

# Categorical Enumerative Invariants, I: String vertices

ANDREI CĂLDĂRARU, KEVIN COSTELLO AND JUNWU TU

ABSTRACT: We define combinatorial counterparts to the geometric string vertices of Sen-Zwiebach and Costello-Zwiebach, which are certain closed subsets of the moduli spaces of curves. Our combinatorial vertices contain the same information as the geometric ones, are effectively computable, and act on the Hochschild chains of a cyclic  $A_\infty$ -algebra.

This is the first in a series of two papers where we define enumerative invariants associated to a pair consisting of a cyclic  $A_\infty$ -algebra and a splitting of the Hodge filtration on its cyclic homology. These invariants conjecturally generalize the Gromov-Witten and Fan-Jarvis-Ruan-Witten invariants from symplectic geometry, and the Bershadsky-Cecotti-Ooguri-Vafa invariants from holomorphic geometry.

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## 1. Introduction

**1.1.** This is the first in a series of two papers whose purpose is to define enumerative invariants associated to a pair  $(A, s)$  consisting of a cyclic  $A_\infty$ -algebra  $A$  and a splitting  $s$  of the Hodge filtration on its periodic cyclic homology.

These invariants conjecturally generalize

- the Gromov-Witten invariants of a symplectic space  $X$  ( $A$  is obtained from the Fukaya category of  $X$ );

- the Bershadsky-Cecotti-Ooguri-Vafa invariants of a complex Calabi-Yau manifold  $\check{X}$  ( $A$  is obtained from the derived category of coherent sheaves on  $\check{X}$ );
- Saito's  $g = 0$  invariants associated to an isolated singularity ( $A$  is obtained from the corresponding category of matrix factorizations);
- the B-model analogue of the Fan-Jarvis-Ruan-Witten invariants (same as previous case).

In this paper we focus on defining combinatorial string vertices and their action on Hochschild chains of  $A$ . These will be used in the follow-up paper [CT20] to give a constructive definition of the categorical enumerative invariants, using the action of the Givental group on the space of string field theories.

**1.2.** String vertices were introduced in physics by Sen-Zwiebach [SZ94] as an essential part of their construction of string field theory. The original approach defined string vertices as certain subsets  $\mathcal{V}_{g,n} \subset M_{g,n}$  parametrizing curves that satisfy a geometric condition in terms of minimal area metrics. Another geometric construction of string vertices, this time using hyperbolic geometry, was recently described by Costello-Zwiebach [CZ19].

Intuitively, the string vertex  $\mathcal{V}_{g,n} \subset M_{g,n}$  is the complement, in the compactified moduli space of curves, of an open tubular neighborhood of the boundary. For example, the string vertex  $\mathcal{V}_{1,1}$  consists of the points in  $M_{1,1}$  which are at least  $\varepsilon$  away (with respect to some metric on  $\overline{M}_{1,1}$ ) from the point corresponding to the nodal curve. As a singular chain  $\mathcal{V}_{1,1}$  is not closed. Its boundary (a real circle) can be thought of as consisting of all elliptic curves that are obtained by *twist-sewing* (see Section 3) on two boundary circles of the unique surface in  $M_{0,3}$ .

**1.3.** This idea was expanded in [SZ94] where it was noted that, appropriately defined, the total string vertex

$$\mathcal{V} = \sum_{g,n \geq 1} \mathcal{V}_{g,n} \hbar^g \lambda^{2g-2+n}$$

should satisfy a form of the quantum master equation (QME)

$$\partial \mathcal{V} + \hbar \Delta \mathcal{V} + \frac{1}{2} \{ \mathcal{V}, \mathcal{V} \} = 0$$

in a Batalin-Vilkovisky (BV) algebra constructed from the spaces of singular chains  $C_*(M_{g,n})$ . Here, and in the sequel,  $C_*$  will denote the functor of normalized singular chains, and  $\lambda$  and  $u$  will be formal variables of homological degree  $-2$ . The variable  $\hbar$  will have degree  $-2$  in this setting, and degree  $-2 + 2d$  in the setting of an  $A_\infty$ -algebra (see Theorem B below).

This observation led Costello to a homological definition of string vertices. Let  $\mathfrak{g}$  denote the differential graded (dg) vector space

$$\mathfrak{g} = \left( \bigoplus_{g,n} C_*(M_{g,n}^{\text{fr}})_{(S^1)^n \times \Sigma_n} [1] \right) \llbracket \hbar, \lambda \rrbracket,$$

where  $M_{g,n}^{\text{fr}}$  is the moduli space of curves of genus  $g$  with  $n$  framed marked points. It is an  $(S^1)^n$  bundle over the usual moduli space of curves  $M_{g,n}$ .

The space  $\mathfrak{g}$  carries geometric operations  $\Delta, \{-, -\}$  making it into (a shift of) the primitive part of a BV algebra. The geometrically-defined string vertices are a solution to the QME. Costello [Cos09] shows that there is a unique solution  $\mathcal{V}$  to the QME, up to homotopy, once we fix the initial condition

$$\mathcal{V}_{0,3} = \frac{1}{6} [M_{0,3}^{\text{fr}}] \lambda^1.$$

Using this result Costello defines the string vertices  $\mathcal{V}_{g,n}$  as the coefficients of the expansion of  $\mathcal{V}$  in terms of  $g$  and  $n$ , see [Cos09, Theorem 1].

Note that we do not need the full BV algebra structure discussed above: all we need is the structure of differential graded Lie algebra (DGLA) on  $\mathfrak{g}$  given by

$$\mathfrak{g} = (\mathfrak{g}, \partial + \hbar\Delta, \{-, -\}).$$

The string vertex  $\mathcal{V}$  is then the unique solution of the Maurer-Cartan equation in this DGLA satisfying the initial condition above.

The goal of this paper and the sequel [CT20] is to construct the string vertices combinatorially, in the ribbon graph model for moduli spaces. We will leverage the fact that string vertices are unique to provide an algorithm to construct them explicitly. The main goal of this paper is to explain how to overcome an important technical difficulty, which we now discuss.

**1.4.** Both the geometric construction of string vertices of Sen-Zwiebach and the homological construction of Costello have two important drawbacks. One is that it is not clear how to compute the string vertices  $\{\mathcal{V}_{g,n}\}$  explicitly. Another is that it is *a priori* difficult to do computations in two-dimensional topological field theory (2d TFT) using either version of string vertices.

The first problem comes from the fact that even though the dg vector spaces  $C_*(M_{g,n}^{\text{fr}})$  admit combinatorial descriptions in terms of ribbon graphs, the operators  $\Delta, \{-, -\}$  do not have direct descriptions in these terms. (They do not respect the cell decomposition of the moduli spaces of curves given by ribbon graphs.) Hence the QME cannot be translated directly into an equation for ribbon graphs.

The second issue is more conceptual. In the process of constructing enumerative invariants of a pair  $(A, s)$  we need to let the string vertices act on the space  $L = CC_*(A)[d]$  of shifted Hochschild chains of  $A$ . (Here  $d$  denotes the Calabi-Yau dimension of  $A$ . With either the geometric or homological string vertices defined above this is not possible.

The problem is that while the algebra  $A$  gives rise to a *positive boundary* 2d TFT, described explicitly in terms of ribbon graphs by Kontsevich-Soibelman [KS09] and Costello [Cos07], the homological string vertices live in the zero-input part of the TFT, so it is not clear how they act on  $L$ . More explicitly, the positive boundary 2d TFT obtained from  $A$  is encoded in a PROP action, given by degree zero operations

$$\rho_{g,k,l}^A : C_* \left( M_{g,k,l}^{\text{fr}} \right) [d \cdot (2 - 2g - 2k)] \longrightarrow \text{Hom} \left( L^{\otimes k}, L^{\otimes l} \right)$$

for every  $g, k \geq 1, l$ . Here  $M_{g,k,l}^{\text{fr}}$  denotes the moduli space of Riemann surfaces of genus  $g$  with  $k$  incoming and  $l$  outgoing framed boundaries. The homological string vertex  $\mathcal{V}_{g,n}$  is naturally interpreted as a chain in  $C_*(M_{g,0,n})$ ; the positive boundary condition on the TFT defined from  $A$  means that it does not act on  $L$  in any obvious way.

