

Categorical Enumerative Invariants, II: Givental formula

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ABSTRACT: To a pair (A, s) consisting of a smooth, cyclic A_∞ -algebra A and a splitting s of the Hodge filtration on its Hochschild homology, Costello (2005) associates an invariant which conjecturally generalizes the total descendant Gromov-Witten potential of a symplectic manifold.

In this paper we give explicit, computable formulas for Costello's invariant, as Feynman sums over partially directed stable graphs. The formulas use in a crucial way the combinatorial string vertices defined earlier by Costello and the authors. Explicit computations elsewhere confirm in many cases the equality of categorical invariants with known Gromov-Witten, Fan-Jarvis-Ruan-Witten, and Bershadsky-Cecotti-Ooguri-Vafa invariants.

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1. Introduction

1.1. Gromov-Witten theory associates to a compact symplectic $2d$ -dimensional manifold X its total descendant potential \mathcal{D}_X [Giv01], [Coa08]. It is a formal expression that encodes the Gromov-Witten invariants of X at all genera and with all possible insertions of cohomology classes from X and ψ -classes from the moduli spaces of curves (descendants). If we denote by $H = H^*(X)[2d]$ the shifted cohomology of X ,

the dual \mathcal{D}^X of the total descendant potential lives in a certain completed symmetric algebra

$$\mathcal{D}^X \in \widehat{\text{Sym}}_{\hbar} (H[u^{-1}]) \llbracket \hbar, \lambda \rrbracket$$

which uses formal variables u, λ, \hbar of even homological degree to keep track of ψ -class insertions, genus, and Euler characteristic, respectively. See Section 2 for a review of the above concepts and of the notations we use.

1.2. In his visionary 1994 address to the International Congress of Mathematicians Kontsevich [Kon95] predicted that Gromov-Witten theory is of categorical nature. More precisely he proposed that it should be possible to extract \mathcal{D}^X directly from the (idempotent completed) Fukaya category $\text{Fuk}(X)$ of X .

The category $\mathcal{C} = \text{Fuk}(X)$ is a Calabi-Yau A_∞ -category. Implicit in Kontsevich's work was the idea that it should be possible to attach an invariant similar to \mathcal{D}^X to any category \mathcal{C} of this type. In particular, if \mathcal{C} were the derived category $\text{D}_{\text{coh}}^b(\check{X})$ of coherent sheaves on a compact Calabi-Yau manifold \check{X} , the resulting genus zero invariants were expected to agree with the solutions of the Picard-Fuchs equation governing the variation of Hodge structures on \check{X} . The higher genus invariants would be new, B-model analogues of Gromov-Witten invariants; their values would, in some cases, be predicted by Bershadsky-Cecotti-Ooguri-Vafa (BCOV) theory [BCOV94].

Costello [Cos09] observed that the proper input for such a construction should include not just the category \mathcal{C} but also a choice of splitting of the non-commutative Hodge filtration on the Hochschild homology of \mathcal{C} . (The reason this was not initially apparent is the fact that the Fukaya category has a canonical choice of splitting; other categories, however, may not.) Costello provided a non-constructive definition of such an invariant in [Cos09].

1.3. Our goal is to extend the works of Kontsevich and Costello to make the definition of the categorical enumerative invariants *explicitly computable*. Let (A, s) be a pair consisting of a smooth, cyclic A_∞ -algebra A and a splitting s of the Hodge filtration on its cyclic homology. If d is the Calabi-Yau dimension of A define

$$H = HH_*(A)[d],$$

the shifted Hochschild homology of A . The main contribution of our paper is to give a new definition of the categorical enumerative total descendant potential of the pair (A, s) ,

$$\mathcal{D}^{A,s} \in \widehat{\text{Sym}}_{\hbar} (H[u^{-1}]) \llbracket \hbar, \lambda \rrbracket.$$

The definition is explicit enough to make direct computations of categorical enumerative invariants possible (within limits of computational power).

1.4. Our invariant agrees with Costello’s (see Section 3) and it is directly inspired by it. The fundamental difference between the two is in the computability aspect. Both Costello’s original construction and our own rely on the existence of certain chains on the moduli spaces of curves, first introduced by Sen-Zwiebach [SZ94] and called *string vertices*. Costello’s original definition uses geometrically defined string vertices, which cannot be explicitly computed. We use instead a new, combinatorial model for string vertices introduced by Costello and the authors in [CCT20]. Using them we are able to bypass certain non-constructive aspects of the construction in [Cos09]. Our resulting formulas for $\mathcal{D}^{A,s}$ are given as Feynman sums over partially directed stable graphs (a concept we introduce), the vertices of which are labeled by combinatorial string vertices. Both of these can be explicitly computed.

1.5. A different approach to categorical enumerative invariants in genus zero was proposed by Barannikov-Kontsevich [Bar01] using the idea of variation of semi-infinite Hodge structures (VSHS). Their work was extending pioneering work of Saito [Sai83], [Sai83] on unfoldings of singularities, and was generalized to arbitrary cyclic A_∞ -algebras by Ganatra-Perutz-Sheridan [GPS15]. Our approach accesses all genera, unlike the VSHS approach which can only recover the genus zero part of the theory.

1.6. We emphasize that the relationship between our theory and classical Gromov-Witten theory is so far speculative; it depends on certain standard conjectures in symplectic geometry which are beyond the scope of this paper. However, all evidence points to the fact that when the algebra A is Morita equivalent to the Fukaya category of a symplectic manifold X the resulting invariant is precisely

$$\mathcal{D}^{A,s} = \mathcal{D}^X.$$

To be more precise, we expect the Fukaya category $\text{Fuk}(X)$ to come endowed with a natural splitting s of the Hodge filtration, arising from the fact that $\text{Fuk}(X)$ is defined over the Novikov ring. It is this choice of splitting s that would be used to get the identification above. Multiple computations of $\mathcal{D}^{A,s}$ confirm this [CT17], [CC20-2].

1.7. Our approach works equally well for \mathbb{Z} - and $\mathbb{Z}/2\mathbb{Z}$ -graded algebras. As a consequence we can use an algebra A that is Morita equivalent to a category of matrix factorizations. Direct computations [CLT19] strongly suggest that in this case the resulting categorical potential agrees with the potential of the B-model Fan-Jarvis-Ruan-Witten (FJRW) theory.

1.8. Our results can be regarded as giving a unified definition of virtually all known enumerative invariants in the literature, by taking the algebra A to be Morita equivalent to

- the Fukaya category, for Gromov-Witten theory;
- the wrapped Fukaya category, for FJRW theory;
- the derived category of a Calabi-Yau manifold, for BCOV theory;
- the category of matrix factorizations, for B-model FJRW theory.

Moreover, all these theories can immediately be generalized to the setting where a finite group of symmetries is present: the input category is simply replaced by a corresponding smash-product construction. In particular this allows us to define orbifold FJRW invariants, for which no direct definition exists.

From this point of view enumerative mirror symmetry follows tautologically from categorical mirror symmetry. This was Kontsevich’s original explanation for the equality of numerical invariants on the two sides of mirror symmetry observed by Candelas-de la Ossa-Green-Parkes [COGP91]. However, we cannot yet claim that categorical mirror symmetry implies enumerative mirror symmetry for all genera – the main difficulty lies in identifying the categorical invariants with the geometric ones.

1.9. Outline of the paper. Section 2 collects definitions and results from the literature that will be needed in the rest of the paper. This material is not new, but we review it here for the convenience of the reader. We present certain aspects of Gromov-Witten theory, in particular the definition of the total descendant potential and Givental’s Lagrangian formalism. We also include a short summary of the construction of combinatorial string vertices from [CCT20] as solutions of the master equation in a certain DGLA $\widehat{\mathfrak{g}}$. A choice of cyclic A_∞ -algebra A gives rise to a similar DGLA $\widehat{\mathfrak{h}}$ and to a map $\rho^A : \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{h}}$.

In Section 3 we clarify Costello’s original definition of categorical enumerative invariants from [Cos07], [Cos09]. For a pair (A, s) consisting of a cyclic A_∞ -algebra A and a splitting s of the non-commutative Hodge filtration we define invariants

$$F_{g,n}^{A,s} \in \text{Sym}^n (H[u^{-1}])$$

for any pair (g, n) such that $2g - 2 + n > 0, n > 0$. The case $n = 0$ can be included by forcing the dilaton equation but we shall not address this technical point in the current paper.

In Section 4 we use a chain-level trivialization of the Connes circle operator B in order to obtain an explicit partial trivialization of the DGLA structure on $\widehat{\mathfrak{h}}$. Trivializations

of circle operators and their relationship with Givental's formula were already studied extensively from the operadic point of view in [Dru14, DSV13, DSV15, KMS13].

In the final section we prove our main result, Corollary 5.7. It gives a Feynman sum formula for the categorical enumerative invariants $F_{g,n}^{A,s}$ as a sum over *partially directed* stable graphs (see Definition 4.19)

$$F_{g,n}^{A,s} = \sum_{G \in \Gamma((g,1,n-1))} \text{wt}(G) \prod_{e \in E_G} \text{Cont}(e) \prod_{v \in V_G} \text{Cont}(v) \prod_{l \in L_G} \text{Cont}(l).$$

The vertex contributions $\text{Cont}(v)$ in this formula are given by the tensors $\widehat{\beta}_{g,k,l}^A = \rho^A(\widehat{Y}_{g,k,l}^{\text{comb}})$ obtained from the string vertex in [CCT20]. The contributions of edges and leaves involve the choice of the splitting s , and the weight $\text{wt}(G)$ of a partially directed stable graph is calculated from the combinatorics of G by a formula involving its automorphism group.

We also prove in Section 5 that our invariants are stable under Morita equivalence.

We conclude the paper with two appendices. The first one, Appendix A, gives several explicit formulas for low genus invariants (for $\chi \geq -3$). The second one, Appendix B, deals with the \mathbb{Z} -graded case. This is needed because, for ease of notation, we assume throughout the body of the paper that all our vector spaces are $\mathbb{Z}/2\mathbb{Z}$ -graded, and we ignore even degree shifts. In Appendix B we outline how our results can be modified to accommodate the \mathbb{Z} -graded case. As a consequence we prove that our invariants satisfy an analogue of the dimension axiom in Gromov-Witten theory.

1.10. Acknowledgments. We thank Kevin Costello, Nick Sheridan, Si Li, and Dima Arinkin for patiently listening to the various problems we ran into at different stages of the project and for providing insight.

Andrei Căldăraru was partially supported by the National Science Foundation through grant number DMS-1811925 and by the Wisconsin Alumni Research Foundation through a Vilas Associate Professorship.

2. Preliminaries

2.1. Notations and conventions. Unless otherwise stated all our vector spaces will be $\mathbb{Z}/2\mathbb{Z}$ -graded over a field \mathbb{K} of characteristic zero. (The \mathbb{Z} -graded case is outlined in Appendix B.) Nevertheless, we think of our chain complexes as being *homologically* graded: differentials decrease degree.

Throughout the paper we will work with a cyclic A_∞ -algebra A of Calabi-Yau dimension d , i.e., the cyclic pairing $\langle -, - \rangle : A \otimes A \rightarrow \mathbb{K}$ has degree d . The algebra A will be assumed to satisfy the conditions below:

(†) A is smooth, finite dimensional, unital, and satisfies the Hodge-de Rham degeneration property.

(The Hodge-de Rham degeneration property is automatic if A is \mathbb{Z} -graded [Kal08].)

We will always use the shifted sign conventions described in [Cho08], [She15]. The shifted degree $|x|'$ of an element x is defined to be

$$|x|' = |x| + 1.$$

All the operations in cyclic A_∞ -algebras, including the pairing, and in L_∞ -algebras, including DGLAs, differ by a sign

$$(-1)^{\sum_{k=1}^n (n-k)|x_k|}$$

from the usual ones, when applied to a tensor $x_1 \otimes \cdots \otimes x_n$. In particular a Maurer-Cartan element in a DGLA has shifted degree zero.

2.2. Circle actions. We shall frequently use complexes with a circle action, together with a compatible pairing on them. A circle action a chain complex (C, ∂) is given by an operator $\delta : C \rightarrow C$ of degree one such that $\delta^2 = 0$, and $[\partial, \delta] = \partial\delta + \delta\partial = 0$. Associated with a chain complex with a circle action, we set

$$\begin{aligned} C^{\text{Tate}} &= (C\langle\langle u \rangle\rangle, \partial + u\delta) \\ C_+ &= (uC[[u]], \partial + u\delta) \\ C_- &= C^{\text{Tate}}/C_+ = (C[u^{-1}], \partial + u\delta), \end{aligned}$$

where u is a formal variable of even degree. Note that for C_+ this is slightly different from the usual definition used in defining negative cyclic homology – we start with u^1 instead of u^0 .

2.3. Pairings. A pairing on a chain complex with a circle action is a symmetric bilinear chain map of even degree

$$\langle -, - \rangle : C \otimes C \rightarrow \mathbb{K}$$

such that the circle operator δ is self-adjoint, i.e., we have

$$\langle \delta x, y \rangle = (-1)^{|x|} \langle x, \delta y \rangle \text{ for all } x, y \in C.$$

A pairing like the one described above induces a so-called *higher residue pairing* on the associated Tate complex, with values in $\mathbb{K}((u))$, defined for $x = \sum x_k \cdot u^k$ and $y = \sum x_l \cdot u^l$ by

$$\langle x, y \rangle_{\text{hres}} = \sum_{k,l} (-1)^k \langle x_k, y_l \rangle \cdot u^{k+l+2}.$$

(The choice of shift by u^2 is motivated by our desire for homogeneity in the construction of the Weyl algebra in (2.9).) Its residue at $u = 0$ is called the residue pairing, valued in \mathbb{K} and defined by

$$\langle x \cdot u^k, y \cdot u^l \rangle_{\text{res}} = \begin{cases} (-1)^k \langle x, y \rangle & \text{if } k + l = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Due to the sign $(-1)^k$ the residue pairing $\langle -, - \rangle_{\text{res}}$ is anti-symmetric and of degree zero.

Note that the residue pairing induces a natural even map

$$C_- \rightarrow \text{Hom}^c(C_+, \mathbb{K})$$

which will frequently be a quasi-isomorphism. Here the superscript “c” denotes continuous homomorphisms in the u -adic topology. Thus we will think of C_+ as the dual of C_- . We will often think of the above map as the odd map

$$\iota : C_- \rightarrow \text{Hom}^c(C_+[1], \mathbb{K})$$

turning an output into an input as in [CCT20].

2.4. The main example of a chain complex with circle action and pairing is provided by the shifted Hochschild chain complex of the A_∞ -algebra A ,

$$L = C_*(A)[d].$$

Its homology is the shifted Hochschild homology of A ,

$$H = H_*(L) = HH_*(A)[d].$$

For a smooth and proper Calabi-Yau algebra it is isomorphic to the dual of the Hochschild cohomology of A , also known as $HH^*(A, A^*)$. The degree of an element $x_1 | \dots | x_n \in L$ is $\sum_{k=1}^n |x_k|' + (d-1)$. The circle action is given by the Connes operator B , and the symmetric pairing on L is the chain-level Mukai pairing

$$\langle -, - \rangle_{\text{Muk}} : L \otimes L \rightarrow \mathbb{K}.$$

2.5. Gromov-Witten theory. Let X be a compact almost Kähler manifold of real dimension $2d$. We will denote by H the graded vector space $H = H^*(X)[2d]$, the shifted cohomology of X . Thinking of H as a chain complex with trivial differential and circle action, the Poincaré pairing on H extends to a skew-symmetric residue pairing $\langle -, - \rangle_{\text{res}}$ on $H^{\text{Tate}} = H(\!(u)\!)$. With respect to this pairing $H_+ = uH[\![u]\!]$ and $H_- = H[u^{-1}]$ are Lagrangian subspaces.

2.6. Fix a class $\beta \in H_2(X)$ which will be used to define classical Gromov-Witten invariants of X . In the discussion below we will think of β as being fixed and will ignore the dependence of the Gromov-Witten invariants on it.

Denote by $\overline{M}_{g,n}(X)$ the moduli space of stable maps to X from curves of genus g with n marked points and class β . It comes equipped with a virtual fundamental class $[\overline{M}_{g,n}(X)]^{\text{vir}}$. For $\gamma_1, \dots, \gamma_n \in H$ and $i_1, \dots, i_n \geq 0$ the corresponding descendant invariant is defined as

$$\langle \tau_{i_1}(\gamma_1), \dots, \tau_{i_n}(\gamma_n) \rangle_g^X = \int_{[\overline{M}_{g,n}(X)]^{\text{vir}}} \psi_1^{i_1} \text{ev}_1^* \gamma_1 \cdots \psi_n^{i_n} \text{ev}_n^* \gamma_n.$$

The genus g invariants are packaged as the Taylor coefficients of a power series F_X^g , the genus g descendant potential of X , see [Coa08, 2.1]. It is the function

$$F_X^g : H_+ \rightarrow \mathbb{K}[\![\lambda]\!]$$

(λ is another formal even variable) whose value at

$$\gamma = \gamma_0 u + \gamma_1 u^2 + \cdots + \gamma_i u^{i+1} + \cdots$$

is

$$F_X^g(\gamma) = \sum_{n \geq 0} \sum_{i_1, \dots, i_n \geq 0} \frac{1}{n!} \langle \tau_{i_1}(\gamma_{i_1}), \dots, \tau_{i_n}(\gamma_{i_n}) \rangle_g^X \cdot \lambda^{2g-2+n}.$$

The potentials F_X^g for $g \geq 0$ are then assembled into the total descendant potential

$$\mathcal{D}_X(\gamma) = \exp \left(\sum_{g \geq 0} \hbar^{g-1} F_X^g(\gamma) \right),$$

another formal function on H_+ with values in a certain completion of $\mathbb{K}[\hbar, \hbar^{-1}, \lambda]$; see (2.12) below. The variable \hbar , used to keep track of the genus, is also even.

