The cup product in orbifold Hochschild cohomology

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ABSTRACT: We study the multiplicative structure of orbifold Hochschild cohomology in an attempt to generalize the results of Kontsevich and Calaque-Van den Bergh relating the Hochschild and polyvector field cohomology rings of a smooth variety.

We introduce the concept of linearized derived scheme, and we argue that when X is a smooth algebraic variety and G is a finite abelian group acting on X, the derived fixed locus $\tilde{X}^G$ admits an HKR linearization. This allows us to define a product on the cohomology of polyvector fields of the orbifold $[X/G]$. We analyze the obstructions to associativity of this product and show that they vanish in certain special cases. We conjecture that in these cases the resulting polyvector field cohomology ring is isomorphic to the Hochschild cohomology of $[X/G]$.

Inspired by mirror symmetry we introduce a bigrading on the Hochschild homology of Calabi-Yau orbifolds. We propose a conjectural product which respects this bigrading and simplifies the previously introduced product.

Contents

1. Introduction

1.1. Let X be a smooth algebraic variety over a field of characteristic zero. The HKR map is a graded vector space isomorphism

$$\mathsf{HT}^*(X) \xrightarrow{\sim} \mathsf{HH}^*(X)$$

between the *polyvector field cohomology* $\mathsf{HT}^*(X)$ of X,

$$\mathsf{HT}^*(X) = \bigoplus_{p+q=*} H^p(X, \wedge^q T_X),$$

and the Hochschild cohomology $\mathsf{HH}^*(X)$ of X.

Both spaces above are graded commutative rings: polyvector field cohomology classes can be multiplied using the wedge product on $\wedge^* TX$ and cup product on cohomology, while Hochschild cohomology classes can be composed using the Yoneda product. However, the HKR isomorphism is not a ring map in general.

Kontsevich [K03] claimed that the rings $\mathsf{HT}^*(X)$ and $\mathsf{HH}^*(X)$ *are* in fact isomorphic, via a modification of the HKR isomorphism. This result was later proved by Calaque and Van den Bergh [CV10].

1.2. The ring $\mathsf{HT}^*(X)$ is bigraded, and the product respects this bigrading. Moreover, the Hochschild cochain complex carries a filtration given by order of polydifferential operators, which in turn induces a filtration on Hochschild cohomology. Kontsevich's claim (along with the explicit formula for the corrected ring isomorphism) can be interpreted as saying that this filtration admits a multiplicative splitting, yielding a bigrading on $\mathsf{HH}^*(X)$ which refines the usual grading.

1.3. The problem of understanding an analogue of Kontsevich's claim for orbifolds has been open for at least 20 years. The most recent (negative) progress is due to Negron-Schedler [NS20] who argue that the Hochschild cochain complex of an orbifold does not satisfy a formality result similar to the one Kontsevich used. However, this does not rule out the possibility that an analogue of the Kontsevich claim holds for a *corrected* filtration from the one they study.

This problem is particularly interesting in view of its connections to Ruan's crepant resolution conjecture. For example, getting a good understanding of the orbifold Hochschild cohomology product would explain the matching between the cohomology ring of the Hilbert scheme of n-points on a K3 surface S and the Chen-Ruan orbifold cohomology ring of $[S^n/\Sigma_n]$, as observed by Fantechi-Göttsche [FG03].

1.4. Consider a global quotient orbifold $[X/G]$, where X is a smooth algebraic variety and G is a finite group acting on X. Arinkin, Căldăraru, and Hablicsek [ACH19] gave an explicit decomposition of the Hochschild cohomology of $[X/G]$ as a graded vector space

$$\mathsf{HH}^*([X/G]) \xleftarrow{\sim} \left(\bigoplus_{g \in G} \bigoplus_{p+q=*} H^{p-c_g}(X^g, \wedge^q T_{X^g} \otimes \omega_g) \right)^G,$$

where X^g is the fixed locus of $g \in G$, c_g is the codimension of X^g in X, and ω_g is the dualizing sheaf of the inclusion $X^g \hookrightarrow X$. The right hand side is the natural analogue of polyvector field cohomology for orbifolds, and the above map can be regarded as the natural generalization of the HKR isomorphisms for global quotient orbifolds.

We define

$$\mathsf{HT}^*(X;G) = \left(\bigoplus_{g \in G} \bigoplus_{p+q=*} H^{p-c_g}(X^g, \wedge^q T_{X^g} \otimes \omega_g) \right).$$

Note that $\mathsf{HT}^*(X;G)$ carries a natural G action, and we set

$$\mathsf{HT}^*([X/G]) = \mathsf{HT}^*(X;G)^G.$$

1.5. As stated above, for a smooth variety X there is an obvious associative product on $\mathsf{HT}^*(X)$. However, when G is non-trivial, it is not at all obvious what the analogous product structure should be on $\mathsf{HT}^*([X/G])$. Understanding candidates for such a product is the goal of this paper.

We will define two operations on $\mathsf{HT}^*(X;G)$. The first one is inspired closely by the construction of the product of distributions on a Lie algebra. We can prove the associativity of this operation when certain cohomology classes vanish. In particular, this product will be associative when X is affine. However, in general we can only define this operation for abelian G, and we are not able to construct a multiplicative bigrading. A special case of this construction, which works with a non-abelian group G, will be discussed in Section 7.

The second definition mimics a construction of Fantechi-Göttsche, and the resulting formulas conjecturally simplify the first construction above. The operation we construct respects a natural bigrading, and it behaves well in all the examples we can compute. However, we cannot prove a general criterion for associativity of this operation.

1.6. Theorem A. *Suppose $[X/G]$ is a global quotient orbifold, where X is a smooth algebraic variety and G is a finite* abelian *group acting on X. Then the construction in Section 3 defines an operation on $\mathsf{HT}^*(X;G)$ which recovers the wedge product on $\mathsf{HT}^*(X)$ when G is trivial.*

This operation is associative if the Bass-Quillen class (5.5) associated to the sequence of closed embeddings $X^{g,h} \hookrightarrow X^g \hookrightarrow X$ vanishes for all $g,h \in G$. (The abstract Bass-Quillen class was introduced by the second author in [H20].)

In particular, the product is associative when X is affine, or when X is an abelian variety and $G = \mathbb{Z}/2\mathbb{Z}$ acting by negation.

1.7. Conjecture A. *Let $[X/G]$ be a global quotient orbifold with G abelian. Then the HKR isomorphism can be corrected to an isomorphism of rings*

$$\mathsf{HH}^*([X/G]) \cong \mathsf{HT}^*([X/G]).$$

We will give in Section 7 an explicit conjectural formula for the corrected isomorphism.

1.8. The relationship to bigradings can be understood from the point of view of mirror symmetry. Recall that under mirror symmetry the Hochschild cohomology of a smooth Calabi-Yau variety X corresponds to the singular cohomology of its mirror $\check{X}$. Since the latter admits a multiplicative bigrading (given by the Hodge decomposition) it is natural to expect that $\mathsf{HH}^*(X)$ should also carry a multiplicative bigrading.

For orbifolds, Chen and Ruan [CR04] used ideas from quantum cohomology to define an analogue of singular cohomology for orbifolds. The Chen-Ruan orbifold cohomology space is defined to be

$$H^*([X/G],\mathbb{C}) = \left(\bigoplus_{g\in G} H^{*-2\iota(g)}(X^g,\mathbb{C}\right))^G,$$

where $\iota(g) \in \mathbb{Q}$ is the age of an element $g \in G$. This space is endowed with a natural, graded commutative product. Fantechi and Göttsche [FG03] later gave an explicit description of this product and noted that it has a natural bigrading induced by the Hodge decompositions on the fixed loci X^g:

$$\left(\bigoplus_{g\in G} H^{*-2\iota(g)}(X^g,\mathbb{C})\right)^G = \left(\bigoplus_{g\in G}\bigoplus_{p,q} H^{p-\iota(g)}(X^g,\wedge^{q-\iota(g)}\Omega_{X^g})\right)^G.$$

The mirror symmetry intuition above suggests that the Hochschild cohomology of Calabi-Yau global quotient orbifolds should also carry a multiplicative bigrading.

1.9. We define a new bigrading on the cohomology of orbifold polyvector fields by setting

$$\mathsf{HT}^{p,q}(X;G) = \bigoplus_{g\in G} H^{p-\iota(g)}(X^g, \wedge^{q+\iota(g)-c_g} T_{X^g} \otimes \omega_g).$$

Based on examples we compute in this paper we propose a conjectural way to simplify the product in Theorem A for Calabi-Yau global quotient orbifolds. This simplified product will preserve the above bigrading.

1.10. Plan of the paper. In Section 2 we briefly summarize Kontsevich's Theorem for smooth algebraic varieties. Then we turn to the global quotient orbifold case, and recall the identification of orbifold polyvector fields with orbifold Hochschild cohomology via the HKR isomorphism.

Section 3 is devoted to the definition of our product structure on orbifold polyvector fields.

Section 4 contains the technical details based on derived intersection and formality of derived schemes. We need to use them to make the definition work. The definition of our product works at least when G is abelian, and also works for examples of Hilbert schemes of n points as will be described in Section 7.

Section 5 presents the strategy for proving the associativity of our product when the Bass-Quillen class vanishes. We explain how to reduce the question of associativity to two statements, Propositions 5.6 and 5.7, which are proved in Section 6.

In Section 7 we propose an ansatz which allows us to simplify the formulas for our product. We compute several examples of the simplified product, leading us to a definition of the bigrading described above on the cohomology of orbifold polyvector fields. The simplified product preserves this bigrading. At the end of this paper, we list several remaining problems directly related to Conjecture A.

1.11. Conventions. All the algebraic varieties in this paper are smooth over a field of characteristic zero.

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2. Background

In this section we review the theorem of Kontsevich and Calaque-Van den Bergh for smooth algebraic varieties, and the construction of the orbifold HKR isomorphism. A few new definitions and interpretations in this section will be important throughout the paper.

2.1. The HKR isomorphism. Let X be a smooth algebraic variety. There is an HKR isomorphism

$$\mathsf{HH}^*(X) \cong \bigoplus_{p+q=*} H^p(X, \wedge^q T_X)$$

that identifies the Hochschild cohomology of X with the cohomology of polyvector fields as graded vector spaces. More precisely, we have a sheaf-level HKR isomorphism in the derived category of X

$$\Delta^*\Delta_*\mathscr{O}_X \cong \mathsf{Sym}_{\mathscr{O}_X} \Omega_X[1],$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding. We get the desired isomorphism on cohomology by applying $\mathsf{Hom}(-, \mathscr{O}_X)$ to this isomorphism of sheaves.

2.2. The HKR isomorphism has an interpretation in the language of derived schemes. The complex $\Delta^*\Delta_*\mathscr{O}_X$ admits a graded commutative product, which makes it into the structure complex of a derived scheme, the derived self-intersection $X \times^R_{X\times X} X$. This is also known as LX, the derived loop space of X.