**1.5.** We note that Lurie [Lur09] shows that one can remove the constraint that  $k \geq 1$  under the hypothesis that the algebra  $A$  is smooth as well as proper. However, we do not know how to make Lurie's construction *explicit* in any combinatorial model for the moduli of Riemann surfaces.

**1.6.** Both issues mentioned above arise from a common root problem, namely that the calculations we want to carry out (finding string vertices, and computing their action on Hochschild chains) are described in terms of the spaces  $C_*(M_{g,0,n})$ . These spaces do not have an effective combinatorial description, even though for  $k \geq 1$  the spaces  $C_*(M_{g,k,l})$  do. The problems would be resolved if we had a mechanism for replacing computations for  $M_{g,0,n}$  by computations for  $M_{g,k,l}$  for  $k \geq 1$ .

The solution we propose is to use a resolution  $\widehat{\mathfrak{g}}$  of the DGLA  $\mathfrak{g}$ . We shall call  $\widehat{\mathfrak{g}}$  the *Koszul resolution* of  $\mathfrak{g}$ . Since finding the string vertices amounts to solving the Maurer-Cartan equation in  $\mathfrak{g}$ , solving it in  $\widehat{\mathfrak{g}}$  gives equivalent information.

**1.7.** First we need to introduce some notation. Let  $k$  and  $l$  be non-negative integers, let  $n = k + l$ , and let  $V$  be a dg vector space endowed with an action of  $(S^1)^n \rtimes (\Sigma_k \times \Sigma_l)$ . Examples of such spaces include  $C_*(M_{g,k,l}^{\text{fr}})$  and  $L^{\otimes n}$ . (We refer the reader to Section 2 for details on circle actions on dg vector spaces.)

The homotopy quotient of  $V$  by the group  $(S^1)^n \rtimes (\Sigma_k \times \Sigma_l)$  will be denoted by  $V_{\text{hS}}$ . This notation is ambiguous, but  $k$  and  $l$  will be evident from context. In this quotient we let  $\Sigma_k$  and  $\Sigma_l$  act via the alternating and trivial representations, respectively. If the original differential on  $V$  is  $d$ , we will denote the differential on  $V_{\text{hS}}$  by  $d + uB$ .

The following are the main theorems of this paper.

**1.8. Theorem A.** *For any  $g, k, l$  non-negative integers, including  $k = 0$ , let  $V_{g,k,l} = C_*(M_{g,k,l}^{\text{fr}})_{\text{hS}}$  and define*

$$\mathfrak{g} = \left( \bigoplus_{g,n \geq 1} V_{g,0,n}[1] \right) \llbracket \hbar, \lambda \rrbracket$$

and

$$\widehat{\mathfrak{g}} = \left( \bigoplus_{g,k \geq 1, l} V_{g,k,l}[2-k] \right) \llbracket \hbar, \lambda \rrbracket.$$

Then there are operators

$$\Delta : V_{g,k,l} \rightarrow V_{g+1,k,l-2}[-1]$$

$$\{-, -\} : V_{g,0,n} \otimes V_{g',0,n'} \rightarrow V_{g+g',0,n+n'-2}[-1]$$

$$\iota : V_{g,k,l} \rightarrow V_{g,k+1,l-1}$$

$$\{-, -\}_i : V_{g,k,l} \otimes V_{g',k',l'} \rightarrow V_{g+g'+i-1,k+k'-i,l+l'-i}[-i] \text{ for } k \geq 1, k' \geq 1, i \geq 1$$

which endow  $\mathfrak{g}$  and  $\widehat{\mathfrak{g}}$  with DGLA structures given by

$$\mathfrak{g} = (\mathfrak{g}, \partial + uB + \hbar\Delta, \{-, -\})$$

and

$$\widehat{\mathfrak{g}} = \left( \widehat{\mathfrak{g}}, \partial + uB + \iota + \hbar\Delta, \{-, -\}_1 + \frac{1}{2!}\hbar\{-, -\}_2 + \frac{1}{3!}\hbar^2\{-, -\}_3 + \dots \right).$$

The map  $\iota : \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$  is a quasi-isomorphism of DGLAs. Hence the Maurer-Cartan equation in  $\widehat{\mathfrak{g}}$  admits a solution

$$\widehat{\mathcal{V}} = \sum_{g,k \geq 1, l} \widehat{\mathcal{V}}_{g,k,l} \hbar^g \lambda^{2g-2+k+l}$$

which is unique up to homotopy once we fix

$$\widehat{\mathcal{V}}_{0,1,2} = \iota(\mathcal{V}_{0,3}) = \frac{1}{2} [M_{0,1,2}^{\text{fr}}] \lambda^1.$$

**1.9. Remark.** The difference of a factor of three between the initial conditions of the Maurer-Cartan equations in  $\mathfrak{g}$  and in  $\widehat{\mathfrak{g}}$  is explained by the fact that  $\iota$  relabels outputs into inputs, and there are three outputs in  $\mathcal{V}_{0,3}$  to be relabeled.

**1.10. Theorem B.** *Let  $A$  be a smooth and proper cyclic  $A_\infty$ -algebra for which the Hodge-de Rham degeneration property holds. If  $d$  is the Calabi-Yau dimension of  $A$  denote by  $L = CC_*(A)[d]$  the  $d$ -shifted Hochschild chain complex of  $A$ . The variable  $\hbar$  will be of degree  $-2 + 2d$ .*

For any  $k, l$  non-negative integers, including  $k = 0$ , let  $W_{k,l}$  be the dg vector space

$$W_{k,l} = \text{Hom} (L^{\otimes k}, L^{\otimes l})_{\text{hS}}.$$

Define dg vector spaces

$$\mathfrak{h} = \left( \bigoplus_n W_{0,n}[1 - 2d] \right) \llbracket \hbar, \lambda \rrbracket$$

and

$$\widehat{\mathfrak{h}} = \left( \bigoplus_{k \geq 1, l} W_{k,l}[2 - 2d + 2dk - k] \right) \llbracket \hbar, \lambda \rrbracket.$$

Then there are operators

$$\begin{aligned} \Delta &: W_{k,l} \rightarrow W_{k,l-2}[2d - 1] \\ \{-, -\} &: W_{0,n} \otimes W_{0,n'} \rightarrow W_{0,n+n'-2}[2d - 1] \\ \iota &: W_{k,l} \rightarrow W_{k+1,l-1}[2d] \\ \{-, -\}_i &: W_{k,l} \otimes W_{k',l'} \rightarrow W_{k+k'-i, l+l'-i}[-i] \text{ for } k \geq 1, k' \geq 1, i \geq 1 \end{aligned}$$

which endow  $\mathfrak{h}$  and  $\widehat{\mathfrak{h}}$  with the structure of DGLAs

$$\begin{aligned} \mathfrak{h} &= (\mathfrak{h}, b + uB + \hbar\Delta, \{-, -\}), \\ \widehat{\mathfrak{h}} &= \left( \widehat{\mathfrak{h}}, b + uB + \iota + \hbar\Delta, \{-, -\}_1 + \frac{1}{2!}\hbar\{-, -\}_2 + \frac{1}{3!}\hbar^2\{-, -\}_3 + \dots \right). \end{aligned}$$

The map  $\iota : \mathfrak{h} \rightarrow \widehat{\mathfrak{h}}$  is a quasi-isomorphism of DGLAs.

**1.11.** The DGLA structures on  $\mathfrak{g}$  and  $\mathfrak{h}$  are well known, having already appeared in [Cos09]. These structures arise from the fact that the spaces  $V_{g,0,n}$  and  $W_{0,n}$  admit actions of the *category* of annuli: sewing the degree one annulus with one input and one output gives the circle action, and sewing the one with two inputs induces the  $\Delta$  operator.

The new idea in this paper is that we have a bit more structure, because the collection of annuli forms a *PROP* which acts. Sewing just one input of the degree zero annulus with two inputs defines the  $\iota$  operator. It effectively relabels an output into an input, in all possible ways; on Hochschild chains it acts by a formula that is extremely similar to that of the Koszul differential. This operator is the crucial new ingredient in the definition of the resolutions  $\widehat{\mathfrak{g}}$  and  $\widehat{\mathfrak{h}}$ .

