1. Find the limit or show that it is infinite or not defined

   (a) $\displaystyle\lim_{x\to 1} \frac{2\sqrt{x}-2x}{1-x}$

   (b) $\displaystyle\lim_{x\to 1} \frac{2\sqrt{x}-x}{(1-x)^2}$

   (c) $\displaystyle\lim_{x\to 0} \frac{1-\cos(3x)}{2x\sin x}$ (no L'Hopital allowed in this one).

   (d) $\displaystyle\lim_{x\to \pi/2} \frac{\cos x}{x-\pi/2}$

   (e) $\displaystyle\lim_{x\to \infty} \sqrt{x+1}-\sqrt{x}$

   (f) $\displaystyle\lim_{x\to \infty} (x+1)^2-x^2$

   (g) $\displaystyle\lim_{x\to -\infty} \frac{x^3+2x}{(x^2+1)(2x-1)}$

   (h) $\displaystyle\lim_{t\to \infty} \frac{2t+3}{\ln t+t}$.

2. The function f is defined on $\mathbb{R}$ and it is given by $f(x) = \dfrac{1-\cos(x^2)}{x^n}$ when $x \neq 0$. If f is continuous at $x=0$, what are the possible values of n and how much is $f(0)$?

3. Use the definition of derivative to find the derivative of $f(x) = x^2+2$.

4. Show that $f(x) = |x+1|-1$ is not differentiable.

5. (a) A particle is constrained to move along a parabola whose equations is $y = x^2$. At what point on the curve are the x and y coordinates changing at the same rate?

   (b) A 6 foot man walks away from a 10 foot high lamp at the rate of 3 ft per sec. How fast is the tip of his shadow moving?

6. Find the derivatives of

   (a) $f(x) = \dfrac{\tan(ax)}{1+x^2} + \ln a$, where a is a constant;

   (b) $f(x) = \ln(x^2+e^x)$;

   (c) $f(x) = \sin(\ln(x))$;

   (d) $f(x) = \displaystyle\int_{\ln x}^{x^2} a\sin^3(t)dt$, with a constant;

   (e) $f(x) = \arcsin(2x^2+3)$.

   Find the second derivative of (d) above.

7. For what values of a and b is it true that f is a differentiable function?

(a) $f(x) = \begin{cases} ax & \text{if } x \leq 1 \\ x^2 + b & \text{if } x > 1 \end{cases}$;

(b) $f(x) = \begin{cases} ax & \text{if } x \leq -1 \\ x^2 + b & \text{if } x > -1 \end{cases}$;

(c) $f(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ ax + b & \text{if } x > 0 \end{cases}$;

8. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel.

9. The processing of raw sugar has a step called *inversion* that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining.

   If 1000Kg of raw sugar reduces to 800Kg of raw sugar during the first 10 hours, how much raw sugar will remain after another 14 hours?

10. (a) You are designing a mural and you would like to have a margin of 5 feet on the left, right, and top of the artwork, but none on the bottom. If you allow 800 ft^2 for the area containing the artwork itself, what dimensions should the wall have if you want to minimize the total area of the wall (i.e., of the artwork and the margins.)

    (b) A man has 112 miles of fencing for inclosing two separate lots, one of which is to be a square and the other a rectangle which is three times as long as it is wide. Find the dimensions of each lot so that the total area which is inclosed shall be a minimum.

11. Find intervals where the function below is positive or negative, increasing or decreasing; find local maxima and minima; inflection points and intervals where it is concave or convex; find all asymptotes and sketch the graph. Do they have an absolute maximum or minimum?

    (a) $f(x) = \dfrac{\ln(x)}{x}$; on $(0, \infty)$

    (b) $f(x) = \dfrac{e^x}{x}$;

    (c) $f(x) = \dfrac{x^2 - 1}{x}$.

12. Find the following integrals

    (a) $\displaystyle\int_0^{\ln 5} e^r (3e^r + 1)^{-3/2} dr$;

    (b) $\displaystyle\int_e^{e^2} \frac{1}{x\sqrt{\ln x}} dx$;

    (c) $\displaystyle\int \sec^2 x e^{\tan x} dx$;

    (d) $\displaystyle\int \frac{x^2}{1 + x^6} dx$;

    (e) $\displaystyle\int_0^{2\pi} \frac{\cos z}{\sqrt{4 + 3\sin z}} dz$;

If you need to practice more, there are many integrals in the text.

13. (a) Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x = y$.

    (b) Find the area of the region bounded by the curves $y = 4 - x^2$, $y = 2 - x$ and the lines $x = -2$ and $x = 3$.

14. (a) Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis (use both washers and cylindrical shells just to practice). Find the volume when you rotate it about the x-axis. Draw the region.

    (b) Find the volume of the solid generated by the triangle with vertices $(1,0),(2,0)$ and $(2,2)$ when rotated around: the x-axis; the y axis (use both washers and cylindrical shells to practice); the line $x = -1$. Be sure to draw the region.