2.7. Since the dual of H_- with respect to the residue pairing is H_+ , it is natural to ask if F_X^g satisfies the finiteness condition needed to ensure that it is the dual of an element $F_g^X \in H_-[[\lambda]]$, in other words if there exists an element

$$F_g^X = \sum_n F_{g,n}^X \cdot \lambda^{2g-2+n} \in H_-[[\lambda]]$$

such that

$$F_X^g(\gamma) = \sum_n \langle F_{g,n}^X, \gamma \rangle \text{ for all } \gamma \in H_+.$$

Such F_g^X exists, and we will call it the *dual genus g descendant potential* of X .

A formula for F_g^X is most easily written by choosing a basis for H , though the result is independent of this choice. Let t_1, \dots, t_N be a basis of H , and let $t^1, \dots, t^N \in H$ be the dual basis with respect to the Poincaré pairing. Define $F_g^X \in H_-[[\lambda]]$ as

$$F_g^X = \sum_{n \geq 0} \sum_{\substack{i_1, \dots, i_n \\ k_1, \dots, k_n}} \frac{1}{n!} \langle \tau_{i_1}(t_{k_1}), \dots, \tau_{i_n}(t_{k_n}) \rangle_g^X \cdot (u^{-i_1} t^{k_1}) \cdots (u^{-i_n} t^{k_n}) \cdot \lambda^{2g-2+n}.$$

Here the sum is over all ψ -class powers $i_1, \dots, i_n \geq 0$ and all indices $k_1, \dots, k_n \in \{1, \dots, N\}$.

The dual total descendant potential \mathcal{D}^X is then defined as before:

$$\mathcal{D}^X = \exp \left(\sum_{g \geq 0} \hbar^{g-1} F_g^X \right).$$

It lives in the localized and completed symmetric algebra $\widehat{\text{Sym}}_{\hbar}(H_-)[[\hbar, \lambda]]$ of H_- defined below (2.12). It is easy to check that all the expressions $\hbar^{g-1} F_g^X$ have homological degree zero. (This is one of the advantages of defining the residue pairing the way we do.)

2.8. The dual descendant potentials encode all the Gromov-Witten invariants of X . For example, for targets $X = \text{pt}$ and $X = T^2$ (a two-torus) we have for $\beta = 0$

$$\begin{aligned} F_0^{\text{pt}} &= \frac{1}{6}([X])^3 \lambda + \frac{1}{24}(u^{-1}[X])([X])^3 \lambda^2 + \cdots \\ F_1^{\text{pt}} &= \frac{1}{24}(u^{-1}[X]) \lambda + \cdots \\ F_1^{T^2} &= -\frac{1}{24}([X]) \lambda + \cdots \\ F_2^{T^2} &= \frac{7}{5760}(u^{-2}[X]) \lambda^3 + \cdots \end{aligned}$$

Here we write cohomology classes as their Poincaré duals: for example $[X] \in H^0(X)$ is the identity of this ring. It corresponds to the insertion of a point in the classical (non-dual) invariants.

2.9. The Weyl algebra and Fock space formalism. We now review the relationship, discovered by Costello [Cos09], between Batalin-Vilkovisky (BV) algebras and Fock spaces for dg-vector spaces with circle action and pairing. This work builds on earlier work of Givental [Giv01]. For simplicity, in this exposition we will ignore issues of completion, which will be addressed in (2.12).

Let $(C, \partial, \delta, \langle -, - \rangle)$ be a complex with circle action and pairing, as in (2.2) and (2.3). Associated with the symplectic vector space $(C^{\text{Tate}}, \langle -, - \rangle_{\text{res}})$ is the Weyl algebra $\mathcal{W}(C^{\text{Tate}})$ defined as

$$\mathcal{W}(C^{\text{Tate}}) = T(C^{\text{Tate}})[[\hbar]] / (\alpha \otimes \beta - (-1)^{|\alpha||\beta|} \beta \otimes \alpha = \hbar \langle \alpha, \beta \rangle_{\text{res}}).$$

Here, again, the formal variable \hbar is of even degree. Note that the relation is homogeneous in $\mathbb{Z}/2\mathbb{Z}$.

The positive subspace C_+ is a subcomplex of C^{Tate} . Hence the left ideal $\mathcal{W}(C^{\text{Tate}}) \cdot C_+$ generated by this subspace is a dg-ideal in $\mathcal{W}(C^{\text{Tate}})$. The quotient

$$\mathcal{F} = \mathcal{W}(C^{\text{Tate}}) / \mathcal{W}(C^{\text{Tate}}) \cdot C_+$$

is known as the Fock space of $\mathcal{W}(C^{\text{Tate}})$; it is a left dg-module of the Weyl algebra.

2.10. The linear subspace C_- of C^{Tate} , on the other hand, is *not* a subcomplex: the differential $\partial + u\delta$ of an element u^0x is by definition equal to $u^0\partial(x)$ when computed in C_- , but it equals $u^0\partial(x) + u^1\delta(x)$ in C^{Tate} .

Nevertheless, disregarding differentials, C_- is still isotropic as a graded vector subspace of the symplectic space $(C^{\text{Tate}}, \langle -, - \rangle_{\text{res}})$. This yields an embedding of algebras (without differentials!)

$$\text{Sym}(C_-)[[\hbar]] \hookrightarrow \mathcal{W}(C^{\text{Tate}}).$$

Post-composing with the canonical projection to the Fock space yields an isomorphism of graded vector spaces

$$\text{Sym}(C_-)[[\hbar]] \xrightarrow{\sim} \mathcal{F}.$$

2.11. Costello [Cos09] observed that under the above isomorphism the differential on the Fock space pulls back to a differential of the form $b + uB + \hbar\Delta$ on the symmetric algebra $\text{Sym } C_-[[\hbar]]$. Here the operator Δ is a BV differential: it is a degree one, square-zero, second order differential operator of the symmetric algebra. As such it is uniquely determined by its action on $\text{Sym}^{\leq 2} C_-$; moreover, it vanishes on $\text{Sym}^{\leq 1} C_-$, and on $\text{Sym}^2 C_-$ it is explicitly given by the formula

$$\Delta(x \cdot y) = \Omega(x, y) = \langle Bx_0, y_0 \rangle,$$

for elements $x, y \in C_-$,

$$\begin{aligned} x &= x_0 + x_{-1}u^{-1} + x_{-2}u^{-2} + \cdots, \\ y &= y_0 + y_{-1}u^{-1} + y_{-2}u^{-2} + \cdots. \end{aligned}$$

It is well-known (see for example [Get94]) that such a BV differential Δ induces a DGLA structure on the shifted Fock space

$$\mathfrak{h} = (\text{Sym } C_-)[1][[\hbar]].$$

The differential is $b + uB + \hbar\Delta$ and the Lie bracket measures the failure of Δ to be a derivation:

$$\{x, y\} = \Delta(x \cdot y) - (\Delta x \cdot y) - (-1)^{|x|}(x \cdot \Delta y).$$

2.12. We want to relate solutions of the Maurer-Cartan equation in \mathfrak{h} with homology classes of the operator $b + uB + \hbar\Delta$. The relationship involves computing the exponential $\exp(\beta/\hbar)$ for a Maurer-Cartan element β . However, this exponential does not make sense until we impose additional finiteness conditions. These are best expressed by introducing a new formal variable λ of even degree.

To this end we extend the definition of the Weyl algebra $\mathcal{W}(C^{\text{Tate}})$ to include this variable:

$$\mathcal{W}(C^{\text{Tate}}) = T(C^{\text{Tate}})[[\hbar, \lambda]] / (\alpha \otimes \beta - (-1)^{|\alpha||\beta|} \beta \otimes \alpha = \hbar \langle \alpha, \beta \rangle_{\text{res}}).$$

The rest of the definitions in the above discussion are unchanged (but the variable λ is now along for the ride).

We localize the Weyl algebra at \hbar and complete in the λ -adic topology to get the localized and completed algebra

$$\widehat{\mathcal{W}}_{\hbar}(C^{\text{Tate}}) = \varprojlim_n \mathcal{W}(C^{\text{Tate}})[[\hbar^{-1}]]/(\lambda^n).$$

Infinite power series of the form

$$\sum_{k \geq 0} \alpha_k \lambda^k \hbar^{-k}$$

exist in $\widehat{\mathcal{W}}_{\hbar}(C^{\text{Tate}})$ but not in $\mathcal{W}(C^{\text{Tate}})$.

By analogy with the construction of $\widehat{\mathcal{W}}(C^{\text{Tate}})$, we denote by $\widehat{\mathcal{F}}_{\hbar}$ and $\widehat{\text{Sym}}_{\hbar}(C_-)[[\hbar, \lambda]]$ the localized and completed versions of the Fock space and the symmetric algebra.

In particular, if H is a graded vector space which carries a symmetric bilinear pairing, endow it with trivial differential and circle action. The above construction defines the vector space

$$\widehat{\text{Sym}}_{\hbar}(H_-)[[\hbar, \lambda]]$$

where the dual total descendant potential and the categorical enumerative potentials will live.

With this preparation we can state the following result, which is well known.

2.13. Lemma. *An element $\beta \in \lambda \cdot \hbar$ of odd degree satisfies the Maurer-Cartan equation if and only if $\exp(\beta/\hbar)$ is $b + uB + \hbar\Delta$ -closed.*

Moreover, two Maurer-Cartan elements β_1 and β_2 are gauge equivalent if and only if $\exp(\beta_1/\hbar)$ and $\exp(\beta_2/\hbar)$ are homologous. All these identities hold in the algebra $\widehat{\text{Sym}}_{\hbar}(C_-)[[\hbar, \lambda]]$.

2.14. We will now take the complex $(C, \partial, \delta, \langle -, - \rangle)$ of the previous discussion to be the shifted Hochschild chain complex

$$(L = C_*(A)[d], b, B, \langle -, - \rangle_{\text{Muk}})$$

of a fixed cyclic A_{∞} algebra A which satisfies condition (\dagger) . As in the introduction d denotes the Calabi-Yau dimension of A .

Kontsevich-Soibelman [KS09] and Costello [Cos07] sketched the construction of the structure of a two-dimensional topological field theory with target the complex (L, b) . In other words they argued that there exists a map from the dg-PROP of normalized singular chains on the moduli spaces of framed curves $M_{g,k,l}^{\text{fr}}$ to the endomorphism dg-PROP of L . More specifically this field theory is given by explicit even degree chain maps

$$\rho_{g,k,l}^A : C_*(M_{g,k,l}^{\text{fr}}) \rightarrow \text{Hom}(L^{\otimes k}, L^{\otimes l})$$

satisfying natural gluing/composition relations. These maps are constructed by first replacing the source chain complex $C_*(M_{g,k,l}^{\text{fr}})$ by a combinatorial version $C_*^{\text{comb}}(M_{g,k,l}^{\text{fr}})$ of it, and then explicitly describing the corresponding maps combinatorially. A complete description of these maps, following the ideas sketched in [KS09], will be given in full detail (including compatible sign conventions) in the upcoming paper [CC20-1].

2.15. In [CCT20] Costello and the authors constructed two morphisms of DGLAs $\mathfrak{g}^+ \rightarrow \widehat{\mathfrak{g}}$, $\mathfrak{h}^+ \rightarrow \widehat{\mathfrak{h}}$. Both are denoted by ι . The former is defined purely combinatorially, and it is a quasi-isomorphism; the latter is associated to a cyclic A_∞ -algebra A , and it is a quasi-isomorphism if we assume condition (\dagger) . We will call $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{h}}$ the Koszul resolutions of \mathfrak{g}^+ and \mathfrak{h}^+ , respectively.

The first pair of DGLAs is constructed using chains on the moduli spaces of curves with framed incoming and outgoing marked points. While the construction of \mathfrak{g}^+ requires the use of geometric singular chains, $\widehat{\mathfrak{g}}$ can be defined using the ribbon graph, combinatorial model of moduli spaces of curves. More precisely, with notations as in [CCT20] we have

$$\mathfrak{g}^+ = \bigoplus_{g,l \geq 1} C_*(M_{g,0,l})_{\text{hS}}[1][[\hbar, \lambda]]$$

and

$$\widehat{\mathfrak{g}} = \bigoplus_{g,k \geq 1, l} C_*^{\text{comb}}(M_{g,k,l}^{\text{fr}})_{\text{hS}}[2-k][[\hbar, \lambda]].$$

The DGLA \mathfrak{g}^+ is a quotient of a larger DGLA \mathfrak{g} which includes the $l = 0$ case in the direct sum.

The second pair of DGLAs is constructed using the shifted Hochschild chain complex L of the A_∞ -algebra A . They are denoted by

$$\mathfrak{h}^+ = \bigoplus_{l \geq 1} \text{Sym}^l(L_-)[1][[\hbar, \lambda]]$$

and

$$\widehat{\mathfrak{h}} = \bigoplus_{k \geq 1, l} \text{Hom}^c(\text{Sym}^k(L_+[1]), \text{Sym}^l(L_-))[[\hbar, \lambda]].$$

The notation Hom^c stands for the space of u -adically continuous \mathbb{K} -linear homomorphisms. As before, \mathfrak{h}^+ is a quotient of a larger DGLA \mathfrak{h} which includes the case $l = 0$. (We emphasize that we are using the conventions from [CCT20, (1.13)] in terms of what $\widehat{\mathfrak{h}}$ means.)

2.16. The two dimensional field theory structure on L gives a morphism of DGLAs

$$\rho^A : \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{h}}.$$

(We include in this map the sign corrections from [CCT20, (1.13)].) We showed in [CCT20, Theorem 3.6] that in the DGLA $\widehat{\mathfrak{g}}$ there exists a special Maurer-Cartan element $\widehat{\mathcal{V}}^{\text{comb}}$,

$$\widehat{\mathcal{V}}^{\text{comb}} = \sum_{g,k \geq 1, l} \widehat{\mathcal{V}}_{g,k,l}^{\text{comb}} \hbar^g \lambda^{2g-2+k+l}$$

which is unique up to gauge equivalence. This element is called the combinatorial string vertex following its original geometric definition of Sen-Zwiebach [SZ94] and Costello [Cos09].

The push-forward of $\widehat{\mathcal{V}}^{\text{comb}}$ under ρ^A yields a Maurer-Cartan element $\widehat{\beta}^A \in \widehat{\mathfrak{h}}$ of the form

$$\widehat{\beta}^A = \rho_*^A(\widehat{\mathcal{V}}^{\text{comb}}) = \sum_{g,k \geq 1, l} \rho^A(\widehat{\mathcal{V}}_{g,k,l}^{\text{comb}}) \hbar^g \lambda^{2g-2+k+l}.$$

The tensors

$$\widehat{\beta}_{g,k,l}^A = \rho^A(\widehat{\mathcal{V}}_{g,k,l}) \in \text{Hom}^c(\text{Sym}^k(L_+[1]), \text{Sym}^l L_-) \llbracket \hbar, \lambda \rrbracket$$

form the starting point of the current paper. They will be used to define and compute the categorical enumerative invariants of the cyclic A_∞ -algebra A and of a splitting s of the Hodge filtration.

3. Definition of the categorical enumerative invariants

In this section we use the Maurer-Cartan element $\widehat{\beta}^A \in \widehat{\mathfrak{h}}$ to define, for a pair A, s consisting of a cyclic A_∞ algebra and a splitting s of its non-commutative Hodge filtration, the categorical enumerative invariant $\mathcal{D}^{A,s}$.

3.1. Sketch of the construction. We begin by sketching the construction of $\mathcal{D}^{A,s}$, ignoring for the sake of clarity two technical points:

- the distinction between \mathfrak{h}^+ and \mathfrak{h} ; and
- the localization and completion aspect of the construction (2.12).

The pre-image $\beta^A \in \mathfrak{h}$ of $\widehat{\beta}^A \in \widehat{\mathfrak{h}}$ under the quasi-isomorphism $\iota : \mathfrak{h} \rightarrow \widehat{\mathfrak{h}}$ is a Maurer-Cartan element in \mathfrak{h} , unique up to gauge. Lemma 2.13 gives a well-defined homology class

$$\mathcal{D}_{\text{abs}}^A = [\exp(\beta)] \in H_*(\mathfrak{h}, b + uB + \hbar\Delta)$$

because the DGLA \mathfrak{h} is a particular case of the construction in (2.11). We call $\mathcal{D}_{\text{abs}}^A$ the abstract categorical enumerative potential $\mathcal{D}_{\text{abs}}^A$ of A ; it only depends on the algebra A and not on the splitting s .

The construction of \mathfrak{h} in (2.11) depends only on the data of $(L, b, B, \langle -, - \rangle_{\text{Muk}})$. Moreover, it is functorial with respect to homotopies (this statement will be made precise in Section 4). A splitting s of the Hodge filtration is a quasi-isomorphism of mixed complexes $(L, b, B) \cong (L, b, 0)$ which respects pairings. In particular it gives a homotopy-trivialization of the operator B . The operator Δ and the bracket $\{ -, - \}$ are defined using B , so they inherit homotopy trivializations. It follows that the choice of s induces an L_∞ quasi-isomorphism of DGLAs

$$(\mathfrak{h}, b + uB + \hbar\Delta, \{ -, - \}) \cong (\mathfrak{h}, b + uB, 0).$$

This quasi-isomorphism will be constructed explicitly in Section 4. Using the splitting s once again, the latter complex is quasi-isomorphic to $(\mathfrak{h}, b, 0)$. The desired categorical enumerative potential

$$\mathcal{D}^{A,s} \in H_*(\mathfrak{h}, b) = \text{Sym}(H_-)(\hbar, \lambda)$$

is defined as the image of the homology class $\mathcal{D}_{\text{abs}}^A$ under the composite quasi-isomorphism

$$(\mathfrak{h}, b + uB + \hbar\Delta, \{ -, - \}) \cong (\mathfrak{h}, b, 0).$$

3.2. Our construction is essentially the same as the original one of Costello [Cos09], differing from it in the following two aspects:

- we use the Koszul resolution $\widehat{\mathfrak{g}}$ instead of \mathfrak{g} , which allows us to compute the combinatorial string vertex explicitly;
- we use explicit formulas in Section 4 to trivialize the DGLA

$$(\mathfrak{h}, b + uB + \hbar\Delta, \{ -, - \});$$

the original construction used a non-explicit argument relying on deformation theory (rigidity of Fock modules).