On the other hand $\mathsf{Sym}\,\Omega_X[1]$ is the structure complex of the shifted tangent bundle $\mathbb{T}_X[-1]$. We can then restate the HKR isomorphism as an isomorphism of derived schemes over X,

$$\mathbb{T}_X[-1] \xrightarrow{\cong} LX = X \times^R_{X\times X} X.$$

2.3. It was observed by Kapranov and Kontsevich that there is a Lie theoretic interpretation of the HKR isomorphism. The derived loop space LX has the structure of a derived group scheme over X, and the relative normal bundle $N_{X/LX} = \mathbb{T}_X[-1]$ is its Lie algebra [Ka99]. The HKR isomorphism can be thought of as a version of the exponential map $\mathbb{T}_X[-1] \to LX$ [CR11], where $\mathbb{T}_X[-1]$ is the total space of the shifted tangent bundle.

The exponential map relates the derived Lie group LX with its Lie algebra $N_{X/LX} = \mathbb{T}_X[-1]$. In general, suppose $\widetilde{X}$ is an arbitrary derived scheme which is not necessary a derived group. We can still consider the total space $\mathbb{N}_{X/\widetilde{X}}$ of the normal bundle of X in $\widetilde{X}$, where X is the underlying classical scheme $X \hookrightarrow \widetilde{X}$. This leads to the following definition.

2.4. Definition. *For a derived scheme $\widetilde{X}$, the* linearization $\mathbb{L}_{\widetilde{X}}$ *of $\widetilde{X}$ is defined to be the total space of the normal bundle $N_{X/\widetilde{X}}$, where X is the underlying classical scheme $X \hookrightarrow \widetilde{X}$. A choice of isomorphism $\mathbb{L}_{\widetilde{X}} \cong \widetilde{X}$ (if one exists) will be called a* linearization *of $\widetilde{X}$.*

For example, consider $X \hookrightarrow LX$. Then

$$\mathbb{L}_{LX} = \mathbb{N}_{X/LX} = \mathbb{T}_X[-1],$$

and the HKR isomorphism provides a linearization of LX.

2.5. We need to address a technical detail about the above isomorphism. The linearization $\mathbb{L}_{\widetilde{X}}$ of a derived scheme $\widetilde{X}$ is by definition the total space of the normal bundle $N_{X/\widetilde{X}}$, hence it comes with a natural projection which makes it a scheme over X. Moreover, this projection splits the inclusion $X \hookrightarrow \widetilde{X}$. However, in general $\widetilde{X}$ may not admit such a projection. This explains why in general here is no way to define an isomorphism $\mathbb{L}_{\widetilde{X}} \cong \widetilde{X}$ over X. If we hope to define an isomorphism $\mathbb{L}_{\widetilde{X}} \cong \widetilde{X}$, we usually need to find a natural base scheme Y such that both $\widetilde{X}$ and $\mathbb{L}_{\widetilde{X}}$ are affine over Y. Then we can consider the structure complex of $\mathbb{L}_{\widetilde{X}}$ and $\widetilde{X}$ as $\mathscr{O}_Y$-algebras. There is a bijection between the set of isomorphisms $\mathscr{O}_{\mathbb{L}_{\widetilde{X}}} \cong \mathscr{O}_{\widetilde{X}}$ in the derived category of Y and the set of isomorphisms $\mathbb{L}_{\widetilde{X}} \cong \widetilde{X}$ over Y. A choice of such an isomorphism will be called a linearization of $\widetilde{X}$ over Y.

The HKR isomorphism

$$\mathsf{exp} : \mathbb{T}_X[-1] = \mathbb{L}_{LX} \xrightarrow{\cong} LX$$

linearizes the derived loop space LX over X. Here $LX = X \times^R_{X \times X} X$ is to be viewed as a scheme over X via one of the two projection maps onto the left or right factors $X \times^R_{X \times X} X \to X$.

2.6. For most derived schemes $\widetilde{X}$, even if they are affine over a scheme Y, it is not true that $\widetilde{X}$ is isomorphic to $\mathbb{L}_{\widetilde{X}}$ over Y. If one thinks about the structure complexes as $\mathscr{O}_Y$-algebras, the existence of such an isomorphism would say that the structure complex $\mathscr{O}_{\widetilde{X}}$ of $\widetilde{X}$ would be quasi-isomorphic to its cohomology. This is equivalent to saying that the derived scheme $\widetilde{X}$ is formal over Y in the sense of [DGMS75]. See [ACH19] for more discussions.

2.7. The HKR isomorphism on Hochschild cohomology is obtained by dualizing the structure complexes of $\mathbb{T}_X[-1]$ and LX with respect to X, and taking global sections:

$$\begin{aligned}\mathsf{HH}^*(X) &= \mathsf{Hom}_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{O}_X) = \mathsf{Hom}_X(\Delta^*\Delta_*\mathscr{O}_X, \mathscr{O}_X)\\ &= \Gamma(X, (\mathscr{O}_L X)^\vee) \cong \Gamma(X, \mathsf{Sym}\, T_X[-1]) = \bigoplus H^p(X, \wedge^q T_X)\\ &= \mathsf{HT}^*(X).\end{aligned}$$

Duals of functions are distributions. We will need a relative version of this concept, made precise in the following definition.

2.8. Definition. *For a map of spaces $f : Y \to X$ the space of relative distributions is defined by*

$$\mathbb{D}(Y/X) = \mathsf{Hom}(f_*\mathscr{O}_Y, \mathscr{O}_X).$$

We will often omit the space X when it is clear from context.

2.9. For example consider the map $LX \to X$, where X is a smooth algebraic variety. Then the Hochschild cohomology of X is naturally identified with the space of distributions on LX,

$$\mathbb{D}(LX/X) = \mathsf{Hom}(\Delta^*\mathscr{O}_\Delta, \mathscr{O}_X) = \mathsf{HH}^*(X),$$

while the polyvector field cohomology of X is naturally the space of distributions on $\mathbb{T}_X[-1]$, the linearization of LX,

$$\mathbb{D}(\mathbb{T}_X[-1]) = \mathsf{Hom}(\mathsf{Sym}\,\Omega_X[1], \mathscr{O}_X) = \mathsf{HT}^*(X).$$

Therefore we should think of polyvector fields as (invariant) distributions on the Lie algebra $\mathbb{T}_X[-1]$ and Hochschild cohomology as (invariant) distributions on the derived group LX. The HKR isomorphism is then interpreted as the isomorphism on distributions induced by the exponential map. The product structures on the two sides are given by convolution of distributions, where the group structure on $\mathbb{T}_X[-1]$ is given by addition in the fibers.

This interpretation was probably the basis for Kontsevich's claim: for ordinary Lie algebras a theorem of Duflo [D69] asserts that the rings of invariant distributions on a Lie group and on its Lie algebra are isomorphic, after a correction to the exponential map by what is known as the Duflo element.

2.10. The Theorem of Kontsevich and Calaque-Van den Bergh. Calaque and Van den Bergh [CV10], following the claim of Kontsevich [K03], proved that the map

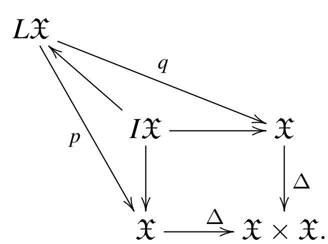
$$\mathsf{HT}^*(X) \xrightarrow{\mathsf{td}^{1/2} \lrcorner} \mathsf{HT}^*(X) \xrightarrow{\mathsf{HKR}} \mathsf{HH}^*(X)$$

is a *ring* isomorphism. Here the analogue of the Duflo element is the characteristic class $\mathsf{td}^{1/2}$.

We are interested in studying generalizations of the above theorem to the case of global quotient orbifolds.

2.11. The orbifold HKR isomorphism. Before we begin we need an analogue of the HKR isomorphism for orbifolds. Let X be a smooth algebraic variety, let G be a finite group acting on X, and denote by $\mathfrak{X} = [X/G]$ the corresponding global quotient orbifold.

We have the following diagram



Here $L\mathfrak{X}$ denotes the loop space of the stack $\mathfrak{X}$ defined by analogy with the case of ordinary spaces as the derived self-intersection

$$L\mathfrak{X} = \mathfrak{X} \times^R_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}.$$

Its underlying underived stack $I\mathfrak{X}$ is the inertia stack of $\mathfrak{X}$,

$$I\mathfrak{X} = \mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}.$$

Unlike the case where X is a smooth space with no group action, the two maps $I\mathfrak{X} \to \mathfrak{X}$ are no longer isomorphisms: it is not hard to see that

$$I\mathfrak{X} = \left[(\coprod_{g\in G} X^g)/G \right],$$

where X^g denotes the fixed locus of the action of $g \in G$ on X.

We can rewrite X^g as $\Delta \times_{X\times X} \Delta^g$, where $\Delta = \{(x,x)\} \subset X \times X$ and $\Delta^g = \{(x,gx)\} \subset X \times X$. We get an explicit formula for the derived loop space $L\mathfrak{X}$ if we replace the above intersection by the corresponding derived intersection:

$$L\mathfrak{X} = \left[(\coprod_{g\in G} \widetilde{X^g})/G \right],$$

where

$$\widetilde{X^g} = \Delta \times^R_{X\times X} \Delta^g.$$

We will call $\widetilde{X^g}$ the derived fixed locus of g.

2.12. The orbifold HKR isomorphism expresses the derived loop space $L\mathfrak{X}$ as the total space of a certain vector bundle over $I\mathfrak{X}$, as explained by Arinkin, Căldăraru and Hablicsek [ACH19].

The derived loop space $L\mathfrak{X}$ decomposes naturally into connected components, so it is better to look at each component $\widetilde{X^g}$ of $L\mathfrak{X}$ individually. The orbifold HKR isomorphism identifies $\widetilde{X^g}$ with the total space of the tangent bundle of X^g. More precisely, for each $g \in G$ [ACH19] construct a linearization isomorphism of derived schemes over X

$$\mathbb{T}_{X^g}[-1] = \mathbb{L}\widetilde{X^g} \xrightarrow{\sim} \widetilde{X^g}.$$

In explicit terms this translates into an isomorphism of commutative $\mathscr{O}_X$-algebras

$$q_* \mathscr{O}_{\widetilde{X^g}} \xrightarrow{\sim} i_{g*} \operatorname{Sym}(\Omega_{X^g}[1]),$$

where $i_g : X^g \hookrightarrow X$ is the inclusion of the fixed locus. Applying $\operatorname{Hom}(-, \mathscr{O}_X)$ to this algebra isomorphism we get an induced isomorphism on distributions

$$\mathbb{D}(\mathbb{T}_{X^g}[-1]/X) \xrightarrow{\sim} \mathbb{D}(\widetilde{X^g}/X).$$

We will denote $\mathbb{D}(\mathbb{T}_{X^g}[-1]/X)$ by $\mathsf{HT}^*(X;g)$, and $\mathbb{D}(\widetilde{X^g}/X)$ by $\mathsf{HH}^*(X;g)$.