**1.12.** The DGLA  $\widehat{\mathfrak{g}}$  and in particular the operators  $\Delta$ ,  $\iota$ ,  $\partial$ ,  $B$ ,  $\{-, -\}_i$  admit combinatorial descriptions in terms of ribbon graphs. Thus the string vertex  $\widehat{\mathcal{V}}$  can be computed recursively as combinations of ribbon graphs, starting from the initial condition. Using the Kontsevich-Soibelman [KS09] action  $\rho^A$  (as described in detail in [CC20]) we obtain explicitly computable tensors

$$\widehat{\alpha}_{g,k,l}^A = \rho^A(\widehat{\mathcal{V}}_{g,k,l}) \in W_{k,l} = \text{Hom}(L^{\otimes k}, L^{\otimes l})_{\text{hS}} = \text{Hom}(\wedge^k L_+, \text{Sym}^l L_-).$$

Here  $L_+ = uL[[u]]$  and  $L_- = L[u^{-1}]$ .

As the image of the Maurer-Cartan element  $\widehat{\mathcal{V}}$ , the element

$$\widehat{\alpha} = \sum_{g,k \geq 1, l} \widehat{\alpha}_{g,k,l}^A \hbar^g \lambda^{2g-2+k+l}$$

also satisfies the Maurer-Cartan equation in  $\widehat{\mathfrak{h}}$ . (One can verify carefully that it has degree  $-1$  in  $\widehat{\mathfrak{h}}$ .)

**1.13.** For the use in [CT20] it will be more convenient to replace the spaces

$$W_{k,l} = \text{Hom}(\wedge^k L_+, \text{Sym}^l L_-)$$

in the definition  $\widehat{\mathfrak{h}}$  by the isomorphic ones (up to a shift by  $-k$ )

$$W'_{k,l} = \text{Hom}(\text{Sym}^k(L_+[1]), \text{Sym}^l L_-).$$

The resulting DGLA will still be denoted by  $\widehat{\mathfrak{h}}$

$$\widehat{\mathfrak{h}} = \left( \bigoplus_{k \geq 1, l} W'_{k,l}[2 - 2d + 2dk] \right) [[\hbar, \lambda]].$$

The old Maurer-Cartan element is identified with the element

$$\widehat{\beta} = \sum_{g,k \geq 1, l} \widehat{\beta}_{g,k,l}^A \hbar^g \lambda^{2g-2+k+l}$$

in the new setting. The individual components  $\widehat{\beta}_{g,k,l}^A \in \text{Hom}(\text{Sym}^k(L_+[1]), \text{Sym}^l L_-)$  are related to the old ones by the signs

$$\widehat{\beta}_{g,k,l}^A(x_1, \dots, x_k) = (-1)^\epsilon \cdot \widehat{\alpha}_{g,k,l}^A(x_1, \dots, x_k)$$

where

$$\epsilon = \sum_{i=1}^k (k-i)|x_i|.$$

The tensors  $\widehat{\beta}_{g,k,l}^A$  will be one of the main ingredients in the definition of the categorical invariants in [CT20].

**1.14. Standing assumptions.** We work over the field  $\mathbb{Q}$  of rational numbers. Whenever a pair  $(g, n)$  or a triple  $(g, k, l)$  of non-negative integers appears (where  $g$  refers to the genus of a curve) it will be assumed that  $2g - 2 + n > 0$ ,  $2g - 2 + k + l > 0$ . In any summation over such triples, unconstrained variables are assumed to be non-negative. Thus  $\sum_{g,n \geq 1}$  means the sum is over all choices of  $g \geq 0$ ,  $n \geq 1$ ,  $2g - 2 + n > 0$ .

We use homological grading conventions: for a graded vector space  $V = \bigoplus V_n$ ,  $V[k]$  is the graded vector space whose  $n$ -th graded piece is  $V_{n-k}$ . Hence a degree zero map  $V \rightarrow W[k]$  decreases degree by  $k$ .

Outside of the introduction we will only work with the reduction of the grading to  $\mathbb{Z}/2\mathbb{Z}$ . In particular we will ignore even shifts (which do not affect signs). The degree shifts chosen in Theorems A and B have been carefully designed to work for the  $\mathbb{Z}$ -grading.

In ribbon graph diagrams an input or output without a label is assumed to have been labeled by  $u^0$ . For a univalent white vertex we do not mark the starting half-edge (since it is uniquely determined).

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## 2. DGLAs induced by actions of the PROP of annuli

In this section we define the PROP of annuli  $Ann^+$  and we give two different constructions of DGLAs associated to an action of  $Ann^+$  onto another PROP  $\mathcal{P}$ . The first construction yields the DGLAs  $\mathfrak{g}, \mathfrak{h}$  in the previous section; the second construction gives rise to their resolutions  $\widehat{\mathfrak{g}}, \widehat{\mathfrak{h}}$ .

Intuitively, both these constructions can be viewed as the results of a two-part process. First, the action of the generator  $S$  of  $Ann^+$  gives rise to homological circle actions on the inputs and outputs of the operations in  $\mathcal{P}(m, n)$ . The composition of  $\mathcal{P}$  does not descend to the homotopy quotient  $\mathcal{P}_{hS}$  by these circle actions, but twisted sewing (composing with a circle action in between) does. Second, the other generator  $M$  of  $Ann^+$  induces two new structures on the quotient  $\mathcal{P}_{hS}$  – twisted sewing of two outputs, and relabeling of an output to an input. These basic structures on  $\mathcal{P}_{hS}$  allow us to define the two types of DGLAs.

**2.1.** We will freely use the language of PROPs in this section. We refer to [Mar06] for generalities on PROPs. If  $\mathcal{P}$  is a PROP we use  $\otimes$  to denote horizontal composition:

$$\otimes : \mathcal{P}(n_1, m_1) \otimes \cdots \otimes \mathcal{P}(n_s, m_s) \rightarrow \mathcal{P}(n_1 + \cdots + n_s, m_1 + \cdots + m_s)$$

and

$$\circ : \mathcal{P}(m, l) \otimes \mathcal{P}(n, m) \rightarrow \mathcal{P}(n, l)$$

the vertical composition. By definition, the space  $\mathcal{P}(n, m)$  is a  $\Sigma_n \times \Sigma_m$ -bimodule. It is convenient to let  $\Sigma_n$  act on the right, while  $\Sigma_m$  will act on the left.

The unit element will be denoted by  $\text{id} \in \mathcal{P}(1, 1)$ . For  $x \in \mathcal{P}(n, m)$ ,  $y \in \mathcal{P}(m', l)$  we define the following notations:

- If  $m' \leq m$ ,

$$y \circ_{(i_1, \dots, i_{m'})} x = (y \otimes \underbrace{\text{id} \otimes \cdots \otimes \text{id}}_{(m-m')\text{-copies}}) \circ (\sigma \cdot x),$$

where  $(i_1, \dots, i_{m'})$  is an ordered  $m'$ -tuple of distinct elements in  $\{1, 2, \dots, m\}$ , and  $\sigma \in \Sigma_m$  is the permutation  $(1, i_1) \cdots (m', i_{m'})$ .

- If  $m' \geq m$ ,

$$y_{(j_1, \dots, j_m)} \circ x = (y \cdot \sigma) \circ (x \otimes \underbrace{\text{id} \otimes \dots \otimes \text{id}}_{(m' - m)\text{-copies}}),$$

where  $(j_1, \dots, j_m)$  is an ordered  $m$ -tuple of distinct elements in  $\{1, 2, \dots, m'\}$ , and  $\sigma \in \Sigma_{m'}$  is the permutation  $(1, j_1) \cdots (m, j_m)$ .

**2.2. The PROP of annuli  $\mathcal{A}nn^+$ .** The dg PROP  $\mathcal{A}nn^+$  is defined by generators and relations as follows. As a (unital) dg PROP it is generated by two operations:

- (1) A degree one element  $S \in \mathcal{A}nn^+(1, 1)$  such that  $S \circ S = 0$ .
- (2) A degree zero element  $M \in \mathcal{A}nn^+(2, 0)$  which is invariant under the action of the symmetric group  $\Sigma_2$ .