3.3. The abstract total descendent potential. We will now carry out the above construction in detail. Condition (\dagger) is assumed to hold and therefore the map

$$\iota : \mathfrak{h}^+ \rightarrow \widehat{\mathfrak{h}}$$

is a quasi-isomorphism of DGLAs ([CCT20, Lemma 4.4]). The Maurer-Cartan moduli space is invariant under such quasi-isomorphisms. Thus there exists a Maurer-Cartan element $\beta^A \in \mathfrak{h}^+$,

$$\beta^A = \sum_{g,n \geq 1} \beta_{g,n}^A \hbar^g \lambda^{2g-2+n},$$

such that $\iota(\beta^A)$ is gauge equivalent to $\widehat{\beta}^A$. It is unique up to homotopy.

Recall that \mathfrak{h} decomposes as a vector space (but not as a DGLA!) as

$$\mathfrak{h} = \mathbb{K}[1][[\hbar, \lambda]] \oplus \mathfrak{h}^+.$$

More precisely, the space of shifted scalars $\mathbb{K}[1][[\hbar, \lambda]] \subset \mathfrak{h}$ forms a central subalgebra in \mathfrak{h} , and \mathfrak{h}^+ is the DGLA quotient of \mathfrak{h} by it, i.e., we have a short exact sequence of DGLAs

$$0 \rightarrow \mathbb{K}[1][[\hbar, \lambda]] \rightarrow \mathfrak{h} \rightarrow \mathfrak{h}^+ \rightarrow 0.$$

The vector space direct sum decomposition

$$\mathfrak{h} = \mathbb{K}[1][[\hbar, \lambda]] \oplus \mathfrak{h}^+$$

allows us to regard the Maurer-Cartan element $\beta^A \in \mathfrak{h}^+$ as an element in \mathfrak{h} , by taking its $\mathbb{K}[1][[\hbar, \lambda]]$ -component to be zero¹. Even though \mathfrak{h}^+ is not a subalgebra of \mathfrak{h} , we will prove in Lemma 4.14 that $\beta^A \in \mathfrak{h}$ still satisfies the Maurer-Cartan equation. Lemma 2.13 then yields a $(b + uB + \hbar\Delta)$ -closed element

$$\exp(\beta^A/\hbar) \in \widehat{\text{Sym}}_{\hbar} L_-[[\hbar, \lambda]]$$

in the localized and completed symmetric algebra. The fact that the string vertex is unique up to homotopy [CCT20, Theorem 3.6] shows that the cohomology class of this element depends only on the cyclic A_∞ -algebra A . We denote it by

$$\mathcal{D}_{\text{abs}}^A = [\exp(\beta^A/\hbar)] \in H_*(\widehat{\text{Sym}}_{\hbar} L_-[[\hbar, \lambda]], b + uB + \hbar\Delta) \cong H_*(\widehat{\mathcal{F}}_{\hbar})$$

and call it the *abstract total descendent potential* of A . It has shifted degree zero.

3.4. Splittings of the non-commutative Hodge filtration. To obtain invariants of A that are similar to those from Gromov-Witten theory we need a further ingredient: a choice of splitting s of the Hodge filtration.

We define a splitting of the (non-commutative) Hodge filtration of A to be a graded vector space map

$$s : HH_*(A) \rightarrow HC_*^-(A)$$

satisfying the following two conditions:

- S1. (*Splitting condition.*) s splits the canonical projection $HC_*^-(A) \rightarrow HH_*(A)$.
- S2. (*Lagrangian condition.*) $\langle s(x), s(y) \rangle_{\text{hres}} = \langle x, y \rangle_{\text{Muk}}$ for any $x, y \in HH_*(A)$.

¹This choice is only made for the sake of proving Proposition 3.11. A more reasonable choice would be to force the dilaton equation. But this would present additional difficulties in Proposition 3.11.

Remark. In many circumstances it is useful to put further restrictions on the allowed splittings. For example one may impose certain homogeneity conditions, or ask for the splitting to be compatible with the cyclic structure of the A_∞ -algebra. In this paper we only need conditions S1 and S2, but we refer the reader to [AT19, Definition 3.7] for more information on these possible restrictions.

3.5. By the Hodge-de Rham degeneration property of A the homology $H = H_*(L)$ is endowed with the trivial circle action. According to our conventions on circle actions we have the following graded vector spaces with trivial differentials:

$$H^{\text{Tate}} = H((u)), \quad H_+ = uH[[u]], \quad H_- = H[u^{-1}].$$

Shifting a splitting s by d allows us to view it as a map

$$s : H \rightarrow u^{-1}H_*(L_+).$$

This map can then be repackaged by extending it u -linearly to an isomorphism of symplectic vector spaces

$$s : (H^{\text{Tate}}, \langle -, - \rangle_{\text{res}}) \rightarrow (H_*(L^{\text{Tate}}), \langle -, - \rangle_{\text{res}})$$

which maps the Lagrangian subspaces H_+ , $H_*(L_+)$ to each other.

3.6. We will need later a chain level analogue of the homology splitting above. Given a splitting s , a chain level lift of s is a chain map

$$R : (L, b) \rightarrow (u^{-1}L_+, b + uB)$$

which associates to $x_0 \in L$ a formal series

$$R(x_0) = x_0 + x_1u + x_2u^2 + \cdots$$

such that its induced map in homology is s (after shifting degrees by d). We will write such a splitting R as

$$R = \text{id} + R_1u + R_2u^2 + \cdots$$

where $R_i : L \rightarrow L$ is an even map.

3.7. Lemma. *Any splitting on homology (3.4) can be lifted to a splitting at chain level (3.6). All liftings of a fixed homology splitting are homotopy equivalent to one another.*

Proof. This is the classical fact that for complexes of vector spaces, the homology of the Hom complex

$$\text{Hom}^*((L, b), (L[[u]], b + uB))$$

computes homomorphisms between the homology groups $\text{Hom}(H, u^{-1}H_+)$. The second statement follows from the fact that the difference $R - R'$ between two chain maps R and R' is exact in the complex $\text{Hom}^*((L, b), (L[[u]], b + uB))$ if and only if R and R' are homotopic. \square

3.8. The enumerative invariants of a pair (A, s) . The symplectomorphism $s : H^{\text{Tate}} \rightarrow H_*(L^{\text{Tate}})$ induces an isomorphism of Weyl algebras

$$\Phi^s : \widehat{\mathcal{W}}_{\hbar}(H^{\text{Tate}}) \rightarrow \widehat{\mathcal{W}}_{\hbar}(H_*(L^{\text{Tate}})).$$

Define Ψ^s to be the composition of the other three maps in the diagram below

$$\begin{array}{ccc} \widehat{\mathcal{W}}_{\hbar}(H^{\text{Tate}}) & \xrightarrow{\Phi^s} & \widehat{\mathcal{W}}_{\hbar}(H_*(L^{\text{Tate}})) = H_*(\widehat{\mathcal{W}}_{\hbar}(L^{\text{Tate}})) \\ \uparrow i & & \downarrow p \\ \widehat{\text{Sym}}_{\hbar} H_-[[\hbar, \lambda]] & \xrightarrow{\Psi^s} & H_*(\widehat{\mathcal{F}}_{\hbar}). \end{array}$$

It is a graded vector space isomorphism. Here the left vertical map is the inclusion of the symmetric algebra generated by the Lagrangian subspace H_- into the Weyl algebra, while the right vertical map is the canonical quotient map from the Weyl algebra to the Fock space. The abstract total descendant potential $\mathcal{D}_{\text{abs}}^A$ lives in the lower right corner.

3.9. Definition. The total descendant potential $\mathcal{D}^{A,s} \in \widehat{\text{Sym}}_{\hbar} H_-[[\hbar, \lambda]]$ of a pair (A, s) is the pre-image of the abstract total descendant potential $\mathcal{D}_{\text{abs}}^A$ under the map Ψ^s ,

$$\mathcal{D}^{A,s} = (\Psi^s)^{-1}(\mathcal{D}_{\text{abs}}^A).$$

The n -point function of genus g , $F_{g,n}^{A,s} \in \text{Sym}^n H_-$, is defined by the identity

$$\sum_{g,n} F_{g,n}^{A,s} \cdot \hbar^g \lambda^{2g-2+n} = \hbar \cdot \ln \mathcal{D}^{A,s}.$$

The right hand term $\hbar \cdot \ln \mathcal{D}^{A,s}$ will be denoted by $F^{A,s}$.

3.10. Compatibility with Givental group action. Denote by \mathfrak{G}_A the Givental group of the pair $(H_+, \langle -, - \rangle_{\text{Muk}})$. Abstractly, it is the subgroup of automorphisms of the symplectic vector space $(H^{\text{Tate}}, \langle -, - \rangle_{\text{res}})$ preserving the Lagrangian subspace H_+ and acting as the identity on H . Explicitly it consists of elements g of the form

$$g = \text{id} + g_1 \cdot u + g_2 \cdot u^2 + \dots$$

with each $g_j \in \text{End}(H)$ required to satisfy

$$\langle g \cdot x, g \cdot y \rangle_{\text{res}} = \langle x, y \rangle_{\text{res}} \quad \text{for any } x, y \in H^{\text{Tate}}.$$

If the set of splittings of the non-commutative Hodge filtration is nonempty, then it is a left torsor over the Givental group, by letting an element $g \in \mathfrak{G}_A$ act on a splitting $s : H_+ \rightarrow u^{-1}H_*(L_+)$ by pre-composing with g^{-1} :

$$g \cdot s : H_+ \xrightarrow{g^{-1}} H_+ \xrightarrow{s} u^{-1}H_*(L_+).$$

The Givental action on the Fock space $\widehat{\text{Sym}}_{\hbar}H_-[[\hbar, \lambda]]$ is by definition the automorphism \widehat{g} of the Fock space induced from the symplectic transformation g :

$$\begin{array}{ccc} \widehat{\mathcal{W}}_{\hbar}(H^{\text{Tate}}) & \xrightarrow{\Phi^s} & \widehat{\mathcal{W}}_{\hbar}(H^{\text{Tate}}) \\ \uparrow i & & \downarrow \pi \\ \widehat{\text{Sym}}_{\hbar}H_-[[\hbar, \lambda]] & \xrightarrow{\widehat{g}} & \widehat{\text{Sym}}_{\hbar}H_-[[\hbar, \lambda]] \end{array}$$

3.11. Proposition. *The construction of the total descendant potential is compatible with the action of the Givental group \mathfrak{G}_A . Explicitly, for a splitting s and an element $g \in \mathfrak{G}_A$ we have*

$$\mathcal{D}^{A,g \cdot s} = \widehat{g}(\mathcal{D}^{A,s}).$$

Proof. Consider the following diagram:

$$\begin{array}{ccccc} \widehat{\mathcal{W}}_{\hbar}(H^{\text{Tate}}) & \xrightarrow{\Phi^{g^{-1}}} & \widehat{\mathcal{W}}_{\hbar}(H^{\text{Tate}}) & \xrightarrow{\Phi^s} & H_*(\widehat{\mathcal{W}}_{\hbar}(L^{\text{Tate}})) \\ \uparrow i & & \downarrow \pi \uparrow i & & \downarrow p \\ \widehat{\text{Sym}}_{\hbar}H_-[[\hbar, \lambda]] & \xrightarrow{\widehat{g^{-1}}} & \widehat{\text{Sym}}_{\hbar}H_-[[\hbar, \lambda]] & \xrightarrow{\Psi^s} & H_*(\widehat{\mathcal{F}}_{\hbar}). \end{array}$$

We claim that

$$p\Phi^{g \cdot s}i = p\Phi^s i \pi \Phi^{g^{-1}}i.$$

The left hand side equals $p\Phi^s \Phi^{g^{-1}}i$. The composition $i\pi$ in the middle is not the identity, but rather the projection onto the negative subspace (image of i). Its kernel is by definition the left ideal generated by the positive subspace H_+ . Thus, we have $i\pi = \text{id} \pmod{H_+}$.

Now observe that Φ^s preserves the ideal H_+ , just because by definition a splitting s only contains non-negative powers in the variable u . Followed by the projection map p gives zero, that is,

$$p\Phi^s i \pi(H_+) = 0.$$

We conclude that $p\Phi^s = p\Phi^s i \pi$, and hence

$$p\Phi^{g \cdot s}i = p\Phi^s \Phi^{g^{-1}}i = p\Phi^s i \pi \Phi^{g^{-1}}i.$$

This implies that

$$\begin{aligned} \mathcal{D}^{A,g,s} &= (p\Phi^{g,s}i)^{-1}(\mathcal{D}_{\text{abs}}^A) \\ &= (\pi\Phi^{g^{-1}}i)^{-1}(p\Phi^s i)^{-1}(\mathcal{D}_{\text{abs}}^A) \\ &= (\pi\Phi^{g^{-1}}i)^{-1}(\mathcal{D}^{A,s}) = \widehat{g}(\mathcal{D}^{A,s}) \end{aligned}$$

□

4. Trivializing the DGLAs associated to a cyclic algebra

One of main difficulties in understanding the categorical invariants $F_{g,n}^{A,s}$ in Definition 3.9 is caused by the fact that the differential $b + uB + \hbar\Delta$ on the Fock space is not homogeneous in the symmetric power degree in $\text{Sym } H_-$. Indeed, $b + uB$ preserves this degree (it acts on each element of a symmetric product individually), while Δ reduces it by 2. In this section we shall explicitly trivialize the operator Δ using an L_∞ quasi-isomorphism. Since the Lie bracket is essentially defined using Δ we shall in fact simultaneously trivialize both the operator Δ and the Lie bracket.

4.1. We need some notation to state this result more precisely. Recall from [CCT20] that there are three DGLA's \mathfrak{h} , \mathfrak{h}^+ and $\widehat{\mathfrak{h}}$ associated to the cyclic A_∞ -algebra A , together with a quasi-isomorphism $\iota : \mathfrak{h}^+ \rightarrow \widehat{\mathfrak{h}}$.

We introduce partial trivializations of \mathfrak{h} , \mathfrak{h}^+ and $\widehat{\mathfrak{h}}$, denoted by $\mathfrak{h}^{\text{triv}}$, $\mathfrak{h}^{\text{triv},+}$ and $\widehat{\mathfrak{h}}^{\text{triv}}$, respectively. Their underlying vector spaces are the same as those of the original Lie algebras, but their differentials are given by $b + uB$ for the first two and $b + uB + \iota$ for the third one (instead of $b + uB + \hbar\Delta$ for the first two and $b + uB + \hbar\Delta + \iota$ for the third one). Moreover, they are all endowed with zero bracket.

Let A be a cyclic A_∞ -algebra satisfying condition (\dagger) and let R be a chain level splitting of its non-commutative Hodge filtration. This data will be fixed for the rest of this section.

4.2. Theorem.

- (A) *There exists an L_∞ quasi-isomorphism $\mathcal{K} : \mathfrak{h} \rightarrow \mathfrak{h}^{\text{triv}}$, depending on R and constructed explicitly in the proof below. The map \mathcal{K} restricts to an L_∞ quasi-isomorphism of the positive parts of both sides. We shall denote this restriction by \mathcal{K} as well.*
- (B) *There exists an L_∞ quasi-isomorphism $\widehat{\mathcal{K}} : \widehat{\mathfrak{h}} \rightarrow \widehat{\mathfrak{h}}^{\text{triv}}$, also depending on R .*

(C) These quasi-isomorphism are compatible with the Koszul resolution maps ι , in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{h}^+ & \xrightarrow{\iota} & \widehat{\mathfrak{h}} \\ \mathcal{K} \downarrow & & \downarrow \widehat{\mathcal{K}} \\ \mathfrak{h}^{\text{triv},+} & \xrightarrow{\iota} & \widehat{\mathfrak{h}}^{\text{triv}}. \end{array}$$

4.3. A first order trivialization. We begin with part (A) of the above theorem. Recall (2.11) that the restriction of the operator $\Delta : \mathfrak{h} \rightarrow \mathfrak{h}$ to $\text{Sym}^2 L_-$ is given by

$$\Omega(x, y) = \langle Bx_0, y_0 \rangle_{\text{Muk}}$$

where

$$x = x_0 + x_{-1}u^{-1} + \cdots \quad \text{and} \quad y = y_0 + y_{-1}u^{-1} + \cdots$$

are elements of L_- . In order to trivialize the operator Δ we will first construct a homotopy operator for the chain map $\Omega : \text{Sym}^2 L_- \rightarrow \mathbb{K}$ using the chain-level splitting R of the Hodge filtration.

Extend R to a u -linear isomorphism of chain complexes

$$R : (L[[u]], b) \rightarrow (L[[u]], b + uB)$$

which we still denote by R . The inverse operator of R is another operator of the form

$$T = \text{id} + T_1u + T_2u^2 + \cdots$$

By definition we have the following identity

$$\sum_{i+j=k} T_i R_j = \begin{cases} \text{id} & \text{if } k = 0 \\ 0 & \text{if } k \geq 1. \end{cases}$$

Solving the above recursively yields formulas for the T_j 's in terms of the R_i 's. For example we have

$$T_1 = -R_1, \quad T_2 = -R_2 + R_1 R_1, \quad \text{etc.}$$

Since R and T are chain maps, we have

$$[b, R_n] = -BR_{n-1}, \quad [b, T_n] = T_{n-1}B \quad \text{for all } n \geq 1.$$

4.4. We now define an even linear map

$$H : L_- \otimes L_- \rightarrow \mathbb{K}.$$

It is the sum, over all $i \geq 0, j \geq 0$, of maps

$$H_{i,j} : u^{-i}L \otimes u^{-j}L \rightarrow \mathbb{K}$$

defined by

$$H_{i,j}(u^{-i}x, u^{-j}y) = \langle (-1)^j \sum_{l=0}^j R_l T_{i+j+1-l} x, y \rangle.$$

4.5. Proposition. For any two elements $\alpha, \beta \in L_-$ we have

$$H((b + uB)\alpha, \beta) + (-1)^{|\alpha|}H(\alpha, (b + uB)\beta) = -\Omega(\alpha, \beta).$$

We write this identity as

$$[b + uB, H] = \Omega.$$

In other words the operator H is a bounding homotopy of Ω .