Grothendieck duality allows us to give an explicit form to the space $\mathbb{D}(\mathbb{T}_{X^g}[-1]/X)$:

$$\mathbb{D}(\mathbb{T}_{X^g}[-1]/X) = \operatorname{Hom}_X(i_{g*} \operatorname{Sym} \Omega_{X^g}[1], \mathscr{O}_X) = \bigoplus_{p+q=*} H^{p-c_g}(X^g, \wedge^q T_{X^g} \otimes \omega_g),$$

where c_g is the codimension of X_g/X and ω_g is the dualizing sheaf of the inclusion $X^g \subseteq X$. Taking G-invariants of the direct sum over $g \in G$ we get the final form of the orbifold HKR isomorphism for $\mathfrak{X}$:

$$\begin{aligned} \mathsf{HH}^*(\mathfrak{X}) &= \left(\bigoplus_{g\in G} \mathsf{HH}^*(X,g) \right)^G = \left(\bigoplus_{g\in G} \mathbb{D}(\widetilde{X^g}/X) \right)^G \\ &\cong \left(\bigoplus_{g\in G} \mathsf{HT}^*(X;g) \right)^G = \left(\bigoplus_{g\in G} \bigoplus_{p+q=*} H^{p-c_g}(X^g, \wedge^q T_{X^g} \otimes \omega_g) \right)^G. \end{aligned}$$

We think of the right hand side above as the definition of the space of polyvector fields on $\mathfrak{X}$,

$$\mathsf{HT}^*(\mathfrak{X}) = \left(\bigoplus_{g \in G} \mathsf{HT}^*(X; g) \right)^G .$$

2.13. We close this section by noting that the above HKR isomorphisms can be assembled to an analogue of the exponential map

$$\exp : \mathbb{T}_{I\mathfrak{X}}[-1] = \mathbb{L}_{L\mathfrak{X}} \xrightarrow{\sim} \mathbb{L}\mathfrak{X}$$

from the Lie algebra $\mathbb{L}_{L\mathfrak{X}}$ to the derived group $L\mathfrak{X}$.

3. Definition of the product on orbifold polyvector fields

In this section we define the product on orbifold polyvector fields. The technical results used in the definition are introduced in this section, but will be proved in Section 4.

As we have explained previously, the Hochschild cohomology and the polyvector fields cohomology of a space X can be viewed as the distributions on the derived loop space (a derived Lie group) and on its Lie algebra, respectively. The product structures on these come from the convolution of distributions. We begin by recalling the definition of the convolution product of distributions on (classical) Lie groups and Lie algebras.

3.1. Distributions on Lie groups and Lie algebras. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The convolution product of distributions $\mathbb{D}(G)$ on G is defined as follows

$$\mathbb{D}(G) \otimes \mathbb{D}(G) \longrightarrow \mathbb{D}(G \times G) \xrightarrow{m_*} \mathbb{D}(G),$$

where m is the multiplication map $G \times G \to G$, and m_* is the induced map on distributions. The Lie algebra $\mathfrak{g}$ of G is a vector space. It is considered as an abelian group under the addition operation of vectors. One can define the convolution product for $\mathbb{D}(\mathfrak{g})$ similarly.

In the derived setting, the convolution product on orbifold Hochschild cohomology is known as the composition of morphisms in the derived category. We hope to define the convolution product on the polyvector fields. Therefore, it is important to know how to recover the convolution product on $\mathbb{D}(\mathfrak{g})$ with the knowledge of the group G only. The following is how we do this.

First there is a multiplication map $m : G \times G \to G$. Taking derivative of this map, we get the induced map on tangent spaces $\mathbb{L}_m : \mathbb{L}_{G\times G} \to \mathbb{L}_G$, where $\mathbb{L}_G = \mathfrak{g}$ is the tangent space of G at origin. We use the same notation $\mathbb{L}$ as the notation for the linearization of derived schemes in the derived setting.

There is a natural isomorphism $\mathbb{L}_{G\times G} \cong \mathbb{L}_G \times \mathbb{L}_G$. Under this natural identification, the map $\mathbb{L}_m : \mathbb{L}_G \times \mathbb{L}_G \to \mathbb{L}_G$ is nothing but the addition law on the vector space $\mathbb{L}_G$. We can recover the convolution product of $\mathbb{D}(\mathbb{L}_G)$ now

$$\mathbb{D}(\mathbb{L}_G) \otimes \mathbb{D}(\mathbb{L}_G) \longrightarrow \mathbb{D}(\mathbb{L}_G \times \mathbb{L}_G) \xrightarrow{\cong} \mathbb{D}(\mathbb{L}_{G\times G}) \xrightarrow{\mathbb{L}_{m*}} \mathbb{D}(\mathbb{L}_G).$$

3.2. A non-trivial isomorphism in the derived setting. We can try to do exactly the same thing in the derived setting. However, there is a technical issue. The natural isomorphism $\mathbb{L}_{G\times G} \cong \mathbb{L}_G \times \mathbb{L}_G$ is not at all obvious for the derived loop space. The analogous statement would be

$$\mathbb{L}_{L\mathfrak{X}\times^R_{\mathfrak{X}} L\mathfrak{X}} \cong \mathbb{L}_{L\mathfrak{X}} \times^R_{\mathfrak{X}} \mathbb{L}_{L\mathfrak{X}}$$

for the derived loop space of an orbifold $\mathfrak{X}$. The left hand side is obviously linear: it is a total space of a vector bundle over the inertia stack $I\mathfrak{X}$. On the other hand, it is not at all obvious that the right hand side can be linearized.

The following two propositions will be proved in the next sections.

3.3. Proposition. *Let $\mathfrak{X} = [X/G]$ be a global quotient orbifold of a finite group G acting on a smooth algebraic variety. If we further assume G is abelian, then there is an isomorphism*

$$\mathbb{L}_{(L\mathfrak{X}\times^R_{\mathfrak{X}} L\mathfrak{X})} \cong \mathbb{L}_{L\mathfrak{X}} \times^R_{\mathfrak{X}} \mathbb{L}_{L\mathfrak{X}}.$$

The derived loop space $L\mathfrak{X}$ decomposes naturally into connected components, so we can restate the above proposition on components.

3.4. Proposition. *In the same setting as Proposition 3.3, there is an isomorphism*

$$\mathbb{L}_{\widetilde{X^g}\times^R_X \widetilde{X^h}} \cong \mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$$

for any $g, h \in G$.

3.5. The definition of the convolution product in the derived setting. The multiplication map for Lie groups plays an important role in the case of Lie groups and Lie algebras. We need to know what the multiplication map is for the derived loop space

$L\mathfrak{X} = \mathfrak{X} \times^R_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}$ of $\mathfrak{X}$. It is the projection map $p_1 \times p_3$ onto the first and the third factors

$$\begin{array}{ccc} L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \xrightarrow{m} & L\mathfrak{X}. \\ \downarrow = & & \downarrow = \\ \mathfrak{X} \times^R_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X} \times^R_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X} & \xrightarrow{p_1\times p_3} & \mathfrak{X} \times^R_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}. \end{array}$$

We need three lemmas for derived groups which are generalizations of well-known results from classical Lie group theory.

3.6. Lemma. *A map $f : X \longrightarrow Y$ between derived schemes induces a map on linearizations $\mathbb{L}_f : \mathbb{L}_X \longrightarrow \mathbb{L}_Y$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} X^0 & \xrightarrow{g} & Y^0 \\ \downarrow i & & \downarrow j \\ X & \xrightarrow{f} & Y, \end{array}$$

where X^0 and Y^0 are the classical schemes of X and Y respectively. Then there is a commutative diagram of derived tangent complexes

$$\begin{array}{ccc} T_{X^0} & \longrightarrow & g^*T_{Y^0} \\ \downarrow & & \downarrow \\ i^*T_X & \longrightarrow & i^*f^*T_Y = g^*j^*T_Y. \end{array}$$

Passing to the quotient, we get an induced map

$$N_{X^0/X} = i^*T_X/T_{X^0} \to g^*(j^*T_Y)/g^*T_{Y^0} = g^*(j^*T_Y/T_{Y^0}) = g^*N_{Y^0/Y}.$$

The map above is equivalent to a map $\mathbb{N}_{X^0/X} \to \mathbb{N}_{Y^0/Y} \times_{Y^0} X^0$ in terms of total spaces. □

Applying the above lemma to the multiplication map of derived loop space yields an induced map $\mathbb{L}_m : \mathbb{L}_{L\mathfrak{X}\times^R_{\mathfrak{X}}L\mathfrak{X}} \to \mathbb{L}_{L\mathfrak{X}}$.

3.7. Lemma. *Suppose there is a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle j} & \\ & S & \end{array}$$

of (derived) schemes. Then there is a pushforward map for relative distributions, i.e., there is a natural induced map $f_* : \mathbb{D}(X/S) \to \mathbb{D}(Y/S)$.

Proof. We have $\mathbb{D}(X/S) = \mathsf{Hom}(i_*\mathscr{O}_X, \mathscr{O}_S)$. Applying j_* to the map $\mathscr{O}_Y \to f_*\mathscr{O}_X$, we get a map $j_*\mathscr{O}_Y \to j_*f_*\mathscr{O}_X = i_*\mathscr{O}_X$. Composing it with $i_*\mathscr{O}_X \to \mathscr{O}_S$, we get the desired pushforward map. □

3.8. Lemma. *Suppose there is a commutative diagram*

$$\begin{array}{ccc} W = X \times_S^R Y & \longrightarrow & Y \\ \downarrow & \overset{\pi}{\searrow} & \downarrow{\scriptstyle j} \\ X & \xrightarrow{i} & S \end{array}$$

of (derived) schemes. Then there is a natural map $\mathbb{D}(X/S) \otimes \mathbb{D}(Y/S) \to \mathbb{D}(W/S)$.

Proof. We have

$$\begin{aligned} \mathsf{Hom}(i_*\mathscr{O}_X, \mathscr{O}_S) \otimes \mathsf{Hom}(j_*\mathscr{O}_Y, \mathscr{O}_S) &\to \mathsf{Hom}(i_*\mathscr{O}_X \otimes_{\mathscr{O}_S} j_*\mathscr{O}_Y, \mathscr{O}_S) \\ &= \mathsf{Hom}(\pi_*\mathscr{O}_W, \mathscr{O}_S). \end{aligned}$$ □

With the three lemmas above we are able to define our desired product.

3.9. Definition. *Under the assumptions in Proposition 3.3 we define the following binary operation on* $\mathbb{D}(\mathbb{L}_{L\mathfrak{X}}/\mathfrak{X})$, *which is our proposed definition for a product on orbifold polyvector fields:.*

$$\begin{aligned} \mathbb{D}(\mathbb{L}_{L\mathfrak{X}}/\mathfrak{X}) \otimes \mathbb{D}(\mathbb{L}_{L\mathfrak{X}}/\mathfrak{X}) \longrightarrow \mathbb{D}(\mathbb{L}_{L\mathfrak{X}} \times_{\mathfrak{X}}^R \mathbb{L}_{L\mathfrak{X}}/\mathfrak{X}) &\xrightarrow{\sim} \mathbb{D}(\mathbb{L}_{(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X})}/\mathfrak{X}) \\ &\xrightarrow{\mathbb{L}_{m*}} \mathbb{D}(\mathbb{L}_{\mathfrak{X}}/\mathfrak{X}), \end{aligned}$$

where the first arrow is due to Lemma 3.8, the second arrow is the non-trivial isomorphism in Proposition 3.3, and the last map is due to Lemmas 3.6 and 3.7.