We put the zero differential on  $\mathcal{A}nn^+$ . The superscript “+” is used to indicate that we only allow annuli with positive number of inputs.

Since  $\Sigma_2$  acts trivially on  $M$  we have

$$M \circ_1 S = M \circ (S \otimes \text{id}) = M \circ (\text{id} \otimes S) = M \circ_2 S.$$

For this reason we shall simply write  $\mathbb{M} = M \circ S$  for either one of them.

**2.3.** One can think of this PROP as the sub-PROP of the PROP of chains on moduli spaces of curves which only includes annuli with positive boundary. Since this PROP is formal, we can replace it by its homology, which is what we have described above.

**2.4.** According to the above description of  $\mathcal{A}nn^+$ , to give an  $\mathcal{A}nn^+$ -algebra structure on a dg vector space  $(V, b)$ , i.e., a PROP morphism

$$\lambda : \mathcal{A}nn^+ \rightarrow \text{End}(V),$$

is equivalent to giving the following data:

- A degree one map  $B = \lambda(S) : V \rightarrow V$  such that  $B \circ B = 0$ , and  $bB + Bb = 0$ .
- A degree zero symmetric pairing  $\langle -, - \rangle = \lambda(M) : V \otimes V \rightarrow \mathbb{C}$  such that for any  $x, y \in V$  we have

$$\begin{aligned} \langle bx, y \rangle + (-1)^{|x|} \langle x, by \rangle &= 0, \\ \langle Bx, y \rangle - (-1)^{|x|} \langle x, By \rangle &= 0. \end{aligned}$$

We emphasize that not all PROPs with  $\mathcal{A}nn^+$  action are of this form, only  $\mathcal{A}nn^+$ -algebras. For example the PROP  $\mathcal{S}$  in the next section is not of this type.

**2.5. Circle actions.** We now recall some basic constructions regarding circle actions. Let  $X$  be a topological space endowed with an  $(S^1)^n$ -action. For  $1 \leq i \leq n$  let  $B_i : C_*(X) \rightarrow C_{*+1}(X)$  denote the map that assigns to an  $m$ -chain  $\sigma : \Delta^m \rightarrow X$  the  $m+1$ -chain defined by

$$S^1 \times \Delta^m \xrightarrow{\text{id} \times \sigma} S^1 \times X \rightarrow X,$$

with the second map given by the  $i$ -th circle action. Since we use the normalized chain complex, we have  $B_i^2 = 0$ . We shall refer to each operator  $B_i$  as a homological circle action, or simply circle action.

The  $(S^1)^n$ -equivariant homology of  $X$  is computed by the chain complex

$$C_*(X)_{(S^1)^n} = (C_*(X)[u_1^{-1}, \dots, u_n^{-1}], \partial + \sum_{i=1}^n u_i \cdot B_i)$$

which we will denote by  $C_*(X)_{(S^1)^n}$ . There is a map of complexes

$$\pi \circ \text{Res} : C_*(X)_{(S^1)^n} \rightarrow C_*(X/(S^1)^n)$$

which takes the modified residue  $\text{Res}$  of an equivariant chain (the coefficient of  $(u_1 \cdots u_n)^0$ ) and applies to it the canonical projection map  $\pi : C_*(X) \rightarrow C_*(X/(S^1)^n)$ . If the  $(S^1)^n$ -action is free, then this map is a quasi-isomorphism.

Note the slightly unusual convention where the equivariant homology is computed by complexes starting with  $u^0$ , not  $u^{-1}$ . Throughout this paper the operator of taking residues means taking the coefficient of  $u^0$  and not of  $u^{-1}$ . This is done in order to match notation with [CT20].

**2.6. The twisted sewing operation.** For the rest of this section we will place ourselves in the setting where we have a dg PROP  $\mathcal{P}$  endowed with an  $\text{Ann}^+$  action; in other words we have a morphism of PROPs  $\lambda : \text{Ann}^+ \rightarrow \mathcal{P}$ .

For any operation in  $\mathcal{P}(m, n)$  we will think of the operation of composing with  $\lambda(S)$ , on either the inputs or the outputs, as acting by a circle rotation. (This is justified by the fact that these compositions give rise to homological circle actions on  $\mathcal{P}(m, n)$ .) Moreover, in addition to the usual composition of  $\mathcal{P}$  which sews an output of an operation to an input of another, the  $\text{Ann}^+$  action also allows us to sew two outputs of the same operation of  $\mathcal{P}$ . This is accomplished by composing these two outputs with the two inputs of  $\lambda(M)$ . We obtain a structure similar in nature to that of a modular operad (though not exactly the same, because we do not allow sewing of two inputs).

Our goal is to pass to the quotient  $\mathcal{P}_{\text{hS}}$  of  $\mathcal{P}$  by the circle actions induced from  $\text{Ann}^+$ , in such a way that the two sewing operations described above would descend

to the quotient. We take our cue from the theory of moduli spaces of curves. Sewing operations are well defined on  $M_{g,n}^{\text{fr}}$ , but after passing to the quotient  $M_{g,n}$  of  $M_{g,n}^{\text{fr}}$  by the circle actions these operations are no longer well defined. However, there is a modification of these sewing operations that does descend, at least at the level of singular chains: the *twisted sewing* operation, which first performs an  $S^1$  twist before sewing.

This suggests that on the quotient  $\mathcal{P}_{\text{hS}}$  there will be two fundamental types of new compositions which we will call twisted sewings. The twisted self-sewing  $\Delta$  is obtained by sewing two outputs of an operation with  $\lambda(\mathbb{M})$ . The twisted sewing between two operations is obtained by composing an output of one with a circle action  $\lambda(S)$  and then with an input of the other.

These twisted sewings are the building blocks of the DGLAs that we construct in the rest of this section from a PROP with  $\text{Ann}^+$  action.

**2.7. A first DGLA associated to a PROP with  $\text{Ann}^+$  action.** We now make the above ideas precise. Let  $\mathcal{P}$  be a PROP with  $\text{Ann}^+$  action, and consider the space  $\mathcal{P}(0, n)$  of operations with zero inputs and  $n$  outputs. For each  $1 \leq j \leq n$  we obtain a homological circle action on  $\mathcal{P}(0, n)$  by composing with  $\lambda(S)$  at the  $j$ -th output. Denote this circle action by

$$B_j : \mathcal{P}(0, n) \rightarrow \mathcal{P}(0, n), \quad B_j(x) = \lambda(S) \circ_j x.$$

The associated  $(S^1)^n$ -equivariant chain complex is of the form

$$\mathcal{P}(0, n)_{(S^1)^n} = \left( \mathcal{P}(0, n)[u_1^{-1}, \dots, u_n^{-1}], \partial + \sum_{i=1}^n u_i \cdot B_i \right),$$

where  $\partial$  is the boundary map of  $\mathcal{P}(0, n)$ . The symmetric group  $\Sigma_n$  still acts by the PROP structure on  $\mathcal{P}(0, n)_{(S^1)^n}$  while also permuting the indices of the circle parameters  $u_1, \dots, u_n$ . Following our conventions we denote by  $\mathcal{P}(0, n)_{\text{hS}}$  the homotopy quotient of  $\mathcal{P}(0, n)$  by the semidirect product  $(S^1)^n \rtimes \Sigma_n$ -action.