Proof. We begin with the first case, when $i = j = 0$. Then

$$H_{0,0}(u^0x, u^0y) = \langle T_1x, y \rangle.$$

We need to verify that

$$H_{0,0}(bx, y) + (-1)^{|x|}H_{0,0}(x, by) = -\langle Bx, y \rangle.$$

This is a straight-forward computation:

$$\begin{aligned} H_{0,0}(bx, y) + (-1)^{|x|}H_{0,0}(x, by) &= \\ &= \langle T_1bx, y \rangle + (-1)^{|x|}\langle T_1x, by \rangle \\ &= \langle T_1bx, y \rangle - \langle bT_1x, y \rangle \\ &= -\langle [b, T_1]x, y \rangle \\ &= -\langle Bx, y \rangle. \end{aligned}$$

In the general case we would like to prove that

$$\begin{aligned} H_{i,j}(u^{-i}bx, u^{-j}y) + (-1)^{|x|}H_{i,j}(u^{-i}x, u^{-j}by) + \\ + H_{i-1,j}(u^{-i+1}Bx, u^{-j}y) + (-1)^{|x|}H_{i,j-1}(u^{-i}x, u^{-j+1}By) = 0. \end{aligned}$$

Again, this is a straightforward computation using the commutator relations of the R 's and the T 's. We have

$$\begin{aligned} H_{i,j}(u^{-i}bx, u^{-j}y) &= \langle (-1)^j \sum_{l=0}^j R_l T_{i+j+1-l} bx, y \rangle \\ &= \langle (-1)^j \sum_{l=0}^j R_l b T_{i+j+1-l} x, y \rangle - \langle (-1)^j \sum_{l=0}^j R_l T_{i+j-l} Bx, y \rangle. \end{aligned}$$

We also have

$$\begin{aligned} (-1)^{|x|}H_{i,j}(u^{-i}x, u^{-j}by) &= -\langle (-1)^j b \sum_{l=0}^j R_l T_{i+j+1-l} x, y \rangle \\ &= -\langle (-1)^j \sum_{l=0}^j R_l b T_{i+j+1-l} x, y \rangle + \langle \sum_{l=0}^j (-1)^j B R_{l-1} T_{i+j+1-l} x, y \rangle. \end{aligned}$$

Adding together these two equations yields

$$\begin{aligned}
 H_{i,j}(u^{-i}bx, u^{-j}y) + (-1)^{|x|}H_{i,j}(u^{-i}x, u^{-j}by) &= \\
 &= -\langle (-1)^j \sum_{l=0}^j R_l T_{i+j-l} Bx, y \rangle + \langle (-1)^j \sum_{l=0}^l BR_{l-1} T_{i+j+1-l} x, y \rangle \\
 &= -H_{i-1,j}(u^{-i+1}Bx, u^{-j}y) - (-1)^{|x|}H_{i,j-1}(u^{-i}x, u^{-j+1}By).
 \end{aligned}$$

This proves the proposition. \square

4.6. The homotopy operator H induces a first order trivialization of Δ . Indeed, since B is self-adjoint with respect to the Mukai pairing, we can symmetrize the homotopy operator H to obtain a homotopy operator $H^{\text{sym}} : \text{Sym}^2 L_- \rightarrow \mathbb{K}$,

$$H^{\text{sym}}(xy) = \frac{1}{2} \left(H(x, y) + (-1)^{|x||y|} H(y, x) \right).$$

The degrees here are the degrees in the complex L_- .

The operator H^{sym} bounds $\Delta : \text{Sym}^2 L_- \rightarrow \mathbb{K}$. Extending it as a second order differential operator to the full symmetric algebra $\text{Sym} L_-$ yields an operator Δ^H . To see that $[b + uB, \Delta^H] = \Delta$ we note that both sides are second order differential operators. Thus it suffices to prove that they are equal on $\text{Sym}^{\leq 2} L_-$, in which case the statement follows from Proposition 4.5.

4.7. Recollections on graphs. The construction of the morphisms \mathcal{H} and $\widehat{\mathcal{H}}$ involves summing over certain types of graphs. We need to introduce some terminology about these graphs. We refer to [GK98, Section 2] for details.

A labeled graph shall mean a graph G endowed with a genus labeling $g : V_G \rightarrow \mathbb{Z}_{\geq 0}$ on its set of vertices V_G . The genus of a labeled graph is defined to be

$$g(G) = \sum_{v \in V_G} g(v) + \dim H^1(G).$$

We will use the following notations for a labeled graph G :

- V_G denotes the set of vertices of G ;
- L_G denotes the set of leaves of G ;
- the valency of a vertex $v \in V_G$ is denoted by $n(v)$;
- the graph G is called stable if $2g(v) - 2 + n(v) > 0$ holds at every vertex $v \in V_G$.

- if G has m vertices, a marking of G is a bijection

$$f : \{1, \dots, m\} \rightarrow V_G.$$

An isomorphism between two marked and labeled graphs is an isomorphism of the underlying labeled graphs that also preserves the markings.

We will denote various classes of graphs as follows:

- $\Gamma(g, n)$ denotes the set of isomorphism classes of graphs of genus g and with n leaves;
- using double brackets as in $\Gamma((g, n))$ requires further that the graphs be stable;
- a subscript m as in $\Gamma(g, n)_m$ or $\Gamma((g, n))_m$ indicates that the graphs in discussion have m vertices;
- adding a tilde as in $\widetilde{\Gamma}(g, n)_m$ or $\widetilde{\Gamma}((g, n))_m$ signifies that we are looking at *marked* graphs and isomorphisms.

4.8. The construction of the L_∞ -morphism \mathcal{H} . We will now construct the L_∞ -morphism $\mathcal{H} : \mathfrak{h} \rightarrow \mathfrak{h}^{\text{triv}}$ claimed in part (A) of Theorem 4.2 by “exponentiating” the first order trivialization H^{sym} .

The morphism \mathcal{H} will be a collection of even linear maps

$$\mathcal{H}_m : \text{Sym}^m(\mathfrak{h}[1]) \rightarrow \mathfrak{h}^{\text{triv}}[1]$$

defined for each $m \geq 1$. Each of these maps will be defined as a Feynman-type sum over all the graphs in $\widetilde{\Gamma}(g, n)_m$ for arbitrary g, n . (Recall that m denotes the number of vertices in a given graph, so graphs with m vertices contribute to the m -th map in the L_∞ -morphism \mathcal{H} .)

Since $\mathfrak{h} = \text{Sym}(L_-)[1][[\hbar, \lambda]]$ already includes a shift by one, the parity of the elements in $\mathfrak{h}[1]$ is the same as in $\text{Sym} L_-$. Hence in order to define \mathcal{H}_m we will associate to a marked graph (G, f) with m vertices a \mathbb{K} -linear map

$$\mathcal{H}_{G,f} : (\text{Sym} L_-[[\hbar]])^{\otimes m} \rightarrow \text{Sym} L_-[[\hbar]].$$

The map \mathcal{H}_m will be the λ -linear extension of the map

$$\mathcal{H}_m = \sum_{g,n} \sum_{(G,f) \in \widetilde{\Gamma}(g,n)_m} \frac{1}{|\text{Aut}(G,f)|} \cdot \mathcal{H}_{G,f}.$$

4.9. Let $(G, f) \in \widetilde{\Gamma}(g, n)_m$ be a marked graph. We will denote the i -th marked vertex of G by $v_i = f(i)$, its genus marking by $g_i = g(v_i)$, and its valency by $n_i = n(v_i)$. For elements $\gamma_1, \dots, \gamma_m \in \text{Sym } L_-$ the expression

$$\mathcal{K}_{G,f}(\gamma_1 \cdot \hbar^{g_1}, \dots, \gamma_m \cdot \hbar^{g_m})$$

will be non-zero only if $t_i = g_i$ and $\gamma_i \in \text{Sym}^{n_i} L_-$ for all i . If this is the case, the result will be an element in $\text{Sym}^n(L_-) \cdot \hbar^g$.

For an element $\gamma = x_1 x_2 \cdots x_i \in \text{Sym}^i L_-$ we let $\tilde{\gamma} \in (L_-)^{\otimes i}$ be its de-symmetrization,

$$\tilde{\gamma} = \sum_{\sigma \in \Sigma_i} \epsilon \cdot x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)} \in L_-^{\otimes i}.$$

Here ϵ is the Koszul sign for permuting the elements x_1, \dots, x_i (with respect to the degree in L_-).

4.10. For elements $\gamma_1 \in (L_-)^{\otimes n_1}, \dots, \gamma_m \in (L_-)^{\otimes n_m}$ we define

$$\mathcal{K}_{G,f}(\gamma_1 \cdot \hbar^{g_1}, \dots, \gamma_m \cdot \hbar^{g_m}) = \gamma \cdot \hbar^g$$

where γ is computed by the following Feynman-type procedure:

- (1) Decorate the half-edges adjacent to each vertex v_i by $\tilde{\gamma}_i$. Tensoring together the results over all the vertices we obtain a tensor of the form

$$\bigotimes_{i=1}^m \tilde{\gamma}_i \in L_-^{\otimes (\sum n_i)}.$$

The order in which these elements are tensored is the one given by the marking.

- (2) For each internal edge e of G contract the corresponding components of the above tensor using the even symmetric bilinear form

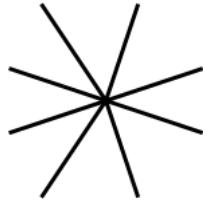
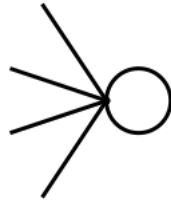
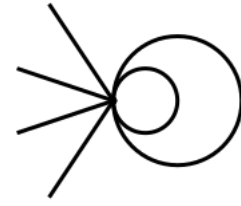
$$H^{\text{sym}} : L_-^{\otimes 2} \rightarrow \mathbb{K}.$$

When applying the contraction we always permute the tensors to bring the two terms corresponding to the two half-edges to the front, and then apply the contraction map. The ordering of the set E_G does not matter since the operator H^{sym} is even; also the ordering of the two half-edges of each edge does not matter since the operator H^{sym} is (graded) symmetric.

- (3) Read off the remaining tensor components (corresponding to the leaves of the graph) in any order, and regard the result as the element γ in $\text{Sym}^n L_-$ via the canonical projection map $L_-^{\otimes n} \rightarrow \text{Sym}^n L_-$.

With these assignments one can check that the maps \mathcal{K}_m are even.

4.11. Example: the map \mathcal{K}_1 . The following figures illustrate a few graphs that contribute to the map \mathcal{K}_1 . Their common feature is that they have only one vertex; in this case the marking f is unique.

The star graph in $\Gamma(g, n)$ A one-loop graph in $\Gamma(g, n)$ A two-loop graph in $\Gamma(g, n)$

The contribution of the first type of graphs (the star graphs) is the identity map

$$\mathfrak{h}[1] = \text{Sym } L_- \llbracket \hbar, \lambda \rrbracket \rightarrow \mathfrak{h}^{\text{triv}} = \text{Sym } L_- \llbracket \hbar, \lambda \rrbracket.$$

To see this, note that de-symmetrization followed by projection produces a factor of $n!$, which is canceled by the size of the automorphism group of the star graph. We also note that it is necessary to use the set $\Gamma(g, n)$ of all labeled graphs, as opposed to using just stable graphs. For instance, the $(g, n) = (0, 0)$ star graph (just a vertex with a genus label of 0) contributes to the identity map $\text{Sym}^0 L_- = \mathbb{K} \rightarrow \text{Sym}^0 L_- = \mathbb{K}$.

In a similar way one checks that the middle type of graph acts precisely by the operator $\hbar\Delta^H$, while the third type of graph contributes $\frac{1}{2}(\hbar\Delta^H)^2$. Here the extra factor $1/2$ comes from the automorphism that exchanges the two loops. In general we have

$$\mathcal{K}_1 = \sum_{l \geq 0} \frac{1}{l!} (\hbar\Delta^H)^l = e^{\hbar\Delta^H}.$$

And indeed, the first L_∞ -morphism identity is $(b + uB)\mathcal{K}_1 = \mathcal{K}_1(b + uB + \hbar\Delta)$, which can be verified directly using the above formula.

4.12. Proof of part (A) of Theorem 4.2. First we note that the maps \mathcal{K}_m are symmetric in their inputs. Indeed, if we permute two inputs γ_i with γ_j and switch the corresponding entries in the marking of the graph, it results in a Koszul sign change in the formation of the tensor $\bigotimes_k \tilde{\gamma}_k$, precisely showing that \mathcal{K}_m is graded symmetric. Furthermore, the contraction maps associated with the edges are all even maps, so \mathcal{K}_m is also even. Therefore the maps $\{\mathcal{K}_m\}_{m \geq 1}$ form a pre- L_∞ homomorphism.

In order to show that the collection of maps $\{\mathcal{K}_m\}_{m \geq 1}$ form an L_∞ -morphism $\mathfrak{h} \rightarrow \mathfrak{h}^{\text{triv}}$

we need to check that for every $m \geq 1$ we have

$$\begin{aligned} [b + uB, \mathcal{K}_m](\gamma_1 \cdots \gamma_m) &= \hbar \sum_i (-1)^{\epsilon_i} \mathcal{K}_m(\gamma_1 \cdots \Delta \gamma_i \cdots \gamma_m) \\ &\quad + \sum_{i < j} (-1)^{\epsilon_{ij}} \mathcal{K}_{m-1}(\{\gamma_i, \gamma_j\} \gamma_1 \cdots \widehat{\gamma}_i \cdots \widehat{\gamma}_j \cdots \gamma_m), \end{aligned}$$

where the signs are Koszul signs

$$\begin{aligned} \epsilon_i &= |\gamma_1| + \cdots + |\gamma_{i-1}|, \\ \epsilon_{ij} &= |\gamma_i|(|\gamma_1| + \cdots + |\gamma_{i-1}|) + |\gamma_j|(|\gamma_1| + \cdots + |\gamma_{i-1}| + |\gamma_{i+1}| + \cdots + |\gamma_{j-1}|). \end{aligned}$$

(Recall that we use shifted sign conventions, and the shifted degrees in our algebras are the ordinary degrees in $\text{Sym } L_-$.)

For each marked graph (G, f) consider the commutator $[b + uB, \mathcal{K}_{G,f}]$. Using the identity

$$[b + uB, H^{\text{sym}}] = \Omega$$

from Proposition 4.5 it is easy to check that

$$[b + uB, \mathcal{K}_{G,f}] = \sum_e \mathcal{K}_{G,f}^e$$

where the sum is over the internal edges e of G , and the operator $\mathcal{K}_{G,f}^e$ is defined in the same way as $\mathcal{K}_{G,f}$, but the contraction corresponding to e uses the operator Ω instead of H^{sym} .

We will see below that when e is a loop we will obtain the first type of term on the right hand side of the L_∞ identities above, while when e is a non-loop we will obtain the second type of term. In the remainder of the proof we will explain this in more detail, primarily in order to match up the coefficients given by graph automorphisms.

First we deal with the loop case. Let e denote a loop at a vertex v_i , and write $G \setminus e$ for the result of deleting e from G . Assume that in a computation of $\mathcal{K}_{G,f}$ we need to insert at v_i a symmetric tensor $\gamma_i = x_1 \cdots x_{N+2} \in \text{Sym}^{N+2} L_-$. (Here $N+2$ is the valency n_i of vertex i .) The total number of terms in $\widetilde{\gamma}_i$ is $(N+2)!$, given by all permutations of the tensors in γ_i .

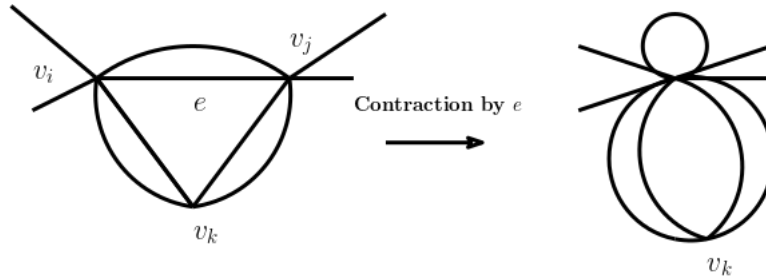
Now consider the result of inserting $\Delta(\gamma_i)$ at v_i in $G \setminus e$. The expression $\Delta(\gamma_i)$ has $\binom{N+2}{2}$ terms, corresponding to the number of choices of picking the two x_j 's to apply Δ to. Thus the total number of terms in $\widetilde{\Delta \gamma}_i \in \text{Sym}^N L_-$ (to be inserted at v_i) is

$$\binom{N+2}{2} \cdot N! = \frac{1}{2}(N+2)!.$$

The extra factor of $1/2$ between the two computations is precisely accounted for by the ratio $|\text{Aut}(G, f)|/|\text{Aut}(G \setminus e, f)| = 2$ between the two automorphism groups, since the loop e has an extra non-trivial automorphism that switches its two ends.

Next we deal with the non-loop case. The combinatorics in this case are more involved. Let e be a non-loop edge in a marked graph (G, f) , connecting vertices v_i and v_j . The result of contracting e will be denoted by $(G/e, f')$, where the marking f' is obtained from the original marking of f by marking the new vertex (the result of collapsing v_i and v_j) to be vertex “1”, and relabeling the remaining vertices to have the original order, skipping v_i and v_j .

