that it was not $D(t)$ that we were after anyways. Instead, it was the function giving the *proportion* of the rope already traversed by time t,

$$P(t) = \frac{D(t)}{1+10t}.$$

5. Problem

1. Show that

$$P'(t) = \frac{D'(t)}{1+10t} - \frac{10D(t)}{(1+10t)^2}.$$

Then use (9) to conclude that

$$P'(t) = \frac{1}{1+10t}. \tag{10}$$

[5]There is one. However, you will need more math courses to be able to solve these types of equations!

It is important to note that equation (10) is much simpler than equation (9) since the right hand side does not depend on anything except t. Thus, equation (10) can be read in the following manner:

$$\text{“Find a function } P(t) \text{ whose derivative is } \frac{1}{1+10t}.\text{”}$$

Note also that we require $P(0) = 0$. Finding such a P is actually the exact *opposite* of what you have learned to do so far, which is how to *take* a derivative.

6. What to do?

Later in the course, you will find an explicit representation for $P(t)$ (if you must know now, the solution is $P(t) = (1/10)\ln(1 + 10t)$, where "ln" is the natural logarithm). For now, it is sufficient to know that there is a solution, and it yields a t^* at which

$$P(t^*) = 1.$$

For example, the value of t^* for our specific choice of constants is

$$t^* = \frac{e^{10} - 1}{10} \approx 2,202.5.$$

7. Problems

1. Redo the analysis of the project except assume that the ant is walking at a speed of s meters per hour (as opposed to one meter per hour) and that the end of the rope is being pulled at a rate of r meters per hour. Of course, assume that $s > 0$ and $r > 0$. You should conclude that in this (general) case

$$P'(t) = \frac{s}{1+rt}.$$

2. Assume now that the ant is walking at 1 meter per hour and the rope is being pulled at a rate of 2 meters per hour. Using the result from the previous exercise, argue that for small h,

$$P(t+h) \approx P(t) + \frac{1}{1+2t}h. \tag{11}$$

Using this approximation for P, fill in the following tables to approximate t^* for which $P(t^*) = 1$:

t	$P(t)$ ($h = \frac{1}{2}$)
0.0	0.00
0.5	0.50
1.0	0.75
1.5	0.92
2.0	1.04
2.5	1.14

t	$P(t)$ ($h = \frac{1}{3}$)
0.00	0.00
0.333	0.33
0.666	0.53
1.000	
1.333	
1.666	
2.000	
2.666	
3.000	
3.333	

t	$P(t)$ ($h = \frac{1}{4}$)
0.00	0.00
0.25	0.25
0.50	0.42
0.75	
1.00	
1.25	
1.50	
1.75	
2.00	
2.25	
2.50	
2.75	
3.00	
3.25	
3.50	

For example, the first chart giving the computation with $h = 1/2$ yields an approximation

$$t^* = 2.$$

Report Instructions

Using complete sentences, write out solutions to all of the exercises.