We shall define a DGLA structure on the graded vector space

$$\mathfrak{g}^{\mathcal{P}} = \left( \bigoplus_{n \geq 0} \mathcal{P}(0, n)_{\text{hS}}[1] \right) \llbracket \hbar \rrbracket.$$

First, for an element

$$x = \sum_{k_1, \dots, k_n \geq 1} x_{k_1 \dots k_n} u_1^{-k_1} \cdots u_n^{-k_n} \in \mathcal{P}(0, n)[u_1^{-1}, \dots, u_n^{-1}]$$

we set

$$\begin{aligned}\Delta(x) &= \sum_{1 \leq i < j \leq n} \text{Res}_{u_i=0, u_j=0} \lambda(\mathbb{M}) \circ_{(i,j)} x \\ &= \sum_{\substack{1 \leq i < j \leq n \\ k_i=k_j=0}} \lambda(\mathbb{M}) \circ_{(i,j)} x_{k_1 \dots k_n} u_1^{-k_1} \dots \widehat{u_i^0} \dots \widehat{u_j^0} \dots u_n^{-k_n}.\end{aligned}$$

In other words the operator  $\Delta$  performs a twisted sewing operation on all pairs of outputs of  $x$  that are labeled by  $u^0$ . The operator  $\Delta$  is  $\Sigma_n$ -invariant, hence it induces a map on quotients which we still denote by  $\Delta$ ,

$$\Delta : \mathcal{P}(0, n)_{\text{hS}} \rightarrow \mathcal{P}(0, n-2)_{\text{hS}}[-1].$$

**2.8. Lemma.** *The sum  $\partial + \sum_{i=1}^n u_i \cdot B_i + \hbar \Delta$  is a differential, i.e. we have  $(\partial + \sum_{i=1}^n u_i \cdot B_i + \hbar \Delta)^2 = 0$ .*

**Proof.** We need to prove that  $[\partial, \Delta] = 0$ ,  $[B_i, \Delta] = 0$ , and  $\Delta^2 = 0$ . The first identity follows from the fact that  $\mathbb{M}$  is closed. Since  $(M \circ S) \circ S = M \circ (S \circ S) = 0$  the second identity holds. The last identity follows from the fact that the degree of  $\mathbb{M}$  is odd.  $\square$

**2.9.** Next we define a Lie bracket on  $\mathfrak{g}^{\mathcal{P}}$ . The definition is very similar to that of  $\Delta$  above: we perform the twisted sewing operation between an output of one element and an output of another one. More formally, for two elements of the form

$$\begin{aligned}x &= \alpha \cdot u_1^{-k_1} \dots u_n^{-k_n} \in \mathcal{P}(0, n)[u_1^{-1}, \dots, u_n^{-1}], \\ y &= \beta \cdot u_1^{-l_1} \dots u_m^{-l_m} \in \mathcal{P}(0, m)[u_1^{-1}, \dots, u_m^{-1}]\end{aligned}$$

we define their Lie bracket by

$$\{x, y\} = (-1)^{|x|} \sum_{k_i=l_j=0} \lambda(\mathbb{M}) \circ_{(i,n+j)} (\alpha \otimes \beta) u_1^{-k_1} \dots \widehat{u_i^0} \dots u_n^{-k_n} u_{n+1}^{-l_1} \dots \widehat{u_{n+j}^0} \dots u_{n+m}^{-l_m}.$$

The sum is over all the pairs  $(i, j)$  such that  $k_i = l_j = 0$ . We extend the bracket to all elements in  $\mathfrak{g}^{\mathcal{P}}$  by linearity. The result is  $\Sigma_n \times \Sigma_m$ -invariant, hence it induces a Lie bracket of degree one

$$\{-, -\} : \mathcal{P}(0, n)_{\text{hS}} \otimes \mathcal{P}(0, m)_{\text{hS}} \rightarrow \mathcal{P}(0, n+m-2)_{\text{hS}}[-1].$$

**2.10. Theorem.** *The triple  $(\mathfrak{g}^{\mathcal{P}}, \partial + uB + \hbar\Delta, \{-, -\})$  forms a DGLA.*

**Proof.** We shall only prove the Jacobi identity. The Leibniz rule can be proved similarly. To avoid tedious formulas it will be useful to depict  $\{x, y\}$  as

$$\{x, y\} = \begin{array}{c} \boxed{x} \quad \boxed{y} \\ \text{---} \end{array}$$

where we put  $\lambda(\mathbb{M})$  on the connecting arc between  $x$  and  $y$ . With this notation the terms in the Jacobi identity can be depicted as

$$\begin{aligned} \{x, \{y, z\}\} &= \begin{array}{c} \boxed{x} \quad \boxed{y} \quad \boxed{z} \\ \text{---} \end{array} + \begin{array}{c} \boxed{x} \quad \boxed{y} \quad \boxed{z} \\ \text{---} \end{array} \\ \{\{x, y\}, z\} &= \begin{array}{c} \boxed{x} \quad \boxed{y} \quad \boxed{z} \\ \text{---} \end{array} + \begin{array}{c} \boxed{x} \quad \boxed{y} \quad \boxed{z} \\ \text{---} \end{array} \\ \{y, \{x, z\}\} &= \begin{array}{c} \boxed{y} \quad \boxed{x} \quad \boxed{z} \\ \text{---} \end{array} + \begin{array}{c} \boxed{y} \quad \boxed{x} \quad \boxed{z} \\ \text{---} \end{array} \end{aligned}$$

Let us illustrate with one of the terms above how the signs work out. The first term in  $\{x, \{y, z\}\}$  is given by

$$\begin{aligned} & (-1)^{|x|+|y|} \lambda(\mathbb{M}) \circ_{(i,j)} (x \otimes \lambda(\mathbb{M}) \circ_{(k,l)} (y \otimes z)) \\ &= (-1)^{|y|+1} \lambda(\mathbb{M}) \circ_{(k,l)} (\lambda(\mathbb{M}) \circ_{(i,j)} (x \otimes y) \otimes z) \\ &= (-1)^{|x|+|y|+1+|x|} \lambda(\mathbb{M}) \circ_{(k,l)} (\lambda(\mathbb{M}) \circ_{(i,j)} (x \otimes y) \otimes z) \end{aligned}$$

The last expression is precisely the first term in  $\{\{x, y\}, z\}$ . The other two signs can be checked similarly.  $\square$

**2.11. A DGLA associated with an  $\mathcal{A}nn^+$ -algebra.** Let  $V$  be an  $\mathcal{A}nn^+$ -algebra given by a dg PROP morphism  $\lambda : \mathcal{A}nn^+ \rightarrow \text{End}(V)$ . As an example of the previous construction we describe explicitly the DGLA  $\mathfrak{g}^V$ . We denote by  $b$  the internal differential of the dg vector space  $V$ , by  $B$  the image of  $S$  under  $\lambda$ , and by  $\langle -, - \rangle$  the image of  $M$ .

Using the circle action  $B$  form the Tate complex

$$V^{\text{Tate}} = (V((u)), b + uB).$$

Denote by  $V_+$  the subcomplex  $(uV[[u]], b + uB)$  of  $V^{\text{Tate}}$ , and by  $V_-$  the quotient complex  $V^{\text{Tate}}/V_+ = (V[u^{-1}], b + uB)$ .

As a graded vector space the DGLA  $\mathfrak{g}^V$  constructed by Theorem 2.10 is given by

$$\mathfrak{g}^V = ((\text{Sym } V_-)[1])[[\hbar]].$$

Define a pairing

$$\Omega : V_- \otimes V_- \rightarrow \mathbb{C}$$

which maps  $x = u^0 x_0 + \dots$  and  $y = u^0 y_0 + \dots$  to

$$\Omega(x, y) = \langle Bx_0, y_0 \rangle.$$

The pairing  $\Omega$  is antisymmetric because  $B$  is self-adjoint with respect to the pairing, which is itself symmetric. One can also check that  $\Omega$  is a chain map. Since  $B$  has degree 1 and  $\langle -, - \rangle$  has degree 0, the degree of  $\Omega$  is 1.

The twisted self-sewing operator  $\Delta$  acts on  $\mathfrak{g}^V = (\text{Sym } V_-)[1][[\hbar]]$  by

$$\Delta(x_1 \cdots x_n) = \sum_{1 \leq i < j \leq n} (-1)^{\epsilon} \Omega(x_i, x_j) x_1 \cdots \widehat{x}_i \cdots \widehat{x}_j \cdots x_n.$$

The sign

$$(-1)^{\epsilon} = (-1)^{|x_1| + \cdots + |x_{i-1}| + (|x_i| + 1)(|x_{i+1}| + \cdots + |x_{j-1}|)}$$

is simply the Koszul sign obtained from passing  $\Omega$  (an odd operator) past the first  $i - 1$  elements, and then the pair  $(\Omega, x_i)$  past the next  $j - i - 1$  elements in the sequence  $x_1 x_2 \cdots x_n$ .