We illustrate such a contraction in the following figure.



Since an automorphism of (G, f) must preserve the marking of the vertices, the cardinality of $\text{Aut}(G, f)$ is given by

$$|\text{Aut}(G, f)| = \prod_v l_v! \cdot 2^{c(v)} \cdot c(v)! \prod_{v \neq w} n(v, w)!,$$

where l_v is the number of leaves at v , $c(v)$ is the number of loops at v , and $n(v, w)$ is the number of edges between two different vertices v and w . From this formula we will deduce the matching of coefficients between the expressions arising in $\mathcal{K}_{G, f}$ and $\mathcal{K}_{G/e, f'}$.

We will compare the combinatorial coefficients when there is only one extra vertex v_k , as illustrated in the picture, leaving the general case to the reader. Let us consider decorations $\gamma_i \in \text{Sym}^{N_i+1} L_-$ and $\gamma_j \in \text{Sym}^{N_j+1}$ of the vertices v_i and v_j . Let us also fix a marked graph (G', f') with $m - 1$ internal vertices. Its contribution to the right hand side of the L_∞ -morphism identities is given by

$$I = \frac{1}{|\text{Aut}(G', f')|} \mathcal{K}_{(G', f')}(\{\gamma_i, \gamma_j\} \gamma_k).$$

It suffices to match the above term with

$$II = \sum_{(G, f), e} \frac{1}{|\text{Aut}(G, f)|} \mathcal{K}_{(G, f)}^e(\gamma_i \cdot \gamma_j \cdot \gamma_k)$$

where the sum is over labeled and marked graphs (G, f) such that $(G/e, f) = (G', f')$ with e an edge between v_i and v_j .

For the graph (G, f) let us write c_i, l_i for the number of loops and leaves at v_i , and similarly for v_j . We also write n_{ij}, n_{jk}, n_{ik} for the number of edges between two vertices as indicated by the indices. Using the above formula of the automorphism group, we deduce that the ratio

$$\frac{\text{Aut}(G', f')}{\text{Aut}(G, f)} = \frac{2^{n_{ij}-1}}{n_{ij}} \binom{n_{ik} + n_{jk}}{n_{ik}} \cdot \binom{l_i + l_j}{l_i} \cdot \binom{c_i + c_j + n_{ij} - 1}{c_i, c_j, n_{ij} - 1}.$$

Using this to compute II we obtain

$$II = \frac{1}{\text{Aut}(G', f')} \sum_{(G, f), e} \frac{2^{n_{ij}-1}}{n_{ij}} \binom{n_{ik} + n_{jk}}{n_{ik}} \cdot \binom{l_i + l_j}{l_i} \cdot \binom{c_i + c_j + n_{ij} - 1}{c_i, c_j, n_{ij} - 1} \cdot \mathcal{H}_{(G, f)}^e(\gamma_i \cdot \gamma_j \cdot \gamma_k)$$

Since choosing e in (G, f) has exactly n_{ij} choices, the above simplifies to

$$II = \frac{1}{\text{Aut}(G', f')} \sum_{(G, f)} \binom{n_{ik} + n_{jk}}{n_{ik}} \cdot \binom{l_i + l_j}{l_i} \cdot 2^{n_{ij}-1} \cdot \binom{c_i + c_j + n_{ij} - 1}{c_i, c_j, n_{ij} - 1} \cdot \mathcal{H}_{(G, f)}^{ij}(\gamma_i \cdot \gamma_j \cdot \gamma_k)$$

Here $\mathcal{H}_{(G, f)}^{ij}$ means we choose any edge between v_i and v_j , and contract it by Ω .

Next we use the combinatorial identity (with $n_{ik} + l_i + 2c_i + n_{ij} - 1 = N_i$ and $n_{jk} + l_j + 2c_j + n_{ij} - 1 = N_j$):

$$\sum_{n_{ij}} 2^{n_{ij}-1} \binom{c_i + c_j + n_{ij} - 1}{c_i, c_j, n_{ij} - 1} = \binom{N_i + N_j - n_{ik} - n_{jk} - l_i - l_j}{N_i - n_{ik} - l_i},$$

which is proved in Lemma 4.13. This implies that

$$II = \frac{1}{\text{Aut}(G', f')} \sum_{(G, f)} \binom{n_{ik} + n_{jk}}{n_{ik}} \cdot \binom{l_i + l_j}{l_i} \cdot \binom{N_i + N_j - n_{ik} - n_{jk} - l_i - l_j}{N_i - n_{ik} - l_i} \cdot \mathcal{H}_{(G, f)}^{ij}(\gamma_i \cdot \gamma_j \cdot \gamma_k)$$

This expression is exactly I , since the combinatorial product

$$\binom{n_{ik} + n_{jk}}{n_{ik}} \cdot \binom{l_i + l_j}{l_i} \cdot \binom{N_i + N_j - n_{ik} - n_{jk} - l_i - l_j}{N_i - n_{ik} - l_i}$$

is precisely the number of ways to split the tensor $\{\gamma_i, \gamma_j\} \in \text{Sym}^{N_i + N_j} L_-$ into two parts with $l_i/l_j, n_{ik}/n_{jk}$ prescribed by the graph (G, f) . Thus the proof of Part (A) of Theorem 4.2 is complete. \square

4.13. Lemma. *Let M, N be two positive integers. Assume that $M \equiv N \pmod{2}$ and that $M \geq N$. Denote by $\delta = \frac{M-N}{2}$. Then we have*

$$\sum_{k \geq 0, k+2l+\delta = \frac{M+N}{2}} 2^k \cdot \binom{\frac{M+N}{2}}{k, l, l+\delta} = \binom{M+N}{M}.$$

Proof. The left hand side is the coefficient of $z^{-\delta}$ in the product $(2 + z + \frac{1}{z})^{\frac{M+N}{2}}$. But observe that

$$(2 + z + \frac{1}{z})^{\frac{M+N}{2}} = (\sqrt{z} + \frac{1}{\sqrt{z}})^{M+N}.$$

The coefficient of $z^{-\delta}$ in the latter expression is given by $\binom{M+N}{M}$. \square

As an application of Part (A) of Theorem 4.2 we prove the following lemma which was used in the previous section to define categorical enumerative invariants.

4.14. Lemma. *The element β^A of (3.3) is a Maurer-Cartan element in \mathfrak{h} , via the canonical inclusion $\mathfrak{h}^+ \subset \mathfrak{h}$.*

Proof. The construction of the L_∞ -morphism \mathcal{K} makes it obvious that it restricts to an L_∞ -morphism $\mathcal{K}^+ : \mathfrak{h}^+ \rightarrow (\mathfrak{h}^+)^{\text{triv}}$. We have a commutative diagram

$$\begin{array}{ccccc} \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}^+ & \longrightarrow & 0 \\ \mathcal{K} \downarrow & & \mathcal{K}^+ \downarrow & & \\ \mathfrak{h}^{\text{triv}} & \longrightarrow & (\mathfrak{h}^+)^{\text{triv}} & \longrightarrow & 0 \end{array}$$

For the bottom row, since the Maurer-Cartan equation is simply $(b + uB)x = 0$ and the map is surjective at chain level, it is obvious that there exists a lift in $\mathfrak{h}^{\text{triv}}$ of the Maurer-Cartan element $\mathcal{K}^+(\beta^A)$. The two vertical maps are both L_∞ quasi-isomorphisms, so it follows that there exists a Maurer-Cartan element $\alpha \in \mathfrak{h}$ such that $\pi(\alpha)$ is gauge equivalent to β^A : for some degree zero element $g \in \mathfrak{h}^+$, we have $\exp(g) \cdot \pi(\alpha) = \beta^A$. Since π is surjective, let $\tilde{g} \in \mathfrak{h}$ be a lift of g in \mathfrak{h} . Then the Maurer-Cartan element $\exp(\tilde{g}) \cdot \alpha$ lifts the Maurer-Cartan element β^A . But modifying such an element in its scalar part does not affect the property of it being Maurer-Cartan (since $\mathbb{K}[[1]][[\hbar, \lambda]]$ is central in \mathfrak{h}), so we can choose the lift to have trivial scalar part. \square

4.15. A first order trivialization of the brackets in $\widehat{\mathfrak{h}}$. Our next goal is to prove Part (B) of Theorem 4.2. More precisely, we would like to use a chain-level splitting R of the non-commutative Hodge filtration, as defined in (3.6), to partially trivialize the DGLA

$$\widehat{\mathfrak{h}} = \bigoplus_{k \geq 1, l \geq 0} \text{Hom}^c(\text{Sym}^k(L_+[1]), \text{Sym}^l(L_-)) \llbracket \hbar, \lambda \rrbracket.$$

To simplify notation we will write

$$L_{k,l} = \text{Hom}^c(\text{Sym}^k(L_+[1]), \text{Sym}^l(L_-)).$$

We think of an element $\gamma \in L_{k,l}$ as an operation with k inputs and l outputs. The inputs are elements in $L_+[1]$; the outputs are in L_- .

For a map $\gamma \in L_{k,l}$ we define its de-symmetrization

$$\widetilde{\gamma} \in \text{Hom}^c((L_+[1])^{\otimes k}, L_-^{\otimes l})$$

as the composition

$$(L_+[1])^{\otimes k} \rightarrow \text{Sym}^k(L_+[1]) \xrightarrow{\gamma} \text{Sym}^l(L_-) \rightarrow L_-^{\otimes l}.$$

4.16. Denote by $S : L_- \rightarrow L_+[1]$ the circle action map defined by

$$S(\alpha) = uB(\alpha_0)$$

for $\alpha \in L_-$ of the form $\alpha = \alpha_0 + \alpha_{-1}u^{-1} + \dots$. Note that S is even.

We think of S as the operation of turning an output in L_- to an input in $L_+[1]$, with an S^1 twist. Combining this with composition of maps is the building block of the bracket operations.

4.17. Proposition. *There exists an odd operator $F : L_- \rightarrow L_+[1]$ of degree one which is a bounding homotopy of S . That is, we have*

$$[b + uB, F] = S.$$

Furthermore, it is compatible with the homotopy operator H constructed in Proposition 4.5 in the sense that the following diagram is commutative

$$\begin{array}{ccc} L_- \otimes L_- & \xrightarrow{H} & \mathbb{K} \\ \text{id} \otimes \iota \downarrow & & \uparrow \text{ev} \\ L_- \otimes \text{Hom}^c(L_+[1], \mathbb{K}) & \xrightarrow{F \otimes \text{id}} & L_+[1] \otimes \text{Hom}^c(L_+[1], \mathbb{K}), \end{array}$$

where ev is the evaluation map and ι is the map discussed in (2.3).

Proof. We set the component $F_{i,j}$ of F that maps $L \cdot u^{-i}$ to $\hbar^{-1}L \cdot u^{j+1}$ (for $i \geq 0$, $j \geq 0$) to be

$$F_{i,j}(x \cdot u^{-i}) = (-1)^j \sum_{l=0}^j R_l T_{i+j+1-l} x \cdot u^{j+1}.$$

The above properties of F can be verified as in Proposition 4.5. \square

4.18. The construction of the L_∞ morphism $\widehat{\mathcal{K}} : \widehat{\mathfrak{h}} \rightarrow \widehat{\mathfrak{h}}^{\text{triv}}$ is similar to the construction of \mathcal{K} in that it also involves the use of labeled graphs. However, as is evident from the definition of $\widehat{\mathfrak{h}}$, we need to have extra information to distinguish inputs and outputs on these graphs. The particular type of graphs that are relevant for us will be called *partially directed graphs*, as defined below.

4.19. Definition. A *partially directed graph* of type (g, k, l) is given by a quadruple $(G, L_G^{\text{in}} \amalg L_G^{\text{out}}, E^{\text{dir}}, K)$ consisting of the following data:

- A labeled graph G of type $(g, k + l)$.
- A decomposition

$$L_G = L_G^{\text{in}} \amalg L_G^{\text{out}}$$

of the set of leaves L_G such that $|L_G^{\text{in}}| = k$ and $|L_G^{\text{out}}| = l$. Leaves in L_G^{in} will be called *incoming*, while leaves in L_G^{out} will be called *outgoing*.

- A subset $E^{\text{dir}} \subset E_G$ of edges of G whose elements are called *directed edges*, and a direction is chosen on them. Edges in $E_G - E^{\text{dir}}$ are called *un-directed*.
- A spanning tree $K \subset E^{\text{dir}}$ of the graph G .

Let us denote by G^{dir} the graph obtained from G by removing all un-directed edges. Then we require the following properties to hold:

- The directed graph G^{dir} is connected.
- Each vertex has at least one incoming half-edge.
- For every $e \in E^{\text{dir}}$ there exists a non-empty, directed path in K joining the same vertices as e , in the same direction.

A *partially directed graph* is called *stable* if the underlying labeled graph is.

4.20. Isomorphisms and markings. To simplify notation we sometimes use the notation \mathbb{G} to denote a partially directed graph $(G, L_G^{\text{in}} \amalg L_G^{\text{out}}, E^{\text{dir}}, K)$. An isomorphism $\varphi : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ of partially directed graphs is an isomorphism of the underlying labeled graphs G_1 and G_2 such that the directions of leaves and edges are preserved, and such that $\varphi(K_1) = K_2$. Denote by $\Gamma(g, k, l)$ the set of isomorphism classes of partially directed graphs of type (g, k, l) and by $\Gamma(\widehat{g, k, l})$ the subset of those that are stable.

A marking of a partially directed graph \mathbb{G} is a bijection $f : \{1, \dots, m\} \rightarrow V_G$ onto the set of vertices of G . We shall denote such a marked partially directed graph by $(G, L_G^{\text{in}} \amalg L_G^{\text{out}}, E^{\text{dir}}, K, f)$, or sometimes simply (\mathbb{G}, f) . An isomorphism of marked partially directed graphs is an isomorphism of partially directed graphs that also preserves the markings. Denote by $\Gamma(\widehat{g, k, l})_m$ the set of isomorphism classes of marked partially directed graphs with m vertices.

4.21. Let \mathbb{G} be a partially directed graph. The spanning tree K defines a partial ordering on V_G : we set $v < w$ if there exists a non-empty directed path in K from v to w .

The last condition in the definition of a partially directed graph can be rephrased as saying that an edge in E^{dir} may only join a vertex v with one of its descendants in K . Another way of stating this is as follows. Define a new relation \prec on V_G by declaring $x \prec y$ if there exists a directed path in G^{dir} from x to y . Then $x \prec y$ if and only if $x < y$. In particular \prec is an order relation, i.e., G^{dir} has no directed loops.

Yet a third way to understand this condition is as follows. For $e \in K$ define \mathbb{G}/e to be the result of deleting all directed edges in \mathbb{G} which are parallel to e (have the same starting and ending points as e), and then contracting e . The condition is equivalent to the fact that after contracting all of K (by successively contracting its edges as above) we are left with a graph with no directed loops.

4.22. Construction of the maps $\widehat{\mathcal{H}}_m$. We will define for each $m \geq 1$ a degree zero linear map

$$\widehat{\mathcal{H}}_m : \text{Sym}^m(\widehat{\mathfrak{h}}[1]) \rightarrow \widehat{\mathfrak{h}}^{\text{triv}}[1],$$

in a way similar to the construction of \mathcal{H}_m in (4.8). The map $\widehat{\mathcal{H}}_m$ will be the λ -linear extension of the sum over all partial directed graphs (\mathbb{G}, f) with m vertices of maps $\widehat{\mathcal{H}}_{\mathbb{G}, f}$ taken with weights $\text{wt}(\mathbb{G}, f)$:

$$\widehat{\mathcal{H}}_m = \sum_{\substack{g, k, l \\ (\mathbb{G}, f) \in \Gamma(\widehat{g, k, l})_m}} \text{wt}(\mathbb{G}, f) \cdot \widehat{\mathcal{H}}_{\mathbb{G}, f} : \widehat{\mathfrak{h}}[1]^{\otimes m} \rightarrow \widehat{\mathfrak{h}}^{\text{triv}}[1]$$

The maps $\widehat{\mathcal{H}}_{\mathbb{G},f}$ will be defined in (4.23), while the weight of (\mathbb{G},f) will be defined in (4.25).

4.23. Let $\mathbb{G} \in \Gamma(\widetilde{g, k, l})_m$ be a partially directed graph of type (g, k, l) endowed with a marking f . Let $v \in V_G$. We define two integers $k_v \geq 1$ and $l_v \geq 0$ by counting the number of incoming/outgoing half edges at v , according to the following rules:

- Each directed half-edge of a directed edge $e \in E^{\text{dir}}$ is considered incoming/outgoing as indicated by the direction of e .
- Each leaf at v is considered incoming/outgoing as indicated by its direction.
- Each un-directed half-edge at v is considered outgoing.

For the vertex v_i marked by $i \in \{1, \dots, m\}$ its genus label and numbers of inputs and outputs will be denoted by g_i, k_i and l_i , respectively.

With this preparation we define the operator

$$\widehat{\mathcal{H}}_{\mathbb{G},f} : \bigotimes_{i=1}^m L_{k_i, l_i} \cdot \hbar^{g_i} \rightarrow L_{k, l} \cdot \hbar^g.$$

It maps

$$\gamma_1 \cdot \hbar^{g_1} \otimes \dots \otimes \gamma_m \cdot \hbar^{g_m}$$

to $\gamma \cdot \hbar^g$ according to the following rules:

- (1) Each internal vertex v_i is decorated by the de-symmetrization $\widetilde{\gamma}_i$ of γ_i . The results are tensored together in the order prescribed by the marking, yielding the product

$$\widetilde{\gamma}_1 \otimes \dots \otimes \widetilde{\gamma}_m$$

of de-symmetrized linear maps.