    (c) Chose the simplest method (washers or shells) to calculate the volume generated by rotating the region bounded by $y = 2x - x^2$ and $y = x$ around the y axis. Draw the region.

    (d) Find the volume of revolution generated by the rotation of the graph of $y = \sqrt{x}$, $1 \le x \le 4$ around the y axis and around the x axis. Draw the function you are rotating.

15. (a) Set up the integral for the length of the curves:

    i. $y = \sin x$, $-\pi/2 \le x \le 0$;

    ii. $x = \sin y$, $0 \le y \le \pi$;

    iii. $x = \ln t$, $y = e^{t^2} - 1$, $1 \le t \le 2$.

    (b) Find the length of the curve $x = \cos t$, $y = t + \sin t$, $\pi/2 \le t \le \pi$. (You might want to use the formula $\cos t = 2\cos^2(t/2) - 1$.)

    (c) An object of mass 5 Kg moves along the x-axis with a velocity given by $v(t) = \cos 2t$. If at $t = 0$ the object is located at a distance of 2 meters form the origin, find: the force that is acting upon the object at any time t; the position of the object at any time t; the work done to move the object between the times $t = 0$ and $t = \pi$ seconds.

    (d) A particle of mass m is at $(1,0)$ and it is attracted towards the origin with a force of magnitude $F = k/t^2$. If the particle starts at rest when $t = a > 0$ and is acted on by no other forces, find the work done by the time $t = b$. Find the position of the particle at this time.

Answers

1. (a) $\displaystyle\lim_{x\to1}\frac{2\sqrt{x}-2x}{1-x} = \lim_{x\to1}2\frac{\sqrt{x}-x}{1-x}\frac{\sqrt{x}+x}{\sqrt{x}+x} = \lim_{x\to1}2\frac{x-x^2}{(1-x)(\sqrt{x}+x)} = \lim_{x\to1}2\frac{x(1-x)}{(1-x)(\sqrt{x}+x)} =$ $\displaystyle\lim_{x\to1}2\frac{x}{\sqrt{x}+x} = \frac{2}{2} = 1.$

   (b) $\displaystyle\lim_{x\to1}\frac{2\sqrt{x}-x}{(1-x)^2} = \infty$ since the top goes to 1.

   (c) $\displaystyle\lim_{x\to0}\frac{1-\cos(3x)}{2x\sin x} = \lim_{x\to0}\frac{1}{2x}(1-\cos(3x))\frac{1}{\sin x} = \lim_{x\to0}\frac{1}{2x}\frac{1-\cos(3x)}{(3x)^2}\frac{x}{\sin x}\frac{(3x)^2}{x}$ $\displaystyle= \lim_{x\to0}\frac{1-\cos(3x)}{(3x)^2}\frac{x}{\sin x}\frac{9}{2} = \frac{9}{4}.$

   (d) $\displaystyle\lim_{x\to\pi/2}\frac{\cos x}{x-\pi/2} = \left(u = \frac{\pi}{2} - x\right) = \lim_{u\to0}\frac{\cos(\frac{\pi}{2}-u)}{-u} = -\lim_{u\to0}\frac{\sin u}{u} = -1.$

(e) $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \lim_{x\to\infty} (\sqrt{x+1} - \sqrt{x}) \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to\infty} \frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}}$
$= \lim_{x\to\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = 0.$

(f) $\lim_{x\to\infty} (x+1)^2 - x^2 = \lim_{x\to\infty} 2x+1 = \infty.$

(g) $\lim_{x\to-\infty} \frac{x^3+2x}{(x^2+1)(2x-1)} = \lim_{x\to-\infty} \frac{x^3}{x^3} \frac{1+2x^{-2}}{(1+x^{-2})(2-x^{-1})} = \frac{1}{2}.$

(h) $\lim_{t\to\infty} \frac{2t+3}{\ln t + t} = \lim_{t\to\infty} \frac{t}{t} \frac{2+3t^{-1}}{t^{-1}\ln t + 1} = 2.$

2. Since we know that $\lim_{u\to 0} \frac{1-\cos u}{u^2} = 1/2$, we know that $\lim_{x\to 0} \frac{1-\cos(x^2)}{x^4} = 1/2$ (using $u = x^2$. Therefore, if $n = 4$ and $f(0) = \frac{1}{2}$, we will have a continuous function. Also, we have

$$\lim_{x\to 0} \frac{1-\cos(x^2)}{x^n} = \lim_{x\to 0} \frac{1-\cos(x^2)}{x^4} x^{4-n},$$

so if $n < 4$ the limit is $1/2 \cdot 0 = 0$, otherwise it will be not exist. So we can also have $n = 0, 1, 2, 3$ and $f(0) = 0$ and the function will be continuous.