The operator  $\Delta$  is a BV operator (i.e., a second order differential operator of homological degree and square zero) with respect to the product structure on  $\text{Sym } V_-$ . The difference  $\Delta(x \cdot y) - (\Delta x \cdot y) - (-1)^{|x|}(x \cdot \Delta y)$ , which measures the failure of  $\Delta$  to be a derivation, is a symmetric operation in  $x$  and  $y$  of degree 1. On the shifted symmetric product  $(\text{Sym } V_-)[1]$  this failure is then an anti-symmetric operation of degree zero which is precisely the Lie bracket of the DGLA  $\mathfrak{g}^V$

$$\{x, y\} = (-1)^{|x|} (\Delta(x \cdot y) - (\Delta x \cdot y) - (-1)^{|x|}(x \cdot \Delta y)).$$

**2.12.** The DGLA  $\mathfrak{g}^{\mathcal{P}}$  associated to an  $\text{Ann}^+$ -algebra  $\mathcal{P}$  can be concretely described in an entirely similar way. For each  $n$  let  $\mathcal{P}(0, n)^{\text{Tate}}$  be the Tate complex for the action of  $(S^1)^n$ : if  $b$  is the differential on  $\mathcal{P}(0, n)$  and  $B_i$  is the  $i$ -th circle action, then

$$\mathcal{P}(0, n)^{\text{Tate}} = \left( \mathcal{P}(0, n)(u_1, \dots, u_n), b + \sum u_i B_i \right).$$

We define  $\mathcal{P}(0, n)_-$  to be the quotient of  $\mathcal{P}(0, n)^{\text{Tate}}$  by the subcomplex spanned topologically by expressions of the form  $\alpha \cdot u_1^{k_1} \cdots u_n^{k_n}$  with  $\alpha \in \mathcal{P}(0, n)$  and at least one  $k_i$  is positive.

The operator  $\Omega$  defined above has an analog in this situation, which is a map  $\Omega_{ij} : \mathcal{P}(0, n)_- \rightarrow \mathcal{P}(0, n-2)_-$  for each  $1 \leq i \neq j \leq n$ .

As before, we define  $\mathcal{P}(0, n)_{\text{hS}}$  to be the coinvariants of  $\mathcal{P}(0, n)_-$  by the symmetric group action (which acts simultaneously on the  $u_i$  variables). We build a BV algebra structure on the direct sum of the spaces  $\mathcal{P}(0, n)_{\text{hS}}$ . The BV operator  $\Delta$  is the symmetrization of the operators  $\Omega_{ij}$  and the product comes from the horizontal product in the PROP structure. With the induced bracket  $\{-, -\}$  (which measures the failure of  $\Delta$  to be a derivation) this yields a DGLA

$$\mathfrak{g}^{\mathcal{P}} = \left( \bigoplus \mathcal{P}(0, n)_{\text{hS}}, b + \hbar\Delta, \{-, -\} \right).$$

**2.13. A second construction of a DGLA from an  $\text{Ann}^+$  action.** The previous construction built a DGLA from the zero-input part of a PROP  $\mathcal{P}$  with  $\text{Ann}^+$  action. We will now construct a different DGLA from the positive boundary part of the same PROP.

A dg PROP  $\mathcal{P}$  is said to have positive boundary if  $\mathcal{P}(0, n) = 0$  for all  $n \geq 1$ . If  $\mathcal{P}$  is any dg PROP, its truncation  $\mathcal{P}^+$  to the part with positive inputs is a positive boundary PROP. The PROP of annuli  $\text{Ann}^+$  is also a positive PROP.

Let  $\mathcal{P}$  be a positive boundary PROP, and let  $\lambda : \text{Ann}^+ \rightarrow \mathcal{P}$  be a morphism of dg PROPs. As before, composing with  $\lambda(S)$  along the inputs or outputs defines  $k+l$  commuting homological circle actions on the space  $\mathcal{P}(k, l)$ . The  $(S^1)^{k+l}$ -equivariant chain complex is

$$\mathcal{P}(k, l)_{(S^1)^{k+l}} = \left( \mathcal{P}(k, l)[u_1^{-1}, \dots, u_{k+l}^{-1}], \partial + \sum_{i=1}^{k+l} u_i \cdot B_i \right).$$

Both  $\Sigma_k$  and  $\Sigma_l$  act on this chain complex. We modify the action of the symmetric group  $\Sigma_k$  on the inputs of the operations in  $\mathcal{P}(k, l)$  by twisting it by the sign representation  $\text{sgn}_k$ . We leave the action of  $\Sigma_l$  on the outputs unchanged. As before, these groups act on the variables  $u_1, \dots, u_k, u_{k+1}, \dots, u_{k+l}$  by permuting them.

There are two reasons behind the presence of the  $\text{sgn}_k$  representation in the action on inputs. One is the presence of the shift by  $2-k$  of  $\mathcal{P}(k, l)$ : it can be thought of as having forced the  $k$  inputs to be odd. Another is related to the fact that we want to obtain an analogue of the Koszul complex; the inputs will play the role of exterior tensors, while the outputs will play the role of symmetric tensors. Again, we shall follow our convention to use the notation  $\mathcal{P}(k, l)_{\text{hS}}$  to denote the homotopy quotient of  $\mathcal{P}(k, l)$  by the action of the semidirect product  $(S^1)^{k+l} \rtimes (\Sigma_k \times \Sigma_l)$ .



We shall construct a DGLA structure on the dg vector space

$$\widehat{\mathfrak{g}}^{\mathcal{P}} = \left( \bigoplus_{k \geq 1, l} \mathcal{P}(k, l)_{\text{HS}}[2 - k] \right) \llbracket \hbar \rrbracket.$$

Due to the shifts of degree in the definition of  $\widehat{\mathfrak{g}}^{\mathcal{P}}$  it is necessary to deal with signs before moving forward. Indeed, the degree of an element  $x \in \mathcal{P}(k, l)_{\text{HS}}[2 - k]$  is given by

$$|x| = \deg(x) + 2 - k,$$

where  $\deg(x)$  stands for the chain degree of  $x \in \mathcal{P}(k, l)_{\text{HS}}$ . On the shifted complex  $\mathcal{P}(k, l)_{\text{HS}}[2 - k]$  the differential is given by

$$\partial' + \sum_{i=1}^{k+l} u_i \cdot B'_i = (-1)^k \cdot \left( \partial + \sum_{i=1}^{k+l} u_i \cdot B_i \right).$$

where the notation  $'$  is used to denote an operator after performing the appropriate shifts of degrees. It is also most convenient to work with a shifted version of the original PROP composition. For two elements  $x \in \mathcal{P}(m, l)$ ,  $y \in \mathcal{P}(n, m)$ , and two ordered indices  $I = (i_1, \dots, i_r)$  and  $J = (j_1, \dots, j_r)$  we define the shifted composition to be

$$x_I \circ'_J y = (-1)^{(n+r) \cdot \deg(x)} x_I \circ_J y \in \mathcal{P}(n + m - r, l + m - r).$$

The following lemma can be verified directly from the definitions.

**2.14. Lemma.** *With notations as above the following identities hold:*

- (1)  $\partial'(x_I \circ'_J y) = (\partial'x)_I \circ'_J y + (-1)^{r+|x|} x_I \circ'_J (\partial'y).$
- (2)  $(x_I \circ'_J y)_K \circ'_L z = x_I \circ'_J (y_K \circ'_L z).$

**2.15.** We may now proceed to the construction of a DGLA structure on  $\widehat{\mathfrak{g}}^{\mathcal{P}}$ . First we shall construct a deformed differential as follows. In addition to the existing differential  $\partial' + \sum_{i=1}^{k+l} u_i \cdot B'_i$ , there are two more components. One of them is similar to the previously defined twisted self-sewing operation  $\Delta$ : we set

$$\Delta(x) = \sum_{1 \leq i < j \leq l} \text{Res}_{u_i=0, u_j=0} \lambda(\mathbb{M}) \circ'_{(i,j)} x.$$

In other words we perform the twisted self-sewing operation only among any pair of outputs that are labeled by 0-powers of the  $u$ 's.