- (2) The above product of linear maps is composed according to the directed graph G^{dir} , using the operators S and F of (4.16) and Proposition 4.17 to convert outputs into inputs: F on the edges of K , S on the edges of $E^{\text{dir}} \setminus K$. The result is a single linear map f with $(\sum k_i) - \#E^{\text{dir}}$ inputs and $(\sum l_i) - \#E^{\text{dir}}$ outputs.

The fact that \prec is a partial order ensures that composition according to the “flow” of the graph G^{dir} is well-defined. This is well-known in the literature, mainly as a tool to establish the bar construction for PROPs, see [EE05], [MV09], [Val07].

- (3) Pairs of outputs of f are contracted according to the undirected edges in $E_G \setminus E^{\text{dir}}$, using the operator H^{sym} of (4.6).

(4) The resulting homomorphism is regarded as a map

$$\gamma \in \text{Hom}^c(\text{Sym}^k(L_+[1]), \text{Sym}^l(L_-))$$

by symmetrizing the inputs and the outputs, respectively.

The picture below illustrates how the compositions and contractions are performed. The levels are from the partial ordering of vertices, and the blue directed edges form the spanning tree $K \subseteq G^{\text{dir}}$.

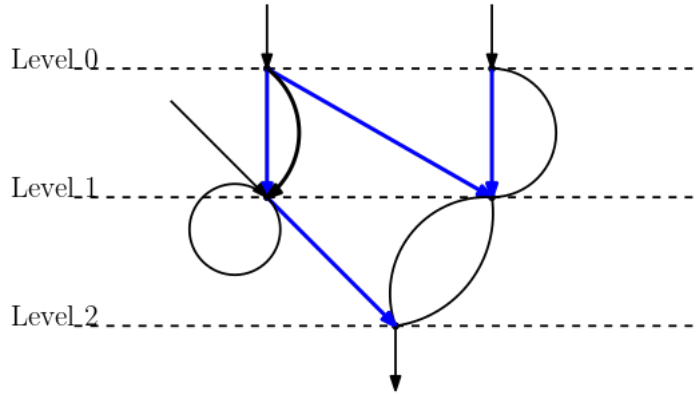


Illustration of composition using a labeled and partially directed graph

4.24. Formally we set

$$\widehat{\mathcal{H}}_{G,f}(\gamma_1 \cdot \hbar^{g_1}, \dots, \gamma_m \cdot \hbar^{g_m}) = \pi \left(\prod_{e \in E_G \setminus E^{\text{dir}}} \tau_e \prod_{e \in G^{\text{dir}}} \tau_e \prod_{i=1}^m \tilde{\gamma}_i \right) \cdot \hbar^g.$$

Here

$$\pi : \text{Hom}^c(L_+[1]^{\otimes k}, L_-^{\otimes l}) \rightarrow \text{Hom}^c(\text{Sym}^k(L_+[1]), \text{Sym}^l L_-)$$

is the natural projection map, and τ_e is the contribution of edge e as above.

We remark that the above expression is purely symbolic, and cannot be thought of as a way to compute the compositions successively, edge by edge. This would be possible if $E^{\text{dir}} \setminus K$ were empty, in which case exactly one output of an operation would be plugged into an input of another at each step. However, edges in $E^{\text{dir}} \setminus K$ indicate further outputs that need to be plugged into inputs of operations that have already been composed; if we were dealing with finite dimensional spaces this would be a trace-like operation, but our spaces of inputs and outputs are infinite dimensional.

Nonetheless one can see that the overall composition is an even operation, because each “step” above is even: for an edge $e \in K$, τ_e is an even operation $\mathfrak{h}[1] \otimes \mathfrak{h}[1] \rightarrow \mathfrak{h}[1]$

because it uses the odd map F in the middle; while the contractions corresponding to edges in $E_G \setminus K$ can be thought of as “maps” $\tau_e : \mathfrak{h}[1] \rightarrow \mathfrak{h}[1]$ using the even maps S or H^{sym} .

4.25. The weight of a graph. Let $\mathbb{G} = (G, L_G^{\text{in}} \amalg L_G^{\text{out}}, E^{\text{dir}}, K)$ be a partially directed graph. We shall now define its associated weight, $\text{wt}(\mathbb{G}) \in \mathbb{Q}$.

To do this we first assign to each vertex $v \in V_G$ its number of inputs and outputs (k_v, l_v) . This gives a map

$$v_{\mathbb{G}} : V_G \rightarrow \mathbb{N}^+ \times \mathbb{N}$$

which we call the vertex type of \mathbb{G} . We define a new set $\text{PD}(\mathbb{G})$ consisting of all the partially directed graph structures on $(G, L_G^{\text{in}} \amalg L_G^{\text{out}})$ which have vertex type $v_{\mathbb{G}}$. Here a partially directed graph structure on $(G, L_G^{\text{in}} \amalg L_G^{\text{out}})$ means a partially directed graph $\mathbb{G}' = (G, L_G^{\text{in}} \amalg L_G^{\text{out}}, E^{\text{dir}}, K)$. We emphasize that we are *not* taking isomorphisms classes of partially directed graphs in the above definition.

The rational weight to \mathbb{G} is defined by the formula

$$\text{wt}(\mathbb{G}) = \frac{1}{|\text{Aut}(\mathbb{G})|} \cdot \frac{1}{|\text{PD}(\mathbb{G})|}.$$

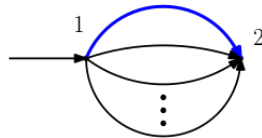
We shall also need the marked version analogue of the above definition. For a marked partially directed graph $(\mathbb{G}, f) \in \widehat{\Gamma}(g, k, l)_m$, we define the set $\text{PD}(\mathbb{G}, f)$ as the set of marked partially directed graph structures on $(G, L_G^{\text{in}} \amalg L_G^{\text{out}})$ which have vertex type $v_{\mathbb{G}}$ and marking f . The associated weight of (\mathbb{G}, f) is given by

$$\text{wt}(\mathbb{G}, f) = \frac{1}{|\text{Aut}(\mathbb{G}, f)|} \cdot \frac{1}{|\text{PD}(\mathbb{G}, f)|}.$$

Note that the sets $\text{PD}(\mathbb{G})$ and $\text{PD}(\mathbb{G}, f)$ are naturally isomorphic by simply keeping track of the marking f . Thus the two weights are related by

$$\text{wt}(\mathbb{G}) = \frac{|\text{Aut}(\mathbb{G}, f)|}{|\text{Aut}(\mathbb{G})|} \text{wt}(\mathbb{G}, f).$$

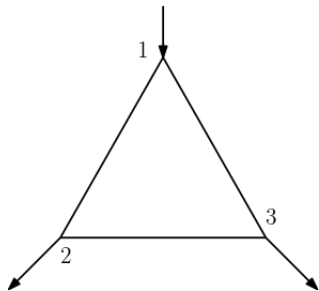
4.26. We illustrate these definitions with a few examples. Consider the following graph:



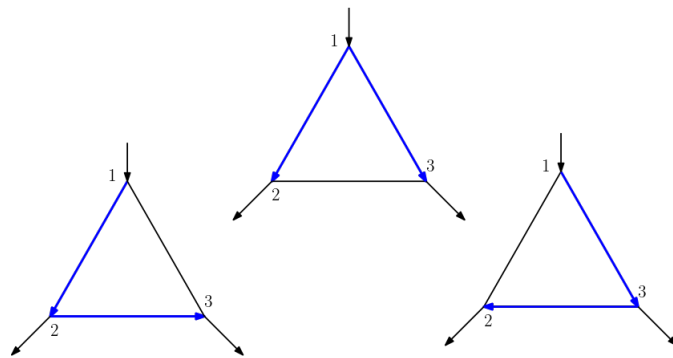
Assume there are d directed edges between the two vertices, with the spanning tree K given by the blue edge. The partially directed graph \mathbb{G} has $k = 1$ input, $l = 0$ outputs, and the two vertices v_1, v_2 are of types $(1, d), (d, 0)$ respectively. The automorphism group $\text{Aut}(\mathbb{G}, f)$ has order $(d - 1)!$ coming from the permutation group of the $d - 1$ directed edges which are not in K .

On the other hand, the set $\text{PD}(\mathbb{G})$ has exactly d elements: all the d edges must point from v_1 to v_2 to preserve the numerical type, but any of them can be chosen as the spanning tree K . Putting the two calculations together, the weight of \mathbb{G} is given by $\text{wt}(\mathbb{G}, f) = 1/d!$. Note that this is precisely the factor needed in the operator that homotopy bounds the d -th Lie bracket $\frac{1}{d!} \{\gamma_1, \gamma_2\}_d$.

As another example consider the marked graph $(G, L^{\text{in}} \amalg L^{\text{out}}, f)$ with $g = 1, k = 1, l = 2$ and $m = 3$; we assume that each vertex has been assigned vertex type $(1, 2)$.



The possible partially directed structures on it are depicted below:



Since these partially directed graphs have no automorphisms, each of them has weight $\text{wt}(\mathbb{G}, f) = 1/3$.

4.27. Proof of Part (B) of Theorem 4.2. By the same reasoning as in the proof of Part (A) the maps $\widehat{\mathcal{H}}_m$ are symmetric. Moreover, we have argued in (4.24) that the maps $\widehat{\mathcal{H}}_{\mathbb{G}, f}$ are even.

Next we need to prove that the maps $\{\widehat{\mathcal{K}}_m\}_{m \geq 1}$ defined above form an L_∞ homomorphism

$$\widehat{\mathcal{K}} : \widehat{\mathfrak{h}} \rightarrow \widehat{\mathfrak{h}}^{\text{triv}}.$$

In other words we need to prove that for each $m \geq 1$ we have

$$\begin{aligned} [b + uB + \iota, \widehat{\mathcal{K}}_m](\gamma_1 \cdots \gamma_m) &= \hbar \sum_{1 \leq i \leq m} (-1)^{\epsilon_i} \widehat{\mathcal{K}}_m(\gamma_1 \cdots \Delta \gamma_i \cdots \gamma_m) \\ &\quad + \sum_{i < j} (-1)^{\epsilon_{ij}} \widehat{\mathcal{K}}_{m-1}(\{\gamma_i, \gamma_j\}_{\hbar} \cdot \gamma_1 \cdots \widehat{\gamma}_i \cdots \widehat{\gamma}_j \cdots \gamma_m), \end{aligned}$$

where the signs are again given by the Koszul rule,

$$\begin{aligned} \epsilon_i &= |\gamma_1| + \cdots + |\gamma_{i-1}| \\ \epsilon_{ij} &= |\gamma_i|(|\gamma_1| + \cdots + |\gamma_{i-1}|) + |\gamma_j|(|\gamma_1| + \cdots + |\gamma_{i-1}| + |\gamma_{i+1}| + \cdots + |\gamma_{j-1}|). \end{aligned}$$

We first consider the term $[b + uB, \widehat{\mathcal{K}}_m](\gamma_1 \cdots \gamma_m)$. By definition, we have $\widehat{\mathcal{K}}_m = \sum \text{wt}(\mathbb{G}, f) \cdot \widehat{\mathcal{K}}_{\mathbb{G}, f}$. Using the identities

$$\begin{aligned} [b + uB, H^{\text{sym}}] &= \Omega, \\ [b + uB, F] &= S, \\ [b + uB, S] &= 0 \end{aligned}$$

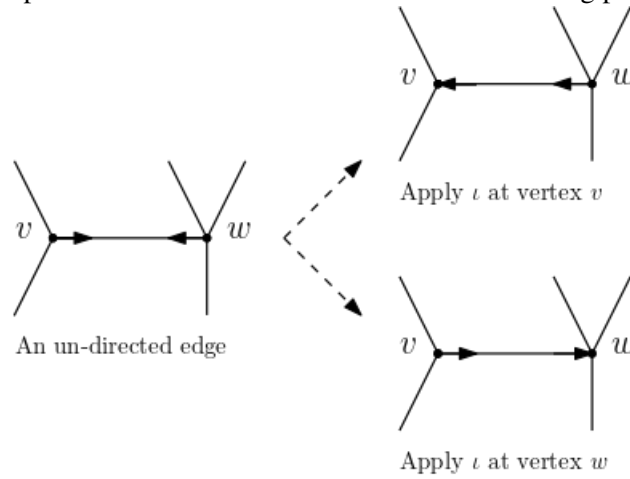
this commutator can be written as the sum of three types of terms:

- (1) the first type is obtained by applying $b + uB$ to a loop edge; the result corresponds to a term $\hbar \cdot \widehat{\mathcal{K}}_m(\gamma_1 \cdots \Delta \gamma_j \cdots \gamma_m)$ on the right hand side.
- (2) the second type is obtained by applying $b + uB$ to an edge in K ; the result corresponds to a term $\widehat{\mathcal{K}}_{m-1}(\{\gamma_i, \gamma_j\}_{\hbar} \cdot \gamma_1 \cdots \widehat{\gamma}_i \cdots \widehat{\gamma}_j \cdots \gamma_m)$ on the right.
- (3) the third type is obtained by applying $b + uB$ to an un-directed, non-loop edge; the result is a similar graph evaluation, but where the contraction operator H^{sym} on that edge is changed to Ω .

The cancellation of the terms of types (1) and (2) with the corresponding terms on the right hand side can be proved in the same way as in the proof of part (A). Thus there are two types of terms left which need to cancel out: the terms of type (3), and the expression $[\iota, \widehat{\mathcal{K}}_m](\gamma_1 \cdots \gamma_m)$.

Let us understand the terms in $[\iota, \widehat{\mathcal{K}}_m](\gamma_1 \cdots \gamma_m)$ in more detail. Observe that the terms in $\iota \cdot \widehat{\mathcal{K}}_m(\gamma_1 \cdots \gamma_m)$ are obtained by switching an output leaf to an input leaf. However, these are only part of the terms in $\sum_{j=1}^m \widehat{\mathcal{K}}_m(\gamma_1 \cdots \iota \gamma_j \cdots \gamma_m)$. Indeed, the extra terms in the latter expression come from switching the in/out label of a half-edge that is part of an edge (i.e., not a leaf). Call e the edge whose end label is being switched. There are three possible cases:

- if the edge e is a loop, since we always consider half-edges of a loop as outgoing, switching either end of the loop results in an invalid graph (now we would have a directed cycle);
- if the edge e is already a directed edge, then switching the outgoing half-edge of e to “incoming” yields a directed edge with two ends that are both incoming, which is also an invalid configuration in a graph;
- this is the essential case: if e is an un-directed edge which is not a loop, then there are two possible contributions illustrated in the following picture:



By the commutativity of the diagram in Proposition 4.17, the resulting term in this case precisely corresponds to a term of type (3).

This finishes the proof of Part (B) of Theorem 4.2. □

4.28. Proof of part (C) of Theorem 4.2. We need to prove that

$$\widehat{\mathcal{H}}_m(\iota\gamma_1 \cdots \iota\gamma_m) = \iota\mathcal{H}_m(\gamma_1 \cdots \gamma_m).$$

We first claim that the left hand side contains no contributions from graphs with more than one input. Indeed, assume that $\mathbb{G} \in \Gamma(g, k, l)$ is a labeled partially directed stable graph with $k \geq 2$. The spanning tree K has $|V_G| - 1$ directed edges. But it also has $k + l$ directed leaves, of which k are incoming, which implies that there are at least $k + |V_G| - 1 \geq |V_G| + 1$ incoming half-edges in \mathbb{G} . However, there are only $|V_G|$ internal vertices. By the pigeonhole principle we conclude that at least one of the vertices must have more than one incoming half-edge. Since all the tensors $\iota\gamma_i$ have only one input, the contribution from such a graph G must be zero.

Thus the left hand side equals

$$\sum_{(\mathbb{G},f)} \frac{1}{|\text{Aut}(\mathbb{G},f)|} \cdot \frac{1}{|\text{PD}(\mathbb{G},f)|} \widehat{\mathcal{H}}_{\mathbb{G},f}(\iota\gamma_1 \cdots \iota\gamma_m)$$

where the summation is over the set of marked, partially directed graphs (\mathbb{G},f) that have only $k = 1$ incoming leaf, and at each vertex v the number of incoming half edges k_v is one. We need to compare this expression with

$$\sum_{(G,f)} \frac{1}{|\text{Aut}(G,f)|} \iota \mathcal{H}_{G,f}(\gamma_1 \cdots \gamma_m)$$

We may rewrite this as

$$\sum_{(G,f,l)} \frac{1}{|\text{Aut}(G,f,l)|} \iota_l \mathcal{H}_{G,f}(\gamma_1 \cdots \gamma_m),$$

where we sum over isomorphism types of triples (G,f,l) where (G,f) is a marked graph and $l \in L_G$ is a fixed leaf (output). In the summand we apply the operator ι only at the leaf l ; this operator is denoted by ι_l . The notation $\text{Aut}(G,f,l)$ stands for the set of automorphisms that also preserve the decomposition $L_G = \{l\} \cup (L_G \setminus \{l\})$.

By the definition of the operators $\mathcal{H}_{G,f}$ and $\widehat{\mathcal{H}}_{\mathbb{G},f}$ and the commutativity of the diagram in Proposition 4.17 the two expressions $\sum \widehat{\mathcal{H}}_{\mathbb{G},f}(\iota\gamma_1 \cdots \iota\gamma_m)$ and $\sum \iota_l \mathcal{H}_{G,f}(\gamma_1 \cdots \gamma_m)$ contain the same terms. Thus it suffices to match up their coefficients; in other words, we need to show that the following identity holds for a fixed triple (G,f,l)

$$\sum_{(\mathbb{G},f)} \frac{1}{|\text{Aut}(\mathbb{G},f)|} \cdot \frac{1}{|\text{PD}(\mathbb{G},f)|} = \frac{1}{|\text{Aut}(G,f,l)|}.$$

The summation on the left hand side is over isomorphism classes of marked partially directed graphs (\mathbb{G},f) whose underlying marked graph is (G,f) , and where l is the only incoming leaf (all others are outgoing) and the vertex type is $(1, l_v)$ at each vertex v . Note that all these graphs have the same $\text{PD}(\mathbb{G},f)$; we will denote this set by $\text{PD}(G,f,l)$.