3.

$$f'(x) = \lim_{h\to 0} \frac{1}{h} \left((x+h)^2 + 2 - (x^2+2)\right) = \lim_{h\to 0} \frac{1}{h} \left(2xh + h^2\right) = \lim_{h\to 0} 2x + h = 2x.$$

4.

$$f(x) = \begin{cases} x+1-1 & \text{if } x+1 \geq 0 \\ -(x+1)-1 & \text{if } x+1 \leq 0 \end{cases} = \begin{cases} x & \text{if } x \geq -1 \\ -x-2 & \text{if } x \leq -1 \end{cases}.$$

The side limits at -1 are both equal to -1 so the function is continuous. But the left slope at -1 is -1 while the right one is 1, and they are different. So it is not differentiable.

5. (a) $(1/2, 1/4)$.
   (b) $15/2$ feet per second.

6. (a) $f'(x) = a(1+\tan^2(ax))\frac{1}{1+x^2} - \tan(ax)\frac{2x}{(1+x^2)^2}$.
   (b) $f'(x) = (2x+e^x)\frac{1}{x^2+e^x}$.
   (c) $f'(x) = \cos(\ln x)\frac{1}{x}$.
   (d) $f'(x) = 2ax\sin^3(x^2) - \frac{a}{x}\sin^3(\ln x)$.
   (e) $f'(x) = \frac{4x}{\sqrt{1-(2x^2+3)^2}}$

   $f''(x) = 2a\sin(x^2) + 4ax^2\cos(x^2) + \frac{a}{x^2}\sin(\ln x) - \frac{a}{x^2}\cos(\ln x)$.

7. (a) $a = 2$, $b = 1$
   (b) $a = -2$, $b = 1$.
   (c) $a = 1$, $b = 1$.

8. The points are $(\pm\sqrt{7}, 0)$ and at those two points $y' = -2$.

9. 585.37 (some 24 hours have passed).

10. (a) 40 feet wide, 20 feet high for the art, 50 wide and 25 high for the wall.
    (b) 12×12, 8×24.

11. (a) Negative on $0 < x < 1$, positive for $x > 1$; increasing for $0 < x < e$, decreasing for $x > e$. It has a local max at e (which is also a global max); an inflection point at $x = e^{3/2}$, and it is concave if $0 < x < e^{3/2}$ and convex if $x > e^{3/2}$; it has a horizontal asymptote at $y = 0$ and a vertical asymptote at $x = 0$ (on the right).
    (b) Negative on $x < 0$, positive for $x > 0$; increasing for $x > 1$, decreasing for $x < 0$ and $0 < x < 1$. It has a local min at 1 (which is not global, there are no global max or min); no inflection point, it is concave if $x < 0$ and convex if $0 < x$; it has a horizontal asymptote at $y = 0$ but only on the left (as $x \to -\infty$) and a vertical asymptote at $x = 0$.
    (c) Negative on $x < -1$ and $0 < x < 1$, positive for $x > 1$ and $-1 < x < 0$; always increasing, so no local max or min; no inflection point, and it is convex if $x < 0$ and concave if $x > 0$; it has no horizontal asymptote and a vertical asymptote at $x = 0$. A slanted asymptote at $y = x$.

12. (a) $1/6$;
    (b) $2(\sqrt{2} - 1)$;
    (c) $e^{\tan x} + C$;
    (d) $\frac{1}{3}\arctan(x^3) + C$;
    (e) 0.

13. (a) $\displaystyle\int_{-1}^{0}(x - x^{1/3})dx + \int_{0}^{1}(x^{1/3} - x)dx = 1/2.$
    (b) $\displaystyle\int_{-2}^{-1}(x^2 - x - 2)dx + \int_{-1}^{2}(2 + x - x^2)dx + \int_{2}^{3}(x^2 - x - 2)dx = 49/6.$

14. (a) 3π around the y-axis, 12π around the x-axis.
    (b) $4\pi/3$ around the x axis; $10\pi/3$ around the y-axis; $16\pi/3$ around $x = -1$.
    (c) The simplest is the shell method because it allows you to integrate in x.The result is $\pi/6$.
    (d) Volume around the x axis is $15\pi/2$ and around the y axis is $124\pi/5$.

15. (a) i. $\displaystyle\int_{-\pi/2}^{0} \sqrt{1 + \cos^2 x}\ dx;$
        ii. $\displaystyle\int_{0}^{\pi} \sqrt{1 + \cos^2 y}\ dy;$
        iii. $\displaystyle\int_{1}^{2} \sqrt{\frac{1}{t^2} + 4t^2e^{2t^2}}\ dt.$
    (b) $4 - 2\sqrt{2}$.
    (c) $F = -10\sin(2t)$; $x(t) = 2 + \frac{1}{2}\sin(2t)$; No work done.
    (d) $W = \frac{1}{2}\frac{k^2}{m}\left(\frac{1}{b} - \frac{1}{a}\right)^2$; $x(b) = 1 + \frac{k}{m}\ln(b/a) - \frac{k}{m}\frac{1}{a}(b - a)$.