The other component of the differential is similar to the usual Koszul differential: it changes an output of an operation  $x$  to an input by sewing that output with one of the inputs of  $\lambda(M) \in \mathcal{P}(2, 0)$ . Formally we define

$$\begin{aligned} \iota : \mathcal{P}(k, l)_{\text{hS}} &\rightarrow \mathcal{P}(k+1, l-1)_{\text{hS}} \\ \iota(x) &= \sum_{1 \leq i \leq l} \lambda(M) \circ'_i x. \end{aligned}$$

**2.16. Lemma.** *The following identity holds*

$$\left( \partial' + \sum_{i=1}^{k+l} u_i \cdot B'_i + \iota + \hbar \Delta \right)^2 = 0.$$

**Proof.** This follows from Lemma 2.14. For example, to prove that  $\iota \partial' + \partial' \iota = 0$ , we have

$$\begin{aligned} \partial' \iota(x) &= \sum_i \partial'(\lambda(M) \circ'_i x) \\ &= \sum_i -\lambda(M) \circ'_i \partial' x \quad (\text{by Lemma 2.14}) \\ &= -\iota \partial' x. \end{aligned}$$

Note that when applying Lemma 2.14 we have used the facts that  $|\lambda(M)| = 0$  and  $r = 1$ . The other commutator identities can be verified in a similar way.  $\square$

**2.17.** The construction of a Lie bracket on  $\widehat{\mathfrak{g}}^{\mathcal{P}}$  is more involved, due to the asymmetry between inputs and outputs. Here are two simple observations:

- To keep the number of inputs positive we should never perform the twisted sewing of two inputs, otherwise the Lie bracket between two elements in  $\mathcal{P}(1, l_1)$  and  $\mathcal{P}(1, l_2)$  would produce elements with zero inputs.
- Since we want the bracket on  $\widehat{\mathfrak{g}}^{\mathcal{P}}$  to have degree zero, we should not perform twisted sewing of two outputs (which would be a degree  $-1$  operation).

**2.18.** We conclude that in the definition of the bracket we should only ever perform twisted sewings between inputs and outputs. As a first attempt, for elements  $x \in \mathcal{P}(l, k)_{\text{hS}}$  and  $y \in \mathcal{P}(n, m)_{\text{hS}}$ , set

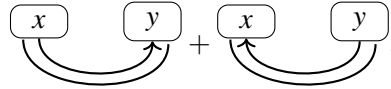
$$\{x, y\}_1 = \sum_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}} \text{Res}(x_i \circ' \lambda(S) \circ'_j y) - (-1)^{|x||y|} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \text{Res}(y_j \circ' \lambda(S) \circ'_i x)$$

where the residue is taken at  $u_i = u_j = 0$ .

It turns out that this operation is indeed a Lie bracket; however,  $\Delta$  fails to be a derivation of the bracket  $\{-, -\}_1$ , so we don't yet get a DGLA:

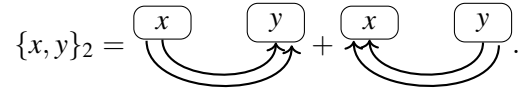
$$\Delta\{x, y\}_1 - \{\Delta x, y\}_1 - (-1)^{|x|}\{x, \Delta y\}_1 \neq 0.$$

In terms of diagrams as in the proof of Theorem 2.10 this discrepancy can be understood as the terms below:



Here the connecting arc with no arrow is the result of applying the operator  $\Delta$ , while the directed arc is what is obtained by applying the Lie bracket  $\{-, -\}_1$ .

To remedy this problem we introduce a new bracket operator  $\{-, -\}_2$ , represented pictorially by



The operator  $\iota$  fails to be a derivation of  $\{-, -\}_2$ , and its failure cancels precisely  $\Delta\{x, y\}_1 - \{\Delta x, y\}_1 - (-1)^{|x|}\{x, \Delta y\}_1$ . But then  $\{-, -\}_2$  again is not compatible with the operator  $\Delta$ . Thus we need to introduce a third bracket  $\{-, -\}_3$ , and so on.

**2.19.** More systematically, for each  $r \geq 1$  we introduce a bracket operation which performs twisted sewing of any  $r$  inputs with any  $r$  outputs. Specifically, for  $x \in \mathcal{P}(l, k)_{\text{HS}}$  and  $y \in \mathcal{P}(n, m)_{\text{HS}}$  we define

$$\{x, y\}_r = \sum_{\substack{I=(i_1, \dots, i_r), \\ J=(j_1, \dots, j_r)}} \text{Res}(x_I \circ' \lambda(S)^{\otimes r} \circ'_J y) - (-1)^{|x||y|} \sum_{\substack{I=(i_1, \dots, i_r), \\ J=(j_1, \dots, j_r)}} \text{Res}(y_J \circ' \lambda(S)^{\otimes r} \circ'_I x)$$

where the residue is taken at  $u_I = 0, u_J = 0$ , and in the first sum  $I = (i_1, \dots, i_r)$  and  $J = (j_1, \dots, j_r)$  are  $r$ -tuples of distinct elements in the sets  $\{1, \dots, l\}$ ,  $\{1, \dots, m\}$ , respectively. The  $r$ -tuples  $I$  and  $J$  in the second summation are defined similarly, but they are subsets of  $\{1, \dots, k\}$ ,  $\{1, \dots, n\}$ .

**2.20. Theorem.** *The following graded vector space forms a DGLA*

$$\mathfrak{g}^{\mathcal{P}} = \left( \bigoplus_{k \geq 1, l} \mathcal{P}(k, l)_{\text{HS}}[2 - k] \right) \llbracket \hbar \rrbracket$$

when endowed with

- differential given by  $\partial' + \sum_{i=1}^{k+1} u_i \cdot B'_i + \iota + \hbar\Delta$ ,
- Lie bracket given by  $\{-, -\}_\hbar = \sum_{r \geq 1} \frac{1}{r!} \{-, -\}_r \cdot \hbar^{r-1}$ .

**Proof.** For the Leibniz rule we need to prove that

$$\begin{aligned} (r+1)(\Delta\{x, y\}_r - \{\Delta x, y\}_r - (-1)^{|x|}\{x, \Delta y\}_r) &= \\ &= -\iota\{x, y\}_{r+1} + \{\iota x, y\}_{r+1} + (-1)^{|x|}\{x, \iota y\}_{r+1}. \end{aligned}$$

Let us begin with the right hand side. The terms in  $-\iota\{x, y\}_{r+1}$  are of the form

$$-\lambda(M) \circ'_p \text{Res}(x_{I'} \circ' \lambda(S)^{r+1} \circ'_{J'} y) + (-1)^{|x||y|} \lambda(M) \circ'_q \text{Res}(y_{K'} \circ' \lambda(S)^{r+1} \circ'_{L'} x),$$

with the multi-indices  $I', J', K'$ , and  $L'$  of cardinality  $r+1$ . This further implies to the following

$$\begin{aligned} &\underbrace{-\text{Res}((\lambda(M) \circ'_p x)_{I'} \circ' \lambda(S)^{r+1} \circ'_{J'} y)}_{(i)} - \underbrace{(-1)^{|x|} \text{Res}(x_{I'} \circ' \lambda(S)^{r+1} \circ'_{J'} (\lambda(M) \circ'_p y))}_{(ii)} \\ &+ \underbrace{(-1)^{|x||y|} \text{Res}((\lambda(M) \circ'_q y)_{K'} \circ' \lambda(S)^{r+1} \circ'_{L'} x)}_{(iii)} \\ &+ \underbrace{(-1)^{|x||y|+|y|} \text{Res}(y_{K'} \circ' \lambda(S)^{r+1} \circ'_{L'} (\lambda(M) \circ'_q x))}_{(iv)} \end{aligned}$$

The summation is over indices  $p, I', J'$  such that  $p \notin I'$  and  $p \notin J'$ , and similarly for  $q, K'$  and  $L'$ . Let us compare these terms with those in  $\{\iota x, y\}_{r+1}$ , which are given by terms of the form

$$\text{Res}((\lambda(M) \circ'_p x)_{I'} \circ' \lambda(S)^{r+1} \circ'_{J'} y) - (-1)^{|x||y|+|y|} \text{Res}(y_{K'} \circ' \lambda(S)^{r+1} \circ'_{L'} (\lambda(M) \circ'_p x)).$$

There are two cases, depending on whether the index  $p$  is in  $I'$  or not.