Looking at each connected component of $L\mathfrak{X}$ individually, the definition gives a map for every $g, h \in G$

$$\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}/X) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}/X) \to \mathbb{D}(\mathbb{L}_{\widetilde{X^g}} \times_X^R \mathbb{L}_{\widetilde{X^h}}/X) \xrightarrow{\sim} \mathbb{D}(\mathbb{L}_{(\widetilde{X^g} \times_X \widetilde{X^h})}/X) \xrightarrow{\mathbb{L}_{m*}} \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}/X).$$

4. The formality of double fixed loci

We begin by studying the cohomology sheaves of the structure complex of $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$. Then we compute the linearization $\mathbb{L}_{\widetilde{X^g}\times^R_X\widetilde{X^h}}$ explicitly. At last, we prove Propositions 3.3 and 3.4, in other words we construct a formality isomorphism

$$\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}} \cong \mathbb{L}_{\widetilde{X^g}\times^R_X\widetilde{X^h}}.$$

The construction will be indirect: we will use known results to show that both sides are isomorphic to

$$\mathbb{T}_{X^g}[-1]|_{X^{g,h}} \oplus \mathbb{T}_{X^h}[-1]|_{X^{g,h}} \oplus \mathbb{E}[-1],$$

where $\mathbb{E}$ is the total space of the excess intersection bundle for the intersection of X^g and X^h in X.

Throughout this section X will be a smooth variety, and g, h will denote commuting elements of a group G which acts on X.

4.1. Before we begin we note that the derived fixed locus $\widetilde{X^g} = \Delta \times^R_{X\times X} \Delta^g$ is not directly a scheme over X. It is more naturally viewed as a scheme over Δ or Δ^g, both of which are isomorphic (in different ways) to X. We will use the latter when computing the fiber product $\widetilde{X^g} \times^R_X \widetilde{X^h}$.

Similarly, the derived fixed locus $\widetilde{X^h} = \Delta \times^R_{X\times X} \Delta^h$ is naturally isomorphic to $\Delta^g \times^R_{X\times X} \Delta^{gh}$, which is also a scheme over Δ^g. Therefore, while the notation $\widetilde{X^g} \times^R_X \widetilde{X^h}$ is imprecise, what we will really mean by it is

$$(\Delta \times^R_{X\times X} \Delta^g) \times^R_{\Delta^g} (\Delta^g \times^R_{X\times X} \Delta^{gh}) = \Delta \times^R_{X\times X} \Delta^g \times^R_{X\times X} \Delta^{gh}.$$

We think of this as the derived fixed locus of g and h, and denote it by $\widetilde{X^{g,h}}$.

4.2. The cohomology sheaves of the structure complex of $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$. It is difficult to compute $\mathscr{O}_{\mathbb{L}_{\widetilde{X^g}}\times^R_X\mathbb{L}_{\widetilde{X^h}}}$ directly, but we can compute its cohomology sheaves more easily, and we begin with this computation.

We hope to compute the cohomology sheaves of

$$\mathscr{O}_{\mathbb{L}_{\widetilde{X^g}}\times^R_X\mathbb{L}_{\widetilde{X^h}}} = \mathsf{Sym}(\Omega_{X^g}[1]) \otimes^L_{\mathscr{O}_X} \mathsf{Sym}(\Omega_{X^h}[1]).$$

The calculation becomes straightforward using the following lemma.

4.3. Lemma. *Suppose i, j are closed embeddings of classical schemes, and $\mathscr{E}$, $\mathscr{F}$ are vector bundles on X and on Y, respectively. Denote by W the fiber product of classical schemes below*

$$\begin{array}{ccc} X \times_S Y = W & \xrightarrow{\ l\ } & Y \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{\ i\ } & S. \end{array}$$

Then

$$\mathcal{H}_q(i_*\mathscr{E} \otimes^L_{\mathscr{O}_S} j_*\mathscr{F}) = j_* l_*(\mathscr{E}|_W \otimes \mathscr{F}|_W \otimes E^\vee),$$

where E is the excess intersection bundle,

$$E = \frac{T_S|_W}{T_X|_W + T_Y|_W}.$$

Proof. [CKS03, Proposition A.6]. □

4.4. The lemma above shows that

$$\mathcal{H}^n(\mathscr{O}_{\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}}) = \bigoplus_{p+q+i=n} (\wedge^p \Omega_{X^g}|_{X^{g,h}} \otimes \wedge^q \Omega_{X^h}|_{X^{g,h}} \otimes \wedge^i E^\vee).$$

In other words, if we knew that $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$ is formal over its underlying classical scheme $X^{g,h}$, the above calculation would imply that

$$\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}} \cong \mathbb{T}_{X^g}[-1]|_{X^{g,h}} \oplus \mathbb{T}_{X^h}[-1]|_{X^{g,h}} \oplus \mathbb{E}[-1].$$

We will prove the formality statement in (4.10).

4.5. The structure complex of $\mathbb{L}_{\widetilde{X^{g,h}}}$. We will now argue that the linearization $\mathbb{L}_{\widetilde{X^{g,h}}}$ is precisely the space that appears on the right hand side of the equality above,

$$\mathbb{L}_{\widetilde{X^{g,h}}} = \mathbb{T}_{X^g}[-1]|_{X^{g,h}} \oplus \mathbb{T}_{X^h}[-1]|_{X^{g,h}} \oplus \mathbb{E}[-1].$$

By definition, $\mathbb{L}_{\widetilde{X^{g,h}}} = \mathbb{N}_{X^{g,h}/\widetilde{X^{g,h}}}$. To compute the normal bundle, we need to know what the derived tangent complex of $\widetilde{X^{g,h}}$ is.

4.6. The derived tangent complex. The standard reference for the derived tangent complex is [I09]. Suppose X and Y are closed subschemes of a scheme S. Let $\widetilde{W} = X \times_S^R Y$ be the derived intersection and $W = X \times_S^R Y$ be the classical intersection. There is an exact sequence

$$0 \to T_X|_W \cap T_Y|_W = T_W \to T_X|_W \oplus T_Y|_W \to T_S|_W \to E \to 0.$$

The complex

$$T_X|_W \oplus T_Y|_W \to T_S|_W = \mathsf{Cone}(T_X|_W \oplus T_Y|_W \to T_S|_W)[-1]$$

is the restriction to W of the derived tangent complex $T_{\widetilde{W}}|_W$ of $\widetilde{W}$. Since only its $\mathcal{H}^0$ and $\mathcal{H}^1$ sheaves are non-zero, the information contained in it is equivalent to the data of the triple $(\mathcal{H}^0, \mathcal{H}^1, \eta)$, where $\mathcal{H}^0(T_{\widetilde{W}}|_W) = T_W$, $\mathcal{H}^1(T_{\widetilde{W}}|_W) = N_{W/\widetilde{W}} = E$, and the class η is an element in $\mathsf{Ext}_S^2(E, T_W)$.

For example, if we consider the situation where $S = X \times X$, $X = \Delta$, $Y = \Delta^g$, so that $W = X^g$, we have $\mathcal{H}^0(T_{\widetilde{X^g}}) = T_{X^g}$ and $\mathcal{H}^1(T_{\widetilde{X^g}}) = E$. Moreover, the excess bundle in this case equals the coinvariant bundle $(T_X|_{X^g})_g$, which in characteristic zero is canonically isomorphic to the invariant bundle T_{X^g}.

The linearization $\mathbb{L}_{\widetilde{W}}$ is by definition the total space of the normal bundle $N_{W/\widetilde{W}}$, the cone of the map $T_W \to T_{\widetilde{W}}|_W$. Since $\mathcal{H}^0(T_{\widetilde{W}}|_W) = T_W$, it follows that the normal bundle $N_{W/\widetilde{W}}$ is the first cohomology $\mathcal{H}^1(T_{\widetilde{W}}|_W)[-1]$ of $T_{\widetilde{W}}|_W$. In the example considered above this shows that $\mathbb{L}_{\widetilde{X^g}} = \mathbb{T}_{X^g}[-1]$.

4.7. The above discussion also works for derived schemes. If we replace X, Y, and S by derived schemes in the commutative diagram at the beginning of (4.6), we have the same formula

$$T_{\widetilde{W}}|_W = \mathsf{Cone}(T_X|_W \oplus T_Y|_W \longrightarrow T_S|_W)[-1],$$

where T_X, T_Y, and T_S are the derived tangent complexes of X, Y, and S. The scheme W is the underlying classical scheme of $\widetilde{W} = X \times_S^R Y$. Since all the complexes are restricted to W, we will omit the restrictions from X, Y, S, and $\widetilde{W}$ to W for simplicity from now on.

It helps us to compute the derived tangent complex $T_{\widetilde{X^{g,h}}}$ if we set $S = X$, $X = \widetilde{X^g}$, $Y = \widetilde{X^h}$, and $\widetilde{W} = \widetilde{X^{g,h}}$ respectively.

4.8. Lemma. *The derived tangent complex of* $\widetilde{X^{g,h}}$ *is quasi-isomorphic to*

$$T_\Delta \oplus T_{\Delta^g} \oplus T_{\Delta^h} \to T_{X\times X} \oplus T_{X\times X} \oplus T_{X\times X} \to T_{X\times X},$$

where the maps are of the form

$$(v_1, v_2, v_3) \to (v_1 - v_2, v_2 - v_3, v_3 - v_1),$$

and

$$(a, b, c) \to a + b + c.$$

We can compute the cohomology of the derived tangent complex of $\widetilde{X^{g,h}}$ using the above lemma. We have $\mathcal{H}^0(T_{\widetilde{X^{g,h}}}) = T_{X^{g,h}}$. To compute the first cohomology it suffices to compute the cokernel of the map below

$$V \oplus V \oplus V \to V \oplus V \oplus V \oplus V,$$

where $(V = T_X \cong T_\Delta \cong T_{\Delta^g} \cong T_{\Delta^h})$ and the maps are $(v, v', v'') \to (v - v', v - gv', v - v'', v - hv'')$. This is done in the lemma below.