To show the above identity consider the action of $\text{Aut}(G,f,l)$ on the set $\text{PD}(G,f,l)$, and observe that its orbits are precisely the isomorphism classes of the marked partially directed graphs (\mathbb{G},f) that appear in the above summation. By the orbit-sum formula we obtain that

$$|\text{PD}(G,f,l)| = \sum_{(\mathbb{G},f)} \frac{|\text{Aut}(G,f,l)|}{|\text{Aut}(\mathbb{G},f)|}$$

which is equivalent to the desired identity. \square

5. Feynman sum formulas for the categorical invariants

In this section we use the L_∞ -morphisms of Theorem 4.2 to derive explicit formulas for the categorical invariants $F_{g,n}^{A,s}$. These formulas are given as summations over partially directed stable graphs, with vertices labeled by the tensors $\widehat{\beta}_{g,k,l}^A$. We also prove that our invariants depend only on the Morita equivalence class of the pair (A, s) .

5.1. Further trivialization. Let $s : H \rightarrow H_*(L_+)$ be a splitting of the Hodge filtration, see (3.4). As in the previous section we fix a chain-level splitting $R : (L, b) \rightarrow (L[[u]], b + uB)$ that lifts s , of the form

$$R = \text{id} + R_1u + R_2u^2 + \cdots, \quad R_j \in \text{End}(L).$$

Denote by $L^{\text{Triv}} = (L, b)$ the same underlying chain complex as L , but endowed with trivial circle action. The chain map R induces an isomorphism $L_-^{\text{Triv}} \rightarrow L_-$, which further induces an isomorphism of DGLA's (which we still denote by R)

$$R : \mathfrak{h}^{\text{Triv}} = \text{Sym}(L_-^{\text{Triv}})[1][[\hbar, \lambda]] \rightarrow \mathfrak{h}^{\text{triv}} = \text{Sym}(L_-)[1][[\hbar, \lambda]].$$

Here the differential on the left hand side is just b while on the right hand side is $b + uB$; they both have zero Lie bracket. As in the previous section we denote by T the inverse of R . It analogously induces an inverse isomorphism of DGLA's

$$T : \mathfrak{h}^{\text{triv}} \rightarrow \mathfrak{h}^{\text{Triv}}.$$

We also have the corresponding hat-version of the morphism

$$\widehat{T} : \widehat{\mathfrak{h}}^{\text{triv}} \rightarrow \widehat{\mathfrak{h}}^{\text{Triv}}.$$

5.2. Consider the following commutative diagram of DGLA's:

$$\begin{array}{ccccc} \mathfrak{h}^+ & \xrightarrow{\mathcal{K}^+} & \mathfrak{h}^{\text{triv},+} & \xrightarrow{T} & \mathfrak{h}^{\text{Triv},+} \\ \iota \downarrow & & \iota \downarrow & & \iota \downarrow \\ \widehat{\mathfrak{h}} & \xrightarrow{\widehat{\mathcal{K}}} & \widehat{\mathfrak{h}}^{\text{triv}} & \xrightarrow{\widehat{T}} & \widehat{\mathfrak{h}}^{\text{Triv}}. \end{array}$$

This induces a commutative diagram of isomorphisms of the associated Maurer-Cartan moduli spaces:

$$\begin{array}{ccccc} \text{MC}(\mathfrak{h}^+) & \xrightarrow{\mathcal{K}_*^+} & \text{MC}(\mathfrak{h}^{\text{triv},+}) & \xrightarrow{T_*} & \text{MC}(\mathfrak{h}^{\text{Triv},+}) \\ \iota_* \downarrow & & \iota_* \downarrow & & \iota_* \downarrow \\ \text{MC}(\widehat{\mathfrak{h}}) & \xrightarrow{\widehat{\mathcal{K}}_*} & \text{MC}(\widehat{\mathfrak{h}}^{\text{triv}}) & \xrightarrow{\widehat{T}_*} & \text{MC}(\widehat{\mathfrak{h}}^{\text{Triv}}). \end{array}$$

Consider the two moduli spaces on the right. Since the Maurer-Cartan equation for both $\mathfrak{h}^{\text{Triv},+}$ and $\widehat{\mathfrak{h}}^{\text{Triv}}$ is the linear equation $bx = 0$, there are natural inclusions

$$\begin{aligned} \text{MC}(\mathfrak{h}^{\text{Triv},+}) &\subseteq \text{Sym } H_-[1][[\hbar, \lambda], \\ \text{MC}(\widehat{\mathfrak{h}}^{\text{Triv}}) &\subseteq \bigoplus_{k \geq 1, l} \text{Hom}^c(\text{Sym}^k(H_+[1]), \text{Sym}^l H_-)[[\hbar, \lambda], \end{aligned}$$

where $H = H_*(L^{\text{Triv}}) = HH_*(A)$. The images of both inclusions are simply the odd homology groups of the corresponding spaces (because Maurer-Cartan elements are odd). The categorical enumerative invariant $F^{A,s}$ of Definition 3.9 lies inside the upper-right corner space $\text{MC}(\mathfrak{h}^{\text{Triv},+})$. The Maurer-Cartan element $\widehat{\beta}^A$ lies inside the lower-left corner $\text{MC}(\widehat{\mathfrak{h}})$.

5.3. Theorem. *We have*

$$\iota_* F^{A,s} = \widehat{T}_* \widehat{\mathcal{K}}_* \widehat{\beta}^A.$$

Before proving this result we need some preparation. In the previous diagram we have by definition $\iota_* \beta^A = \widehat{\beta}^A$, thus the commutativity of the diagram yields

$$\widehat{T}_* \widehat{\mathcal{K}}_* \widehat{\beta}^A = \widehat{T}_* \widehat{\mathcal{K}}_* \iota_* \beta^A = \iota_* T_* \mathcal{K}_*^+ \beta^A.$$

From this, it is clear that Theorem 5.3 would follow from the identity

$$F^{A,s} = T_* \mathcal{K}_*^+ \beta^A.$$

The inverse of T is given by R , thus the above identity is equivalent to $R_* F^{A,s} = \mathcal{K}_*^+ \beta^A$. A remarkable property of the construction of the L_∞ -morphism \mathcal{K} is that one can construct an inverse morphism \mathcal{K}^{-1} in the same way as in the construction of \mathcal{K} (4.8) with the only change that edges in a labeled graph are all contracted by $-H^{\text{Sym}}$ instead of H^{Sym} . Denote the resulting L_∞ morphism by \mathcal{K}^{-1} . The following lemma justifies this notation.

5.4. Lemma. *We have $\mathcal{K}^{-1} \circ \mathcal{K} = \text{id}$.*

Proof. Observe that in the composition $\mathcal{K}^{-1} \circ \mathcal{K}$ we are summing over labeled graphs with edges either decorated by $-H^{\text{Sym}}$ (from \mathcal{K}^{-1}) or by H^{Sym} (from \mathcal{K}). This is equivalent to summing over labeled graphs with decorations of edges by $(H^{\text{Sym}} + (-H^{\text{Sym}}))$, which of course is zero. Thus the only nonzero contribution to the composition $\mathcal{K}^{-1} \circ \mathcal{K}$ is from graphs with no edges at all, i.e., the star graphs. These yield the identity map on Maurer-Cartan spaces as discussed in (4.8). \square

5.5. Proof of Theorem 5.3. The above discussion reduces the problem to proving the following identity:

$$\mathcal{K}_*^{-1} R_* F^{A,s} = \beta^A.$$

According to Definition 3.9, the Maurer-Cartan element $F^{A,s}$ is determined by the identity

$$\Psi^s(\exp(F^{A,s}/\hbar)) = \exp(\beta^A/\hbar).$$

Using the Feynman rules (see [Giv01] and [Pan18]), the term

$$\beta^A = \hbar \cdot \ln \Psi^s(\exp(F^{A,s}/\hbar))$$

is given by a stable graph sum with vertices labeled by $F_{g,n}^{A,s}$'s, legs labeled by R 's, and edge propagator given by Givental's formula:

$$\begin{aligned} \text{Giv} : L_-^{\text{triv}} \otimes L_-^{\text{triv}} &\rightarrow \mathbb{K} \\ \text{Giv}(u^{-i} \cdot x, u^{-j} \cdot y) &= \sum_{l=0}^j (-1)^{j-l} \langle R_{i+j-l+1}(x), R_l(y) \rangle. \end{aligned}$$

(Note that we are re-indexing so that $i, j \geq 0$, rather than using the more classical convention that $i, j \geq 1$.)

On the other hand, observe that the term $\mathcal{K}_*^{-1} R_* F^{A,s}$ is also given by a sum over stable graphs, with vertices labeled by $F_{g,n}^{A,s}$'s, legs labeled by R 's, and edge propagator given by

$$-H^{\text{Sym}} \circ (R \otimes R) : L_-^{\text{triv}} \otimes L_-^{\text{triv}} \rightarrow \mathbb{K}.$$

Thus it remains to identify Givental's propagator Giv with $-H^{\text{Sym}} \circ (R \otimes R)$. The elements $F_{g,n}^{A,s}$ are Maurer-Cartan elements of $\mathfrak{h}^{\text{Triv}}$ so they are all b -closed. The result then follows from the following lemma. \square

5.6. Lemma. *The two propagators above are equal when restricted to the subspace $(\ker b)_- \subset L_-$. In other words we have*

$$\text{Giv}(u^{-i} \cdot x, u^{-j} \cdot y) = -H^{\text{Sym}} \circ (R(u^{-i} \cdot x) \otimes R(u^{-j} \cdot y)) \text{ for } x, y \in \ker b.$$

Proof. By the symplectic property of the splitting s and the fact that R is a chain level lift of s , one can verify that Givental's propagator Giv is symmetric when restricted to $(\ker b)_-$. Thus it suffices to prove that we have

$$\text{Giv}(u^{-i} \cdot x, u^{-j} \cdot y) = -H(R(u^{-i} \cdot x), R(u^{-j} \cdot y)) \text{ for } x, y \in \ker b.$$

Next we explicitly calculate the right hand side. Take two elements $u^{-i} \cdot x$ and $u^{-j} \cdot y$ in L_-^{triv} with $i, j \geq 0$. We have

$$R(u^{-i} \cdot x) = \sum_{k=0}^i u^{-i+k} R_k(x),$$

$$R(u^{-j} \cdot y) = \sum_{l=0}^j u^{-j+l} R_l(y).$$

Plugging in the formula for H from Proposition 4.5 yields

$$\begin{aligned} H(R(u^{-i} \cdot x), R(u^{-j} \cdot y)) &= H\left(\sum_{k=0}^i u^{-i+k} R_k(x), \sum_{l=0}^j u^{-j+l} R_l(y)\right) \\ &= \sum_{k=0}^i \sum_{l=0}^j H(u^{-i+k} R_k(x), u^{-j+l} R_l(y)) \\ &= \sum_{k=0}^i \sum_{l=0}^j (-1)^{j-l} \left\langle \sum_{r=0}^{j-l} R_r T_{i+j-k-l+1-r} R_k(x), R_l(y) \right\rangle \\ &= \sum_{r=0}^j \sum_{l=0}^{j-r} (-1)^{j-l} \left\langle R_r \sum_{k=0}^i T_{i+j-k-l+1-r} R_k(x), R_l(y) \right\rangle \\ &= \sum_{r=0}^j \sum_{l=0}^{j-r} (-1)^{j-l+1} \left\langle R_r \sum_{k=i+1}^{i+j-l-r+1} T_{i+j-k-l+1-r} R_k(x), R_l(y) \right\rangle \\ &= \sum_{l=0}^j (-1)^{j-l+1} \sum_{k=i+1}^{i+j-l-1} \left\langle \sum_{r=0}^{i+j-k-l+1} R_r T_{i+j-k-l+1-r} R_k(x), R_l(y) \right\rangle. \end{aligned}$$

The crucial step is the fifth equality above, which uses the fact that for $n \geq 1$ we have $\sum_k T_k R_{n-k} = 0$. We use this in the form

$$\sum_{k=0}^i T_{i+j-k-l+1-r} R_k = - \sum_{k=i+1}^{i+j-l-r+1} T_{i+j-k-l+1-r} R_k.$$

Using the fact that $R \circ T = \text{id}$ it follows that the summation over r in the last expression is zero unless $r = 0$ and $k = i + j - l + 1$, in which case it is $R_k(x)$. We conclude that

$$H(R(u^{-i} \cdot x), R(u^{-j} \cdot y)) = \sum_{l=0}^j (-1)^{j-l+1} \langle R_{i+j-l+1}(x), R_l(y) \rangle = - \text{Giv}(u^{-i} \cdot x, u^{-j} \cdot y).$$

This completes the proof. \square

Theorem 5.3 yields an explicit formula for the categorical enumerative invariants $F_{g,n}^{A,s}$ as a Feynman sum over partially directed stable graphs, using the formula of $\widehat{\mathcal{H}}$.

5.7. Corollary. *The following formula holds for any $g \geq 0$, $n \geq 1$ such that $2g - 2 + n > 0$:*

$$\iota(F_{g,n}^{A,s}) = \sum_{\mathbb{G} \in \Gamma(g, 1, n-1)} \text{wt}(\mathbb{G}) \prod_{e \in E_{\mathbb{G}}} \text{Cont}(e) \prod_{v \in V_{\mathbb{G}}} \text{Cont}(v) \prod_{l \in L_{\mathbb{G}}} \text{Cont}(l).$$

The contributions of vertices, edges and legs are as follows:

- (i) Vertices are decorated by tensors in $\widehat{\beta}^A$. More precisely, a vertex v is decorated with the tensor $\widehat{\beta}_{g(v), k(v), l(v)}^A$, where the genus $g(v)$ and the number of incoming/outgoing half-edges $k(v)$ and $l(v)$ of v are as defined in (4.23).
- (ii) Incoming leaves are decorated by R , outgoing leaves are decorated by T .
- (iii) Edges are decorated by the contraction operators τ_e defined in (4.24).

Proof. By Theorem 5.3 the left hand side equals

$$\widehat{T} \left(\sum_{m \geq 1} \frac{1}{m!} \widehat{\mathcal{H}}_m(\widehat{\beta}^A, \dots, \widehat{\beta}^A) \right)_{1, n-1} = \widehat{T} \left(\sum_{m \geq 1} \frac{1}{m!} \sum_{(\mathbb{G}, f) \in \Gamma(g, 1, n-1)_m} \text{wt}(\mathbb{G}, f) \cdot \widehat{\mathcal{H}}_{\mathbb{G}, f}(\widehat{\beta}^A, \dots, \widehat{\beta}^A) \right).$$

The fact that we apply \widehat{T} explains the leg contribution as in (ii), and the construction of $\widehat{\mathcal{H}}_{(\mathbb{G}, f)}$ explains the contributions from vertices and edges as in (i) and (iii). We only need to match up the coefficients given by graph weights.

Consider the forgetful map $\pi : \Gamma(g, 1, n-1)_m \rightarrow \Gamma(g, 1, n-1)_m$ that forgets the marking f on the set of vertices. Fix a partially directed graph $\mathbb{G} \in \Gamma(g, 1, n-1)_m$. Denote the set of all possible markings of \mathbb{G} by $\text{Mark}(\mathbb{G})$. The group $\text{Aut}(\mathbb{G})$ acts on $\text{Mark}(\mathbb{G})$. Observe that we have

$$\pi^{-1}(\mathbb{G}) \cong \text{Mark}(\mathbb{G}) / \text{Aut}(\mathbb{G}).$$

Furthermore, the stabilizer of this action is exactly given by $\text{Aut}(\mathbb{G}, f)$ for each marked partially directed graph (\mathbb{G}, f) . This implies that the size of the orbit containing (\mathbb{G}, f) is $|\text{Aut}(\mathbb{G})| / |\text{Aut}(\mathbb{G}, f)|$, which by the formula at the end of (4.25) equals

$\text{wt}(\mathbb{G}, f) / \text{wt}(\mathbb{G})$. We thus obtain

$$\begin{aligned}
& \frac{1}{m!} \sum_{(\mathbb{G}, f) \in \widetilde{\Gamma(g, 1, n-1)}_m} \text{wt}(\mathbb{G}, f) \cdot \widehat{\mathcal{H}}_{(\mathbb{G}, f)}(\widehat{\beta}^A, \dots, \widehat{\beta}^A) = \\
& = \frac{1}{m!} \sum_{\mathbb{G} \in \Gamma(g, 1, n-1)_m} \sum_{(f) \in \pi^{-1}(\mathbb{G})} \text{wt}(\mathbb{G}, f) \cdot \widehat{\mathcal{H}}_{(\mathbb{G}, f)}(\widehat{\beta}^A, \dots, \widehat{\beta}^A) \\
& = \frac{1}{m!} \sum_{\mathbb{G} \in \Gamma(g, 1, n-1)_m} \sum_{f \in \text{Mark}(G)} \text{wt}(\mathbb{G}, f) \cdot \frac{|\text{Aut}(\mathbb{G}, f)|}{|\text{Aut}(\mathbb{G})|} \widehat{\mathcal{H}}_{(\mathbb{G}, f)}(\widehat{\beta}^A, \dots, \widehat{\beta}^A) \\
& = \sum_{\mathbb{G} \in \Gamma(g, 1, n-1)_m} \text{wt}(\mathbb{G}) \cdot \widehat{\mathcal{H}}_{\mathbb{G}}(\widehat{\beta}^A, \dots, \widehat{\beta}^A).
\end{aligned}$$

In the last equality we use the fact that the inputs $\widehat{\beta}^A$ are invariant under permutation. There are exactly $m!$ markings in the set $\text{Mark}(G)$, and this cancels the coefficient $1/m!$. Here the notation $\widehat{\mathcal{H}}_{\mathbb{G}}$ means any $\widehat{\mathcal{H}}_{(\mathbb{G}, f)}$, since the result is independent of the marking f . Finally, we also observe that since the input tensor $\widehat{\beta}^A$ is only non-zero for stable triples (g, k, l) , the terms in the summation are non-zero only when the corresponding graph is stable. \square

5.8. Morita invariance. Let \mathcal{C} be a cyclic A_∞ -category. We assume that \mathcal{C} satisfies the categorical version of condition (†) and that it is compactly generated: there exists a split generator E of \mathcal{C} . These conditions are satisfied, for example, for $\mathcal{C} = \text{D}_{\text{coh}}^b(X)$, the derived category of coherent sheaves on a smooth projective Calabi-Yau variety.