4.9. Lemma. *Suppose* V *is a finite dimensional representation of a finite group* G *over a field of characteristic* 0. *Let* g *and* h *be two elements of* G. *Then the quotient of* $V \oplus V \oplus V \oplus V$ *by the relations* (v, v, v, v), $(v, gv, 0, 0)$, *and* $(0, 0, v, hv)$ *is isomorphic to*

$$V_g \oplus V_h \oplus \frac{V}{V^g + V^h}.$$

Proof. Let L be the linear subspace (v, v, v, v), and note that $L = H_1 \cap H_2 \cap H_3$, where H_1 is defined by $v_1 = v_2$, H_2 is defined by $v_1 = v_3$, and H_3 is defined by $v_3 = v_4$. Then we have an isomorphism

$$\frac{V \oplus V \oplus V \oplus V}{L} \cong \frac{V^{\oplus 4}}{H_1} \oplus \frac{V^{\oplus 4}}{H_2} \oplus \frac{V^{\oplus 4}}{H_3} \cong V \oplus V \oplus V.$$

Under this identification, the second and third relations become $(v - gv, v, 0)$ and $(0, -v, v - hv)$. There is a natural projection to the first and third components,

$$\frac{V \oplus V \oplus V}{(v - gv, v, 0), (0, -v', v' - hv')} \to \frac{V \oplus V}{(v - gv, 0), (0, v' - hv')} = V_g \oplus V_h.$$

It is easy to show that the kernel is $\frac{V}{V^g + V^h}$. So we get a short exact sequence

$$0 \to \frac{V}{V^g + V^h} \to \frac{V \oplus V \oplus V}{(v - gv, v, 0), (0, -v', v' - hv')} \to V_g \oplus V_h \to 0.$$

By the averaging map $v \to \frac{1}{\mathrm{ord}(g)} \sum_{i=1}^{\mathrm{ord}(g)} g^i \cdot v$, the map $V \to V_g$ splits in characteristic 0. We can use the averaging map of g and h to get a canonical splitting of the short exact sequence above. □

The discussion above shows that the first cohomology of the tangent complex of $\widetilde{X^{g,h}}$ is $E \oplus T_{X^g} \oplus T_{X^h}$, where $E = \frac{T_X}{T_{X^g}+T_{X^h}}$. As a consequence we have an isomorphism

$$\mathbb{L}_{\widetilde{X^g} \times_X^R \widetilde{X^h}} = \mathbb{L}_{\widetilde{X^{g,h}}} \cong \mathbb{T}_{X^g}|_{X^{g,h}}[-1] \oplus \mathbb{T}_{X^h}|_{X^{g,h}}[-1] \oplus \mathbb{E}[-1].$$

4.10. Formality of $\mathbb{L}_{\widetilde{X^g}} \times_X^R \mathbb{L}_{\widetilde{X^h}}$. The linearizations $\mathbb{L}_{\widetilde{X^g}}$ and $\mathbb{L}_{\widetilde{X^h}}$ are by definition the total spaces of vector bundles over X^g and X^h respectively, so we study the formality of $X^g \times_X^R X^h$ first. The key tools are [ACH19, Theorem 1.8 and Lemma 4.3].

Proof of Proposition 3.4. The inclusion $X^g \to X$ splits to first order. By [ACH19, Lemma 4.3], the derived scheme $X^g \times_X^R X^h$ is formal over $X^g \times X^h$ if and only if the short exact sequence on $X^{g,h} = X^g \times_X X^h$

$$0 \to \frac{T_{X^h}}{T_{X^g} \cap T_{X^h}} \to \frac{T_X}{T_{X^g}} \to E = \frac{T_X}{T_{X^g} + T_{X^h}} \to 0$$

splits.

Define a map

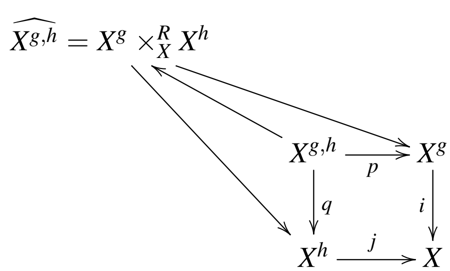
$$\frac{T_X}{T_{X^g}} \to \frac{T_{X^h}}{T_{X^g} \cap T_{X^h}}$$

by the formula

$$v \mapsto \frac{1}{\mathrm{ord}(h)} \sum h^i \cdot v.$$

The map is well-defined because g and h commute under our initial assumptions (or the ones in Proposition 3.4). It splits the short exact sequence above. This shows $X^g \times_X^R X^h$ is formal over $X^g \times X^h$.

Consider the following commutative diagram



By [ACH19, Theorem 1.8] we know that the dg functor $j^* i_*(-)$ is isomorphic to $q_* \left(p^*(-) \otimes \mathsf{Sym}(E^\vee[1])\right)$.

The structure complex of $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$ is

$$i_* \mathsf{Sym}(\Omega_{X^g}[1]) \otimes^L_{\mathscr{O}_X} j_* \mathsf{Sym}(\Omega_{X^h}[1]) = j_* \left(j^* i_* \mathsf{Sym}(\Omega_{X^g}[1]) \otimes \mathsf{Sym}(\Omega_{X^h}[1])\right).$$

Using the isomorphism of the two dg functors above, we see that $j^* i_*(\mathsf{Sym}(\Omega_{X^g}[1])) \cong q_*(p^*(\mathsf{Sym}(\Omega_{X^g}[1])) \otimes \mathsf{Sym}(E^\vee[1]))$. As a consequence

$$\begin{aligned} i_* \mathsf{Sym}(\Omega_{X^g}[1]) \otimes^L_{\mathscr{O}_X} j_* \mathsf{Sym}(\Omega_{X^h}[1]) = \\ &= j_* q_* \left(\mathsf{Sym}(\Omega_{X^g}|_{X^{g,h}}[1]) \otimes \mathsf{Sym}(E^\vee[1]) \otimes \mathsf{Sym}(\Omega_{X^h}|_{X^{g,h}}[1])\right) \\ &= j_* q_* \mathsf{Sym}\left((\Omega_{X^g}|_{X^{g,h}} \oplus \Omega_{X^h}|_{X^{g,h}} \oplus E^\vee)[1]\right). \end{aligned}$$

Therefore $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$ is formal over X (and it is isomorphic to $\mathbb{L}_{\widetilde{X^{g,h}}}$). □

5. Associativity of the product

In this section we explain the strategy for proving Theorem A. In other words we want to show that, under the assumption that certain Bass-Quillen cohomology classes vanish, the product defined in Section 3 is associative. The proof is reduced to Propositions 5.6 and 5.7, which will be proved in Section 6.

5.1. Formality of triple intersections. To prove the associativity it is natural to study the triple intersection $\widetilde{X^{g,h,k}} = \widetilde{X^g} \times^R_X \times \widetilde{X^h} \times^R_X \widetilde{X^k}$ for g, h, and $k \in G$. More precisely we define

$$\begin{aligned} \widetilde{X^{g,h,k}} &= (\Delta \times^R_{X\times X} \Delta^g) \times_{\Delta^g} (\Delta^g \times^R_{X\times X} \Delta^{gh}) \times_{\Delta^{gh}} (\Delta^{gh} \times^R_{X\times X} \Delta^{ghk}) \\ &= \Delta \times^R_{X\times X} \Delta^g \times^R_{X\times X} \Delta^{gh} \times^R_{X\times X} \Delta^{ghk}, \end{aligned}$$

as explained in (4.5). Under the assumption that G is abelian it is not hard to see that $\widetilde{X^{g,h,k}}$ is formal over X, and it is isomorphic to $\mathbb{L}_{\widetilde{X^{g,h,k}}}$. The proof is essentially the same as the one in Section 4.

5.2. The diagram

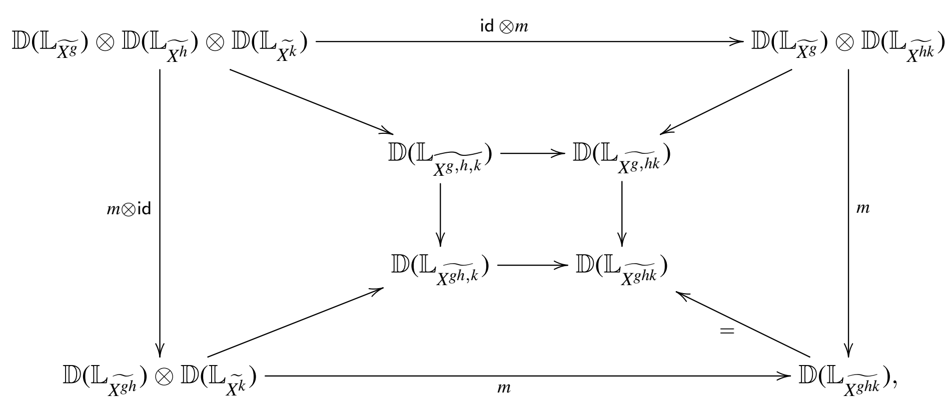
$$\begin{array}{ccc} \widetilde{X^{g,h,k}} & \longrightarrow & \widetilde{X^{g,hk}} \\ \downarrow & & \downarrow \\ \widetilde{X^{gh,k}} & \longrightarrow & \widetilde{X^{ghk}} \end{array}$$

is commutative because it is the associativity of the group law of the loop space $L[X/G]$. Taking distributions over on the corresponding linearizations, we get the following commutative diagram

$$\begin{array}{ccc} \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h,k}}}) & \longrightarrow & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,hk}}}) \\ \downarrow & & \downarrow \\ \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh,k}}}) & \longrightarrow & \mathbb{D}(\mathbb{L}_{\widetilde{X^{ghk}}}). \end{array}$$

For simplicity we have denoted the relative distributions with respect to X as $\mathbb{D}(-)$ instead of $\mathbb{D}(-/X)$.

5.3. What we need to prove. Consider the following diagram:



where m is the product on orbifold polyvector fields in Definition 3.9. Associativity of m is equivalent to commutativity of the outer part of the diagram.

The middle square is commutative by the discussion in the previous paragraph. The squares on the bottom and right are commutative because they are the definitions of our product.

We need to examine the ones on the top and left. The left one is the diagram

$$\begin{array}{ccc} \mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) & \longrightarrow & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h,k}}}) \\ \downarrow{\scriptstyle m\otimes \mathsf{id}} & & \downarrow \\ \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) & \longrightarrow & \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh,k}}}). \end{array}$$

Expand the diagram in detail

$$\begin{array}{ccccccc}
\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) & \longrightarrow & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h}}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) & \longrightarrow & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h}}} \times_X^R \mathbb{L}_{\widetilde{X^k}}) & \xrightarrow{\sim} & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h,k}}}) \\
{\scriptstyle m\otimes \mathsf{id}}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) & \xrightarrow{=} & \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) & \longrightarrow & \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}} \times_X^R \mathbb{L}_{\widetilde{X^k}}) & \xrightarrow{\sim} & \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh,k}}}).
\end{array}$$

Clearly, the left and middle squares of the diagram above are commutative. We have to show commutativity of the square on the right. Note that the maps on distributions are induced from maps on spaces, so we only need to show the commutativity of the diagram below

$$\begin{array}{ccc}
\mathbb{L}_{\widetilde{X^{g,h}}} \times_X^R \mathbb{L}_{\widetilde{X^k}} & \xrightarrow{\sim} & \mathbb{L}_{\widetilde{X^{g,h,k}}} \\
\downarrow & & \downarrow \\
\mathbb{L}_{\widetilde{X^{gh}}} \times_X^R \mathbb{L}_{\widetilde{X^k}} & \xrightarrow{\sim} & \mathbb{L}_{\widetilde{X^{gh,k}}}.
\end{array}$$

Similarly, the commutativity of the top square in the big diagram in (5.2) reduces to the commutativity of the diagram below

$$\begin{array}{ccc}
\mathbb{L}_{\widetilde{X^g}} \times_X^R \mathbb{L}_{\widetilde{X^{h,k}}} & \xrightarrow{\sim} & \mathbb{L}_{\widetilde{X^{g,h,k}}} \\
\downarrow & & \downarrow \\
\mathbb{L}_{\widetilde{X^g}} \times_X^R \mathbb{L}_{\widetilde{X^{hk}}} & \xrightarrow{\sim} & \mathbb{L}_{\widetilde{X^{g,hk}}}.
\end{array}$$

We will only analyze the former diagram; the proof of the commutativity of the latter is entirely similar.

5.4. There is, however, one more compatibility that needs to be discussed. Even though we wrote the top left diagonal map in the big diagram (5.2) as a single map, it is in fact clear from the discussion above that there are two maps here,

$$\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) \rightrightarrows \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h,k}}}).$$

One is the one that appears in the left square in the big diagram in (5.2), and the other one is the one that is in the top square of the big diagram in (5.2). We need to prove that these two maps are the same. This question is easily reduced to the following problem.

As mentioned in the previous section, the linearizations $\mathbb{L}_{\widetilde{X^g}}$, $\mathbb{L}_{\widetilde{X^h}}$, and $\mathbb{L}_{\widetilde{X^k}}$ are vector bundles over the underlying schemes X^g, X^h, and X^k. To prove the formality of derived intersections of linearizations, it suffices to prove the formality of the underlying schemes. There are two ways to define the isomorphism $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}} \times^R_X \mathbb{L}_{\widetilde{X^k}} \cong \mathbb{L}_{\widetilde{X^{g,h,k}}}$. One uses the fact that $\widehat{X^{g,h}} = X^g \times^R_X X^h$ is formal and $\widehat{X^{(g,h),k}} = X^{g,h} \times^R_X X^k$ is formal. The other uses the fact that $\widehat{X^{h,k}} = X^h \times^R_X X^k$ is formal and $\widehat{X^{g,(h,k)}} = X^g \times^R_X X^{h,k}$ is formal. Therefore, we need to prove that the two isomorphisms agree, i.e., that the diagram below is commutative

$$\begin{array}{ccccccccc}
(X^g \times^R_X X^h) \times^R_X X^k & \xrightarrow{\sim} & \mathbb{L}_{\widehat{X^{g,h}}} \times^R_X X^k = \mathbb{L}_{\widehat{X^{g,h}}} \times^R_{X^{g,h}} (X^{g,h} \times^R_X X^k) & \xrightarrow{\sim} & \mathbb{L}_{\widehat{X^{g,h}}} \times^R_{X^{g,h}} \mathbb{L}_{\widehat{X^{(g,h),k}}} & \to & \mathbb{L}_{\widehat{X^{g,h,k}}} \\
\downarrow \text{id} & & & & & & \downarrow \text{id} \\
X^g \times^R_X (X^h \times^R_X X^k) & \xrightarrow{\sim} & X^g \times^R_X \mathbb{L}_{\widehat{X^{h,k}}} = (X^g \times^R_X X^{h,k}) \times^R_{X^{h,k}} \mathbb{L}_{\widehat{X^{h,k}}} & \xrightarrow{\sim} & \mathbb{L}_{\widehat{X^{g,(h,k)}}} \times^R_{X^{h,k}} \mathbb{L}_{\widehat{X^{h,k}}} & \to & \mathbb{L}_{\widehat{X^{g,h,k}}}.
\end{array}$$

5.5. Unfortunately we can not prove the commutativity of the diagrams above without further assumptions. There is a cohomology class which plays an important role in what follows. It appears in a more general setting, which we review now.

Let $X \hookrightarrow Y \hookrightarrow S$ be a sequence of closed embedding of smooth schemes, and assume that there is a fixed first order splitting of the map $X \hookrightarrow Y$, i.e., we have fixed a map $X^{(1)} \to X$ which splits the inclusion $X \to X^{(1)}$ of X into its first order neighborhood $X^{(1)}$ in Y.

The class we need is the Bass-Quillen class associated to the restriction $N_{Y/S}|_{X^{(1)}}$ of the normal bundle $N_{Y/S}$ to the first order neighborhood $X^{(1)}$. It was introduced by the second author in [H20]. In this paper, we call this class the Bass-Quillen class associated to the sequence of embeddings $X \hookrightarrow Y \hookrightarrow S$.

The following two statements, which will be proven in the next section, imply the commutativity of the diagrams above, under the assumption that the Bass-Quillen classes associated to $X^{g,h} \hookrightarrow X^{gh} \hookrightarrow X$ and $X^{g,h} \hookrightarrow X^g \hookrightarrow X$ vanish for all $g, h \in G$. This will complete the proof of Theorem A.

5.6. Proposition. *Under the assumptions of Theorem A, assume that the Bass-Quillen class associated to $X^{g,h} \hookrightarrow X^{gh} \hookrightarrow X$ vanishes. Then the diagram*

$$\begin{array}{ccc}
\mathbb{L}_{\widetilde{X^{g,h}}} \times^R_X \mathbb{L}_{\widetilde{X^k}} & \xrightarrow{\sim} & \mathbb{L}_{\widetilde{X^{g,h,k}}} \\
\downarrow & & \downarrow \\
\mathbb{L}_{\widetilde{X^{gh}}} \times^R_X \mathbb{L}_{\widetilde{X^k}} & \xrightarrow{\sim} & \mathbb{L}_{\widetilde{X^{gh,k}}}.
\end{array}$$

is commutative.

5.7. Proposition. *Under the assumptions of Theorem A, assume that the Bass-Quillen class associated to $X^{g,h} \hookrightarrow X^g \hookrightarrow X$ and $X^{g,h} \hookrightarrow X^h \hookrightarrow X$ vanish. Then the diagram*

$$\begin{array}{ccccc} (X^g \times^R_X X^h) \times^R_X X^k & \xrightarrow{\sim} & \mathbb{L}_{\widehat{X^{g,h}}} \times^R_X X^k & \xrightarrow{\sim} & \mathbb{L}_{\widehat{X^{g,h,k}}} \\ \downarrow{\scriptstyle \mathsf{id}} & & & & \downarrow{\scriptstyle \mathsf{id}} \\ X^g \times^R_X (X^h \times^R_X X^k) & \xrightarrow{\sim} & X^g \times^R_X \mathbb{L}_{\widehat{X^{h,k}}} & \xrightarrow{\sim} & \mathbb{L}_{\widehat{X^{g,h,k}}}. \end{array}$$

is commutative.

5.8. Examples.

- If X is affine, then all the Bass-Quillen classes above are zero.
- Consider the $G = \mathbb{Z}/2\mathbb{Z}$ action on an abelian variety X. We have either $X^{g,h} = X^{gh}$ or $X^{gh} = X$ in this case, so it is easy to show that all the Bass-Quillen classes are zero.

Therefore the product on $\mathsf{HT}^*(X;G)$ defined in Section 3 is associative in the cases above.

6. Consequences of vanishing of Bass-Quillen classes

We prove Propositions 5.6 and 5.7 in this section.

6.1. All the linearizations are total spaces of vector bundles over the underlying schemes, so we can reduce the result of Proposition 5.6 to the commutativity of the following two formality isomorphisms

$$\begin{array}{ccccc} X^{g,h} & \longleftarrow & X^{g,h} \times^R_X X^k & \xrightarrow{\sim} & \mathbb{E}_{(g,h),k}[-1] = \mathbb{L}_{\widehat{X^{(g,h),k}}} \\ \downarrow & & \downarrow & & \downarrow \\ X^{gh} & \longleftarrow & X^{gh} \times^R_X X^k & \xrightarrow{\sim} & \mathbb{E}_{gh,k}[-1] = \mathbb{L}_{\widehat{X^{gh,k}}} \\ \downarrow & & \downarrow & & \\ X & \longleftarrow & X^k & & \end{array}$$

where $E_{(g,h),k} = \frac{T_X}{T_{X^{g,h}} + T_{X^k}}$ and $E_{gh,k} = \frac{T_X}{T_{X^{gh}} + T_{X^k}}$ are excess bundles supported on $X^{g,h,k}$ and $X^{gh,k}$ respectively.

To check the commutativity, we need to look at how the isomorphism is defined in [ACH19]. For simplicity, denote T_X by V. The two isomorphisms are defined based on two splittings of the two short exact sequences

$$\begin{array}{ccccccccc}
0 \longrightarrow & \dfrac{V^k}{V^{g,h} \cap V^k} & \longrightarrow & \dfrac{V}{V^{g,h}} = N_{X^{g,h}/X} & \longrightarrow & \dfrac{V}{V^{g,h} + V^k} = E_{(g,h),k} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \dfrac{V^k}{V^{gh} \cap V^k} & \longrightarrow & \dfrac{V}{V^{gh}} = N_{X^{gh}/X} & \longrightarrow & \dfrac{V}{V^{gh} + V^k} = E_{gh,k} & \longrightarrow 0.
\end{array}$$

The two splittings are compatible in the sense that the diagram above commutes because the two splittings are the averaging map by the element $k \in G$

$$v \to \frac{1}{\mathrm{ord}(k)} \sum_{i=1}^{\mathrm{ord}(k)} k^i \cdot v.$$

Proposition 5.6 is a consequence of the more general result Proposition 6.2 below, by replacing X, Y, Z, and S in by $X^{g,h}$, X^{gh}, X^k, and X. Note that all the assumptions in Proposition 6.2 except for the last one hold trivially for $X^{g,h}$, X^{gh}, X^k, and X.

6.2. Proposition. *Consider a sequence of closed embeddings $X \hookrightarrow Y \hookrightarrow S$, and a separate closed embedding $Z \hookrightarrow S$.*

Assume that all the closed embeddings split to first order in the sense of [ACH19], and that the first order splittings of $X \hookrightarrow Y \hookrightarrow S$ are compatible in the sense of [H20]. We further assume that the Bass-Quillen class associated to $X \hookrightarrow Y \hookrightarrow S$ is zero. Then the diagram

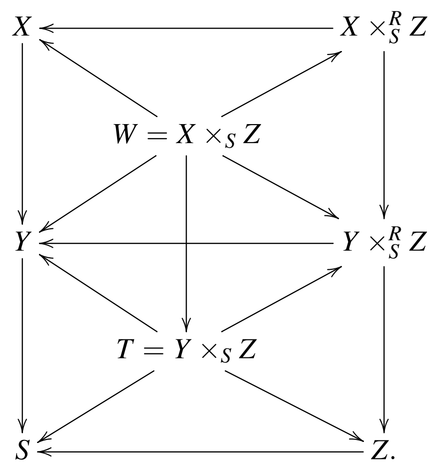
$$\begin{array}{ccc}
X \times^R_S Z & \xrightarrow{\sim} & \mathbb{E}_{X,Z}[-1] = \mathbb{E}_W[-1] = \mathbb{L}_{X \times^R_S Z} \\
\downarrow & & \downarrow \\
Y \times^R_S Z & \xrightarrow{\sim} & \mathbb{E}_{Y,Z}[-1] = \mathbb{E}_T[-1] = \mathbb{L}_{Y \times^R_S Z}
\end{array}$$

is commutative, where $E_{X,Z} = E_W = \frac{T_S}{T_X + T_Z}$ and $E_{Y,Z} = E_T = \frac{T_S}{T_Y + T_Z}$ are the excess bundles supported on

$$W = X \times_S Z \text{ and } T = Y \times_S Z.$$

The horizontal isomorphisms are defined in [ACH19] and will be explained in the proof below. The map $\mathbb{E}_{X,Z}[-1] \to \mathbb{E}_{Y,Z}[-1]$ is induced by the obvious map of vector bundles.