Given the data of such a pair (\mathcal{C}, E) and a choice of splitting $s : H_*(L_{\mathcal{C}}^{\mathcal{C}}) \rightarrow H_*(L_{\mathcal{C}}^{\mathcal{C}})$ (see Definition 3.4) we can define enumerative invariants in two steps: first, we replace the category \mathcal{C} by the cyclic \mathcal{A}_∞ algebra

$$A_E = \text{End}_{\mathcal{C}}(E),$$

and then we compute categorical enumerative invariants of (A_E, s) . (Note that the splitting s induces a splitting, also denoted by s , for the Hodge filtration of A_E .)

Ideally, the above construction should not depend on the choice of E : if this were the case, we could then define

$$F_{g,n}^{\mathcal{C}, s} = F_{g,n}^{A_E, s}$$

for some choice of generator E . Since the algebras A_E and A_F are Morita equivalent for different generators E, F of \mathcal{C} , the problem reduces to the problem of showing that the invariants we defined are constant under Morita equivalences.

Let F be another generator. We then obtain inclusions

$$A_E \hookrightarrow A_E \amalg F \hookleftarrow A_F.$$

We then reduce the problem of proving Morita invariance to the problem of showing that $F_{g,n}^{A_E, s} = F_{g,n}^{A_E \amalg F, s}$.

To see this, observe that these invariants depend on two parts of the data:

- The tensors $\widehat{\beta}_{g,k,l}$'s obtained from the map ρ , see (2.14).
- The chain-level splitting $R : L \rightarrow L_+$ that lifts the splitting s .

To prove that $F_{g,n}^{A_E, s} = F_{g,n}^{A_E \amalg F, s}$ it suffices to match up the above two pieces of data. For the first part, note that the restriction of $\rho^{A_E \amalg F}$ to the subspaces $\text{Sym}^k(L_+^{A_E}[1])$ (for various $k \geq 1$) equals to ρ^{A_E} . As for the second part, let us start with a chain-level splitting for E , a chain map

$$R : (L^{A_E}, b) \rightarrow (L_+^{A_E}, b + uB).$$

Since we are over a field we can extend it to obtain a chain-level splitting

$$\widetilde{R} : (L^{A_E \amalg F}, b) \rightarrow (L_+^{A_E \amalg F}, b + uB).$$

This matches the second part: the restriction of the map \widetilde{R} to the subspace $L^{A_E} \subset L^{A_E \amalg F}$ is given by R . The formula in Corollary 5.7 then shows that indeed we have $F_{g,n}^{A_E, s} = F_{g,n}^{A_E \amalg F, s}$.

A. Explicit formulas

In this appendix we list explicit formulas for categorical invariants of Euler characteristic $\chi \geq -3$.

The formulas will be written in terms of partially directed graphs. Our conventions when drawing such a graph $(G, L_G^{\text{in}} \amalg L_G^{\text{out}}, E^{\text{dir}}, K)$ are as follows:

- we shall omit the genus decoration of a vertex if it is clear from the combinatorics of the graph;
- we shall omit the drawing of the spanning tree $K \subset E^{\text{dir}}$ if there is a unique choice of it; otherwise, the spanning tree K will be drawn in blue;
- when drawing ribbon graphs, vertices decorated by u^0 will not be marked;
- the orientation of ribbon graphs is the one described in [CC20-1].

A.1. The formula for the (0, 1, 2)-component. We begin with the case when $g = 0$ and $n = 3$. In this case there is a unique partially directed stable graph. Thus we have

$$\iota(F_{0,3}^{A,s}) = \frac{1}{2} \begin{array}{c} | \\ \swarrow \downarrow \searrow \end{array}$$

The coefficient $\frac{1}{2}$ is due to the automorphism that switches the two outputs in the stable graph. Its vertex is decorated by the tensor

$$\widehat{\beta}_{0,1,2}^A = \rho^A(\widehat{\mathcal{V}}_{0,1,2}) = -\frac{1}{2}\rho^A(\begin{array}{c} \uparrow \\ \circ \text{---} \circ \end{array})$$

using the action ρ^A on the first combinatorial string vertex $\widehat{\mathcal{V}}_{0,1,2}$, see [CCT20]. Note that the latter “T”-shaped graph is a ribbon graph, not be confused with the first graph which is a (partially directed) stable graph. (The negative sign appears due to our choice of orientation of ribbon graphs.)

A.2. The formula for the (1, 1, 0)-component. In this case there are two stable graphs. We have $\iota(F_{1,1}^{A,s})$ is given by

$$\begin{array}{c} | \\ \bullet \\ g=1 \end{array} + \frac{1}{2} \begin{array}{c} | \\ \bullet \\ \circ \end{array}$$

For the first graph the unique vertex is decorated by the image under ρ^A of the combinatorial string vertex $\widehat{\mathcal{V}}_{1,1,0}^{\text{comb}}$. It was computed in [CT17] and it is explicitly given by the following linear combination of ribbon graphs:

$$\widehat{\mathcal{V}}_{1,1,0}^{\text{comb}} = -\frac{1}{24} \begin{array}{c} \circ \\ \text{---} \times u^{-1} \\ \circ \end{array} + \frac{1}{4} \begin{array}{c} \circ \\ \text{---} \times \\ \circ \end{array}$$

A.3. The formula for the (0, 1, 3)-component. In this case we have $\iota(F_{0,4}^{A,s})$ is equal to

$$\frac{1}{3!} \begin{array}{c} | \\ \bullet \\ \swarrow \downarrow \searrow \end{array} + \frac{1}{2} \begin{array}{c} \swarrow \downarrow \searrow \\ \bullet \text{---} \bullet \\ \swarrow \downarrow \searrow \end{array}$$

Note that the coefficient $\frac{1}{2}$ disappears due to the symmetry of the two outgoing leaves on the right hand side of the stable graph. The combinatorial string vertex $\widehat{\mathcal{V}}_{0,1,3}^{\text{comb}}$ is computed explicitly in [CCT20] and it is given by

$$\widehat{\mathcal{V}}_{0,1,3}^{\text{comb}} = -\frac{1}{2} \begin{array}{c} \times \\ | \\ \circ \text{---} \circ \text{---} \circ \end{array} - \frac{1}{2} \begin{array}{c} \times \\ | \\ \circ \text{---} \circ \text{---} \times \\ | \\ \circ \end{array} + \frac{1}{2} \begin{array}{c} \times \\ | \\ \circ \text{---} \circ \text{---} \times \\ | \\ \circ \text{---} u^{-1} \end{array} + \frac{1}{6} \begin{array}{c} \times \\ | \\ \circ \text{---} \times \\ | \\ \circ \end{array}$$

A.4. The formula for the (1, 1, 1)-component. In this case we get

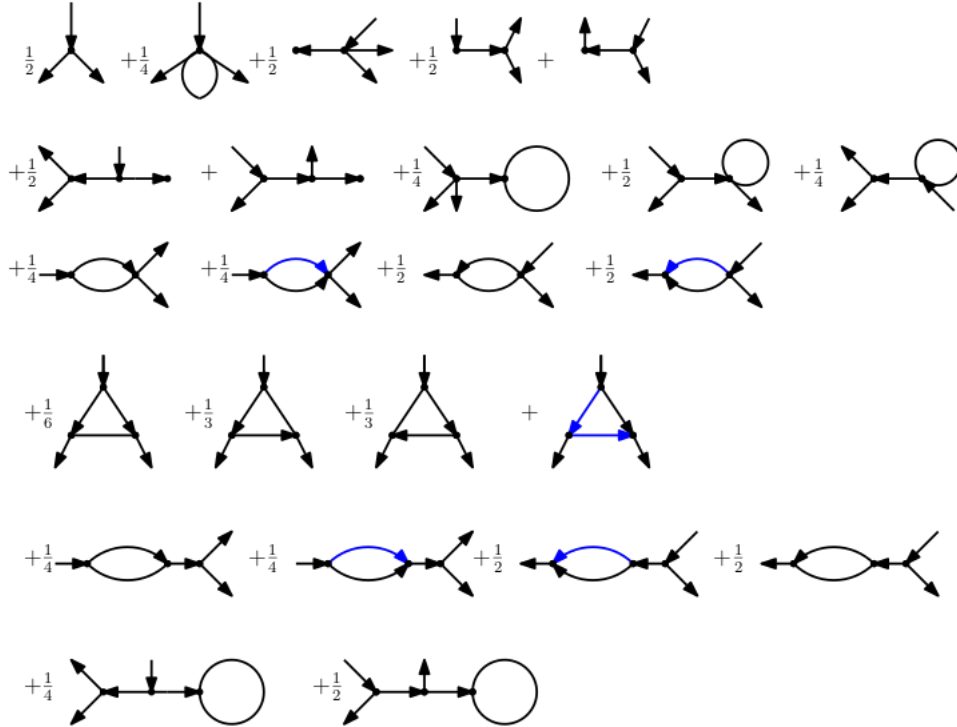
$$\iota(F_{1,2}^{A,s}) = g = 1 \begin{array}{c} | \\ \bullet \\ | \\ | \end{array} + \frac{1}{2} \begin{array}{c} | \\ | \\ \circ \end{array} + \begin{array}{c} | \\ | \\ \bullet \text{---} \bullet \text{---} g = 1 \end{array} + \\ + \frac{1}{2} \begin{array}{c} | \\ | \\ \bullet \text{---} \bullet \end{array} + \frac{1}{2} \begin{array}{c} | \\ | \\ \bullet \text{---} \bullet \end{array} + \frac{1}{2} \begin{array}{c} | \\ | \\ \bullet \text{---} \bullet \\ | \\ \circ \end{array} .$$

Observe that in the first graph of the second line, there are two directed edges between the two vertices. This explains how the tensors $\widehat{\beta}_{g,k,l}^A$ with $k \geq 2$ can contribute to the categorical enumerative invariants.

A.5. The formula for the (0, 1, 4)-component. In this case we have $\iota(F_{0,5}^{A,s})$ is given by

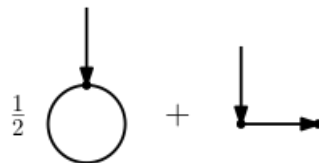
$$\frac{1}{4!} \begin{array}{c} | \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{1}{3!} \begin{array}{c} | \\ | \\ \bullet \text{---} \bullet \end{array} + \frac{1}{4} \begin{array}{c} | \\ | \\ \bullet \text{---} \bullet \end{array} + \\ + \frac{1}{2} \begin{array}{c} | \\ | \\ \bullet \text{---} \bullet \end{array} + \frac{1}{8} \begin{array}{c} | \\ | \\ \bullet \text{---} \bullet \end{array} .$$

A.6. The formula for the $(1, 1, 2)$ -component. In this case we have $\iota(F_{1,3}^{A,s})$ is given by

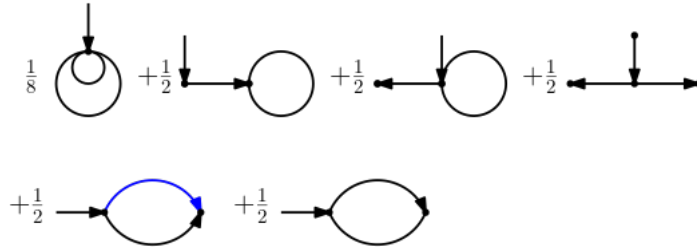


Note that in the above graphs the genus decoration is omitted because it is evident from the graph itself. For example, in the first graph there is a unique vertex of genus one. In the third graph the genus one decoration is forced on the left vertex, otherwise it would not be stable.

A.7. A partial formula for the $(2, 1, 0)$ -component. We list a few formulas in $\iota(F_{2,1}^{A,s})$ according to the number of edges in stable graphs. There is a unique star graph in $\Gamma((2, 1, 0))$ with no internal edges. Then there are terms with only one edge given by



The terms with two edges are



B. The integer-graded case

So far we have assumed that all our vector spaces were $\mathbb{Z}/2\mathbb{Z}$ -graded, and we did not concern ourselves with even graded shifts (as these do not affect signs). In this appendix we will sketch how our results need to be modified in the \mathbb{Z} -graded case, so that all the maps involved will be of a well-defined homogeneous degree. This allows us to obtain an analogue of the dimension axiom of Gromov-Witten theory in this case.

B.1. The choices we make are as follows. The formal variables u and λ have homological degree -2 . If the Calabi-Yau degree (i.e., the homological degree of the cyclic pairing) of the algebra A is d , then \hbar has degree $-2 + 2d$. When the variable \hbar is used in the context of chains on moduli spaces of curves, d will be assumed to be zero (since the chain level operator giving the Mukai pairing has degree zero), so $\deg \hbar = -2$ in that setting.

The standard Mukai pairing has homological degree zero

$$\langle -, - \rangle_{\text{Muk}} : C_*(A) \otimes C_*(A) \rightarrow \mathbb{K}.$$

Thus when viewed as an operator on $L = C_*(A)[d]$ the Mukai pairing has degree $-2d$.

B.2. With the above conventions the Weyl algebra is \mathbb{Z} -graded, because the generators of the ideal defining it are of homogeneous degree $2d - 2$. This is the reason we choose the definition of L_- to start with u^0 and not u^{-1} , and we take the residue pairing to take the coefficient of u^1 and not u^{-1} .

B.3. Taking into account the shifts in the definition of $\widehat{\mathfrak{h}}$ in [CCT20] we can rewrite its definition as

$$\widehat{\mathfrak{h}} = \bigoplus_{k \geq 1, l} \text{Hom}^c(\text{Sym}^k(L_+[1 - 2d]), \text{Sym}^l L_-) [2 - 2d][[\hbar, \lambda]].$$

Now assume that the splitting s , its chain-level lift R , and the inverse T of R all preserve degrees. This implies that

- $H^{\text{Sym}} : \text{Sym}^2 L_- \rightarrow \mathbb{K}$ has degree $2 - 2d$ in Proposition 4.5;
- $S : L_- \rightarrow L_+[1 - 2d]$ has degree $2 - 2d$ in Proposition 4.17;
- $F : L_- \rightarrow L_+[1 - 2d]$ has degree $3 - 2d$ in Proposition 4.17.

Using these facts it is easy to verify that the maps $\widehat{\mathcal{K}}_m$ have degree zero, as desired.

B.4. We can now use Theorem 5.3 to prove that the categorical enumerative invariants $F_{g,n}^{A,s}$ satisfy the dimension axiom of Gromov-Witten theory when the A_∞ -algebra is \mathbb{Z} -graded. More precisely we have the following.

B.5. Theorem. *Assume that A is \mathbb{Z} -graded, of Calabi-Yau dimension d , and assume that the splitting s preserves degrees. Then*

$$\deg F_{g,n}^{A,s} = 2(g-1)(3-d) + 2n$$

as an element of $\text{Sym}^n H_-$.

Proof. The element

$$\widehat{\mathcal{V}} = \sum_{g,k \geq 1, l} \widehat{\mathcal{V}}_{g,k,l} \hbar^g \lambda^{2g-2+k+l}$$

is a solution of the Maurer-Cartan equation in $\widehat{\mathfrak{g}}$, so in particular its (unshifted) degree is -1 . Thus the degree of its component $\widehat{\mathcal{V}}_{g,k,l}$ of genus g , k inputs and l outputs is

$$\deg \widehat{\mathcal{V}}_{g,k,l} = 6g - 7 + 3k + 2l.$$

Its image under ρ^A will have degree

$$\deg \widehat{\beta}_{g,k,l}^A = (6g - 7 + 3k + 2l) + d(2 - 2g - 2k).$$

A simple inspection shows that in $\widehat{\mathfrak{h}}$ (with the further shift by $2 - 2d + 2dk - k$, and with \hbar of degree $2d - 2$) the element $\widehat{\beta}_{g,k,l}^A \hbar^g \lambda^{2g-2+k+l}$ has degree -1 .

We conclude that $\widehat{\beta}^A$ is a Maurer-Cartan element in the \mathbb{Z} -graded DGLA $\widehat{\mathfrak{h}}$. The splitting s is degree preserving, so we can find a chain-level lift of it R that also preserves degrees. This implies that the inverse T of R also preserves degrees, and hence the L_∞ morphism $\widehat{\mathcal{K}}$ has degree zero.

Theorem 5.3 implies that

$$F^{A,s} = \sum_{g,n} F_{g,n}^{A,s} \hbar^g \lambda^{2g-2+n},$$

as a Maurer-Cartan element of $\mathfrak{h}^{\text{Triv}} = \text{Sym} H_- \llbracket \hbar, \lambda \rrbracket [1 - 2d]$, also has degree -1 . When we consider $F_{g,n}^{A,s}$ as an element of $\text{Sym}^n H_-$ its degree is given by

$$\begin{aligned} \deg(F_{g,n}^{A,s}) &= (-1) + (2d - 1) - \deg(\hbar) \cdot g - \deg(\lambda) \cdot (2g - 2 + n) \\ &= (2d - 2) - (2d - 2)g + 2(2g - 2 + n) = 2(g - 1)(3 - d) + 2n. \end{aligned}$$

This indeed matches with the virtual dimension formula in Gromov-Witten theory. \square

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