6.3. Before we begin the proof we note that the setup of the above proposition gives rise to the following diagram of spaces, where W and T are the underived fiber products,



Proof. The compatibility of first order splittings implies that the following two short exact sequences and their splittings are compatible

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{W/Z} & \longrightarrow & N_{X/S}|_W & \longrightarrow & E_W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{T/Z}|_W & \longrightarrow & N_{Y/S}|_W & \longrightarrow & E_T|_W & \longrightarrow & 0. \end{array}$$

The two isomorphisms $\mathbb{E}_W[-1] \cong X \times^R_S Z$, and $\mathbb{E}_T[-1] \cong Y \times^R_S Z$ are defined using the two splittings of short exact sequences above. The three horizontal maps on the left of the diagram below are the splittings of the short exact sequences. The composition of horizontal maps below are the desired isomorphisms $\mathbb{E}_W[-1] \cong X \times^R_S Z$ and $\mathbb{E}_T[-1] \cong Y \times^R_S Z$,

$$\begin{array}{ccccccc} \mathbb{E}_W[-1] & \dashrightarrow & \mathbb{N}_{X/S}[-1]|_W & \xrightarrow{\sim} & X \times^R_S X|_W = X \times^R_S W & \longrightarrow & X \times^R_S Z \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{E}_T[-1]|_W & \dashrightarrow & \mathbb{N}_{Y/S}[-1]|_W & \xrightarrow{\sim} & Y \times^R_S Y|_W = Y \times^R_S W & \longrightarrow & Y \times^R_S Z \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{E}_T[-1] & \dashrightarrow & \mathbb{N}_{Y/S}[-1]|_T & \xrightarrow{\sim} & Y \times^R_S Y|_T = Y \times^R_S T & \longrightarrow & Y \times^R_S Z. \end{array}$$

We only need to prove the commutativity of the isomorphisms in the middle

$$\begin{array}{ccc} \mathbb{N}_{X/S}[-1]|_W & \xrightarrow{\sim} & X \times^R_S X|_W = X \times^R_S W \\ \downarrow & & \downarrow \\ \mathbb{N}_{Y/S}[-1]|_W & \xrightarrow{\sim} & Y \times^R_S Y|_W = Y \times^R_S W. \end{array}$$

because all the others are commutative. We can restrict everything to X first, and then restrict to W. Therefore, it suffices to show the commutativity of

$$\begin{array}{ccc} \mathbb{N}_{X/S}[-1] & \xrightarrow{\sim} & X \times^R_S X \\ \downarrow & & \downarrow \\ \mathbb{N}_{Y/S}[-1]|_X & \xrightarrow{\sim} & Y \times^R_S Y|_X = Y \times^R_S X. \end{array}$$

This is Theorem A in [H20]. □

Proof of Proposition 5.7. For simplicity denote the space X^g, X^h, X^k, and X in Proposition 5.7 by X, Y, Z, and S.

Because of Proposition 6.2 we have the commutativity of

$$\begin{array}{ccc} X \times^R_S Y & \xrightarrow{\sim} & \mathbb{E}_{X,Y}[-1] = \mathbb{L}_{X \times^R_S Y} \\ \uparrow & & \uparrow \\ X \times^R_S T & \xrightarrow{\sim} & \mathbb{E}_{X,T}[-1] = \mathbb{L}_{X \times^R_S T} \end{array} \qquad \begin{array}{ccc} W \times^R_S Z & \xrightarrow{\sim} & \mathbb{E}_{W,Z}[-1] = \mathbb{L}_{W \times^R_S Z} \\ \downarrow & & \downarrow \\ Y \times^R_S Z & \xrightarrow{\sim} & \mathbb{E}_{Y,Z}[-1] = \mathbb{L}_{Y \times^R_S Z}, \end{array}$$

where $T = Y \times_S Z$ and $W = X \times_S Y$. As a consequence we get the commutative diagram

$$\begin{array}{ccccccc} X \times^R_S Y \times^R_S Z & \longrightarrow & \mathbb{E}_{X,Y}[-1] \times^R_S Z = \mathbb{E}_{X,Y}[-1] \times^R_W (W \times^R_S Z) & \longrightarrow & \mathbb{E}_{X,Y}[-1] \times^R_W \mathbb{E}_{W,Z}[-1] & \longrightarrow & \mathbb{E}_{X,Y,Z}[-1] \\ {\scriptstyle id}\downarrow & & \downarrow & & \downarrow & & {\scriptstyle id}\downarrow \\ X \times^R_S Y \times^R_S Z & \longrightarrow & \mathbb{E}_{X,Y}[-1] \times^R_S Z = \mathbb{E}_{X,Y}[-1] \times^R_Y (Y \times^R_S Z) & \longrightarrow & \mathbb{E}_{X,Y}[-1] \times^R_Y \mathbb{E}_{Y,Z}[-1] & \longrightarrow & \mathbb{E}_{X,Y,Z}[-1] \\ {\scriptstyle id}\updownarrow & & & & {\scriptstyle id}\updownarrow & & {\scriptstyle id}\updownarrow \\ X \times^R_S Y \times^R_S Z & \longrightarrow & X \times^R_S \mathbb{E}_{Y,Z}[-1] = (X \times^R_S Y) \times^R_Y \mathbb{E}_{Y,Z}[-1] & \longrightarrow & \mathbb{E}_{X,Y}[-1] \times_Y \mathbb{E}_{Y,Z}[-1] & \longrightarrow & \mathbb{E}_{X,Y,Z}[-1] \\ {\scriptstyle id}\uparrow & & \uparrow & & \uparrow & & {\scriptstyle id}\uparrow \\ X \times^R_S Y \times^R_S Z & \longrightarrow & X \times^R_S \mathbb{E}_{Y,Z}[-1] = (X \times^R_S T) \times^R_T \mathbb{E}_{Y,Z}[-1] & \longrightarrow & \mathbb{E}_{X,T}[-1] \times^R_T \mathbb{E}_{Y,Z}[-1] & \longrightarrow & \mathbb{E}_{X,Y,Z}[-1], \end{array}$$

where all the arrows are isomorphisms, and $E_{X,Y,Z}$ is the excess bundle of the triple intersection $X \times^R_S Y \times^R_S Z$.

The two rightmost squares of the diagram above commute because there are natural isomorphisms $E_{X,Y,Z} \cong E_{X,Y}|_U \oplus E_{W,Z} \cong E_{X,T} \oplus E_{Y,Z}|_U$, where $U = X \cap Y \cap Z = X \times_S Y \times_S Z$. The commutativity of the outer square of the diagram above is the one that we needed to prove in Proposition 5.7. □

7. A possible simplification

This section is more speculative. After we rewrite our product in Definition 3.9 in more concrete terms, we propose a way to simplify the formulas for Calabi-Yau global quotient orbifolds. The simplification is motivated by the definition of Chen-Ruan orbifold cohomology, so we need to first review this definition before proceeding. We then provide several examples comparing the simplified product with the Chen-Ruan orbifold cohomology, via homological mirror symmetry. We end the paper by stating a number of questions that remain open for future research.

7.1. In Section 4 we computed the structure complexes of $\mathbb{L}_{\widetilde{X^{g,h}}}$ and $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$, so we can write the two-step map

$$\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) \to \mathbb{D}(\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}) \cong \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h}}}) \to \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}})$$

in Definition 3.9 in more concrete terms. Explicitly, for $g \in G$ define

$$\mathsf{HT}^{(p,q)}(X;g) = H^{p-c_g}(X^g, \wedge^q T_{X^g} \otimes \omega_g).$$

Remark that this is *not* the same bigrading as the one in the introduction.

The two composition of the maps above can then be written as a direct sum over p, p', q, q' of maps

$$\mathsf{HT}^{(p,q)}(X;g) \otimes \mathsf{HT}^{(p',q')}(X;h) \to \bigoplus_{i=0}^{\mathsf{rk}\, E} \mathsf{HT}^{(p+p'-i,q+q'+i)}(X;gh)$$

factoring through the middle term (coming from $\mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h}}})$)

$$\bigoplus_{i=0}^{\mathsf{rk}\, E} H^{p+p'-c_{g,h}-i}(X^{g,h}, \wedge^q T_{X^g}|_{X^{g,h}} \otimes \wedge^{q'} T_{X^h}|_{X^{g,h}} \otimes \wedge^i E \otimes \omega_{g,h}).$$

Here E is the excess bundle for the intersection of X^g and X^h in X. Note in particular that if G is trivial our definition recovers the classical product on polyvector fields.

Observe that the $\mathsf{HT}^{(p,q)}$ notation does *not* give a bigrading – *a priori* all the maps above, for $0 \leq i \leq \mathsf{rk}\, E$ could be non-zero. The simplification we propose, for the Calabi-Yau case, is to leave only one of these maps, for a specific i. (Conjecturally, all the other maps would be zero anyway.)

7.2. The formulas for Chen-Ruan orbifold cohomology. We now discuss some preparations for the motivation for the simplification of the product we defined. The idea is to draw inspiration from mirror symmetry, and to regard, in the Calabi-Yau case, Chen-Ruan orbifold cohomology as the mirror of orbifold Hochschild cohomology.

Let X be a complex manifold endowed with the action of a finite group G. Chen and Ruan [CR04] defined a version of the singular cohomology *ring* for the orbifold $[X/G]$. Fantechi and Göttsche [FG03] wrote down the formula for the product explicitly as follows. They first constructed an associative product on

$$H^*_{\mathsf{orb}}(X;G) = \bigoplus_{g\in G} H^{*-2\iota(g)}(X^g,\mathbb{C})$$

which maps $\alpha_g \in H^{*-2\iota(g)}(X^g,\mathbb{C})$ and $\beta_h \in H^{*-2\iota(h)}(X^h,\mathbb{C})$ to

$$(\alpha_g,\beta_h) \mapsto i^{gh}_{g,h*}(\alpha_g|_{X^{g,h}} \cdot \beta_h|_{X^{g,h}} \cdot \gamma_{g,h}).$$

Here $\gamma_{g,h}$ is the top Chern class of a certain twist bundle whose rank is $\iota(g)+\iota(h)-\iota(gh)-\mathsf{codim}(X^{g,h},X^{gh})$, where $\iota(g)$ is the so-called *age* of g, see [FG03].

The Chen-Ruan orbifold singular cohomology ring is obtained by taking G-invariants:

$$H^*_{\mathsf{orb}}([X/G]) = H^*_{\mathsf{orb}}(X;G)^G.$$

Note that the above ring is *bigraded* with respect to the orbifold Hodge decomposition [ALR07]

$$H^{n-2\iota(g)}(X^g,\mathbb{C}) = \bigoplus_{p+q=n} H^{p-\iota(g)}(X,\wedge^{q-\iota(g)}\Omega_{X^g}).$$

7.3. Mirror symmetry associates two graded commutative rings to a Calabi-Yau space: the A- and the B-model state spaces, which are interchanged by the mirror operation. When the target space is a compact Calabi-Yau manifold X, the A-space is $H^*(X,\mathbb{C})$, while the B-space is $\mathsf{HH}^*(X)$. When it is an orbifold $[X/G]$, these spaces are naturally the Chen-Ruan orbifold cohomology and the orbifold Hochschild cohomology rings of $[X/G]$.

Since the product in the A-model preserves the p,q bidegree, the yoga of mirror symmetry suggests that the product in the B-model should also preserve *some* bidegree for Calabi-Yau orbifolds.

The proofs of the following two lemmas are left as exercises to the reader.

7.4. Lemma. *There is a natural isomorphism*

$$\omega_g|_{X^{g,h}}[-c_g] \otimes \omega_h|_{X^{g,h}}[-c_h] \cong \wedge^r E[r] \otimes \omega_{g,h}[-c_{g,h}],$$

where r is the rank of the excess bundle E.

7.5. Lemma. *The bundle $T_{X^g}|_{X^{g,h}}$ decomposes naturally into a direct sum as $T_{X^{g,h}} \oplus N_{X^{g,h}/X^g}$, and similarly for $T_{X^h}|_{X^{g,h}}$.*

The class $\gamma_{g,h} \in H^k(X^{g,h}, \wedge^k \Omega_{X^{g,h}})$ in Fantechi and Göttsche's paper [FG03] acts naturally on

$$\bigoplus_{p,q} H^p(X^{g,h}, \wedge^q T_{X^g}|_{X^{g,h}} \otimes \wedge^{q'} T_{X^h}|_{X^{g,h}} \otimes \wedge^r E^\vee \otimes \omega_{g,h}),$$

where

$$k = \iota(g) + \iota(h) - \iota(gh) - \mathsf{codim}(X^{g,h}, X^{gh})$$

and the action is given by the contraction of $\Omega_{X^{g,h}}$ with $T_{X^{g,h}}$.

7.6. We are now ready to give a new construction for an operation on $\mathsf{HT}^*(X;G)$ which mimics more closely the Fantechi-Göttsche product [FG03]. Define the bigraded piece $\mathsf{HT}^{p,q}(X;G)$ of bidegree p,q of $\mathsf{HT}(X;G)$ by

$$\mathsf{HT}^{p,q}(X;G) = H^{p-\iota(g)}(X, \wedge^{q+\iota(g)-c_g} T_{X^g} \otimes \omega_g).$$

The product will be bigraded, being given by maps

$$\mathsf{HT}^{p,q}(X;G) \otimes \mathsf{HT}^{p',q'}(X;G) \to \mathsf{HT}^{p+p',q+q'}(X;G).$$

Note that unlike the product in (7.1), only one of the maps there is non-zero. We conjecture that in Calabi-Yau situations, the two products agree – in other words, all the maps in (7.1) which do not preserve the bigrading are zero. This is the case in all the examples we study below.

The new product is defined as the following composition:

$$\begin{aligned}
&H^p(X^g, \wedge^q T_{X^g} \otimes \omega_g[-c_g]) \otimes H^{p'}(X^h, \wedge^{q'} T_{X^h} \otimes \omega_h[-c_h]) \\
&\quad \to H^{p+p'}(X^{g,h}, \wedge^q T_{X^g}|_{X^{g,h}} \otimes \omega_g|_{X^{g,h}}[-c_g] \otimes \wedge^{q'} T_{X^h}|_{X^{g,h}} \otimes \omega_h|_{X^{g,h}}[-c_h]) \\
&\quad \cong H^{p+p'-r}(X^{g,h}, \wedge^q T_{X^g}|_{X^{g,h}} \otimes \wedge^{q'} T_{X^h}|_{X^{g,h}} \otimes \omega_{g,h}[-c_{g,h}] \otimes \wedge^r E) \\
&\quad \to \bigoplus_{i+j=k} H^{p+p'-r+k}(X^{g,h}, \wedge^{q-i} T_{X^g}|_{X^{g,h}} \otimes \wedge^{q'-j} T_{X^h}|_{X^{g,h}} \otimes \omega_{g,h}[-c_{g,h}] \otimes \wedge^r E) \\
&\quad \to H^{p+p'-r+k}(X^{gh}, \wedge^{q+q'+r-k} T_{X^{gh}} \otimes \omega_{gh}[-c_{gh}]).
\end{aligned}$$

The first arrow is the naive restriction from X^g and X^h to $X^{g,h}$. The isomorphisms in the middle are due to Lemma 7.4. The last arrow is the map $\mathbb{L}_{m*}$ in Definition 3.9. The second arrow in the middle involving k is the action of $\gamma_{g,h}$ in Lemma 7.5. One does indeed verify that this map respects the bigrading defined above.

7.7. Examples. For a first example consider an abelian surface A endowed with the action of $\mathbb{Z}/2\mathbb{Z}$, acting by negation in the group law of A. The tangent bundle of A is trivial, so there are no Duflo correction terms. The mirror of the orbifold $[A/G]$ is expected to be $[A/G]$ itself in this case. This suggests that the product we defined should match with the one on the orbifold cohomology of $[A/G]$, so we expect to find isomorphisms

$$\mathsf{HT}^*([A/G]) \cong \mathsf{HH}^*([A/G]) \cong H^*_{\mathsf{orb}}([A/G], \mathbb{C}),$$

It is known [FG03] that in this case the classes $\gamma_{g,h}$ are trivial. Write $G = \{e, \tau\}$ where e is the identity element. Then we have

$$\mathsf{HH}^*([A/G]) = \big(\mathsf{HH}^*(A,e) \oplus \mathsf{HH}^*(A,\tau)\big)^G = \mathsf{HH}^*(A,e) \oplus \mathsf{HH}^*(A,\tau)^\tau,$$

where for $g \in G$ the notation $\mathsf{HH}^*(A,g)$ was explained in Section 2. The space $\mathsf{HH}^*(A,e)$ is the Hochschild cohomology of A, and its product is well-understood from the Kontsevich and Calaque-Van den Bergh theorem. The only non-trivial product we need to understand is

$$\mathsf{HH}^*(A,\tau) \otimes \mathsf{HH}^*(A,\tau) \to \mathsf{HH}^*(A,e).$$

Note that the space

$$\mathsf{HH}^*(A,\tau) = H^0(A^\tau, \wedge^0 T_{A^\tau} \otimes \omega_\tau) = H^0(A^\tau, \mathbb{C}),$$

is a 16-dimensional vector space in cohomological degree 2. It is of bidegree $(1,1)$ under the new bigrading we defined in (7.6). By the definition of our product, it is also clear that the product of two $(1,1)$-form gives a $(2,2)$-form which lands in $H^2(A, \wedge^2 T_A)$. This matches perfectly with the product on orbifold cohomology [FG03].

7.8. For another example, consider a holomorphic symplectic orbifold $[X/G]$. Again, the mirror of $[X/G]$ is expected to be $[X/G]$, so we expect to get

$$\mathsf{HT}^*([X/G]) \cong \mathsf{HH}^*(X/G) \cong H^*_{\mathsf{orb}}([X/G], \mathbb{C}).$$

The right hand side decomposes into

$$H^{*-2\iota(g)}(X^g, \mathbb{C}) = \bigoplus_{p+q=*} H^{p-\iota(g)}(X^g, \wedge^{q-\iota(g)} \Omega_{X^g})$$

by the Hodge decomposition. The left hand side is

$$\bigoplus_{g \in G} H^{p-\iota(g)}(X^g, \wedge^{q+\iota(g)-c_g} T_{X^g} \otimes \omega_g).$$

Moreover, ω_g is trivial and $\Omega_{X^g} \cong T_{X^g}$ because of the holomorphic symplectic condition. There is a canonical identification between the two sides as vector spaces, and we believe the two products should agree. The bigradings of the two sides match completely because $2\iota(g) = c_g$.

One important example we have in mind is when $X = K^n$ consists of n copies of a K3 surface K and $G = \Sigma_n$ is the symmetric group acting on K^n by permutation. The group is not abelian in this case, but the constructions and results in Sections 3 and 4 still work because one can check directly that Proposition 3.4 holds in this situation. The key point is that in this situation all the tangent bundles and normal bundles involved are copies of direct sums of T_K, so the short exact sequence in the proof of Proposition 3.4 splits naturally.

7.9. Open questions.

(1) We can not prove that the simplified product in Definition 7.6 agrees with the one in Definition 3.9. We conjecture that they agree under the Calabi-Yau assumption.

(2) For any Calabi-Yau orbifold, we believe that our product on orbifold polyvector fields should match with the Chen-Ruan orbifold cohomology of the mirror.

(3) This is our main Conjecture A. For any orbifold $[X/G]$ with an abelian group action, we believe that Kontsevich's Theorem holds, i.e., the orbifold Hochschild cohomology should be isomorphic to the orbifold polyvector fields. More precisely, we conjecture that the diagram

$$\begin{array}{ccc}
\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) & \xrightarrow{HKR\circ(\sqrt{\mathrm{td}(T_{X^g})}\lrcorner -)\otimes HKR\circ(\sqrt{\mathrm{td}(T_{X^h})}\lrcorner -)} & \mathbb{D}(\widetilde{X^g}) \otimes \mathbb{D}(\widetilde{X^h}) \\
\downarrow & & \downarrow \\
\mathbb{D}(\mathbb{L}_{\widetilde{X^g}\times^R_X \widetilde{X^h}}) \cong \mathbb{D}(\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}) & \xrightarrow{HKR\circ(\sqrt{\mathrm{td}(T_{X^{g,h}})}\lrcorner -)} & \mathbb{D}(\widetilde{X^g} \times^R_X \widetilde{X^h}) = \mathbb{D}(\widetilde{X^{g,h}}) \\
\downarrow{\scriptstyle \mathbb{L}_{m*}} & & \downarrow{\scriptstyle m_*} \\
\mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}) & \xrightarrow{HKR\circ(\sqrt{\mathrm{td}(T_{X^{gh}})}\lrcorner -)} & \mathbb{D}(\widetilde{X^{gh}})
\end{array}$$

is commutative, where the horizontal maps are isomorphisms. All the HKR maps that appear in the horizontal isomorphisms are the formality isomorphisms in Sections 2–4. They generalize the classical HKR isomorphism as explained in [ACH19]. As mentioned at the very beginning of this paper, HKR can not be an isomorphism of rings, so we need to add the Duflo correction term in the horizontal isomorphisms.

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