- if  $p \notin I'$  then these terms cancel precisely the terms (i) and (iv).
- if  $p \in I'$  the extra terms are equal to

$$\sum_{\substack{p=i'_\alpha, \\ 1 \leq \alpha \leq r+1}} \lambda(\mathbb{M}) \circ'_{(i'_\alpha, j'_\alpha)} \text{Res}(x_{I' \setminus i'_\alpha} \circ' \lambda(S)^r \circ'_{J' \setminus j'_\alpha} y).$$

A similar argument also works for the terms in  $(-1)^{|x|}\{x, \iota y\}_{r+1}$ , when a part of the terms cancel the terms (ii) and (iii). The extra term is given by

$$(-1)^{|x||y|} \sum_{\substack{q=k'_\alpha, \\ 1 \leq \alpha \leq r+1}} \lambda(\mathbb{M}) \circ'_{(k'_\alpha, l'_\alpha)} \text{Res}(y_{K' \setminus k'_\alpha} \circ' \lambda(S)^r \circ'_{L' \setminus l'_\alpha} x).$$

Adding the above two extra terms gives precisely  $(r + 1)(\Delta\{x, y\}_r - \{\Delta x, y\}_r - (-1)^{|x|}\{x, \Delta y\})$ . This proves the Leibniz rule.

The Jacobi identity is proved diagrammatically as in the proof of Theorem 2.10. We have

$$\begin{aligned} \{x, \{y, z\}\} = & \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} - \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} \\ + & \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} - \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} \\ - & \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} + \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} \\ - & \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} + \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} \\ + & \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} - \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} \\ - & \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} + \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \curvearrowright \quad \curvearrowright \end{array} . \end{aligned}$$

The signs arise from the number of left-to-right sewing operations. Each arrow in the above diagram indicates a twisted sewing operation of (possibly multiple) outputs with inputs. The last four terms in the above sum are due to the fact that when performing the second bracket operation, it is possible to choose inputs/outputs in both  $y$  and  $z$ . Similarly, writing the terms in  $\{\{x, y\}, z\}$  and  $(-1)^{|x||y|}\{y, \{x, z\}\}$  all the 36 terms cancel out.  $\square$

**2.21. An example of the second construction of a DGLA.** We illustrate the second construction of a DGLA in the case of an  $\mathcal{Ann}^+$ -algebra structure on a dg vector space  $(V, b)$ . We continue to use the notations from (2.11).

As a graded vector space we have

$$\widehat{\mathfrak{g}}^V = \left( \bigoplus_{k \geq 1, l} \text{Hom}(\wedge^k V_+, \text{Sym}^l V_-)[2 - k] \right) \llbracket \hbar \rrbracket.$$

The self-sewing operator  $\Delta$  is only defined on the output part of these Hom's, i.e. for  $\varphi \in \text{Hom}(\text{Sym}(V_+[1]), \text{Sym}^l V_-)$  we set

$$(\Delta\varphi)(x_1 \cdots x_k) = \Delta(\varphi(x_1 \cdots x_k)),$$

where on the right hand side the operator  $\Delta$  acts by the formula in (2.11) applied to symmetric tensors.

We next consider the Lie bracket

$$\{-, -\}_{\hbar} = \sum_{r \geq 1} \frac{1}{r!} \{-, -\}_r \cdot \hbar^{r-1}.$$

Denote by  $\theta : V_- \rightarrow V_+$  the map defined by

$$\theta(x_0 + x_{-1}u^{-1} + \cdots) = uBx_0 \in V_+.$$

Since  $\theta$  is odd (it has degree one), for each  $r \geq 1$  we obtain induced maps

$$\theta^{(r)} : \text{Sym}^r V_- \rightarrow \wedge^r V_+$$

given by

$$\theta^{(r)}(x_1 \cdots x_r) = \theta(x_1) \wedge \cdots \wedge \theta(x_r).$$

Both the symmetric algebra  $\text{Sym} V_-$  and the exterior algebra  $\wedge V_+$  are cocommutative coalgebras with respect to the shuffle coproduct. The  $r$ -th Lie bracket of elements  $\varphi \in \text{Hom}(\wedge^k V_+, \text{Sym}^l V_-)$  and  $\psi \in \text{Hom}(\wedge^{k'} V_+, \text{Sym}^{l'} V_-)$  lies inside  $\text{Hom}(\wedge^{k+k'-r} V_+, \text{Sym}^{l+l'-r} V_-)$ . It is given by  $\varphi * \psi - (-1)^{|\varphi||\psi|} \psi * \varphi$  where  $\varphi * \psi$  is defined as the composition

$$\begin{aligned} \wedge^{k+k'-r} V_+ &\longrightarrow \wedge^{k'} V_+ \otimes \wedge^{k-r} V_+ \xrightarrow{\psi \otimes \text{id}} \text{Sym}^{l'} V_- \otimes \wedge^{k-r} V_+ \longrightarrow \\ \text{Sym}^r V_- \otimes \text{Sym}^{l'-r} V_- \otimes \wedge^{k-r} V_+ &\xrightarrow{\theta^{(r)} \otimes \text{id}} \wedge^r V_+ \otimes \text{Sym}^{l'-r} V_- \otimes \wedge^{k-r} V_+ \longrightarrow \\ \xrightarrow{\cong} \wedge^r V_+ \otimes \wedge^{k-r} V_+ \otimes \text{Sym}^{l'-r} V_- &\longrightarrow \wedge^k V_+ \otimes \text{Sym}^{l'-r} V_- \longrightarrow \\ \xrightarrow{\varphi \otimes \text{id}} \text{Sym}^l V_- \otimes \text{Sym}^{l'-r} V_- &\longrightarrow \text{Sym}^{l+l'-r} V_-. \end{aligned}$$

**2.22. A map between the two constructions of DGLAs.** Let  $\mathcal{P}$  be a PROP with  $\text{Ann}^+$  action given by  $\lambda : \text{Ann}^+ \rightarrow \mathcal{P}$ . By the above constructions we obtain two DGLAs  $\mathfrak{g}^{\mathcal{P}}$  and  $\widehat{\mathfrak{g}}^{\mathcal{P}^+}$ . Denote the part of the DGLA  $\mathfrak{g}^{\mathcal{P}}$  with positive number of outputs by

$$(\mathfrak{g}^{\mathcal{P}})^+ = \bigoplus_{n \geq 1} \mathcal{P}(0, n)_{\text{hS}}[1][[\hbar]].$$

Note that this is not a sub-DGLA of  $\mathfrak{g}^{\mathcal{P}}$ , but rather a quotient. Indeed, there exists a short exact sequence

$$0 \rightarrow \mathcal{P}(0, 0)_{\text{hS}}[[\hbar]] \rightarrow \mathfrak{g}^{\mathcal{P}} \rightarrow (\mathfrak{g}^{\mathcal{P}})^+ \rightarrow 0$$

which is a central extension of DGLAs. The relationship between the constructions of Theorems 2.10 and 2.20 is that there exists a natural morphism of DGLAs

$$\iota : (\mathfrak{g}^{\mathcal{P}})^+ \rightarrow \widehat{\mathfrak{g}}^{\mathcal{P}^+}$$

which maps  $x \in \mathcal{P}(0, n)_{\text{hS}}$  to

$$\iota(x) = \sum_{1 \leq i \leq n} \lambda(M) \circ'_i x \in \mathcal{P}(1, n-1)_{\text{hS}}.$$

By construction, the image of this map is in the kernel of the operator  $\iota : P(1, n-1)_{\text{hS}} \rightarrow P(2, n-2)_{\text{hS}}$ . This is why the map  $\iota : (\mathfrak{g}^{\mathcal{P}})^+ \rightarrow \widehat{\mathfrak{g}}^{\mathcal{P}^+}$  is a cochain map. It is a map of Lie algebras for the simple reason that, on the subspace of elements of  $\widehat{\mathfrak{g}}^{\mathcal{P}^+}$  with only one input, the only possible bracket involves connecting only one input to one output.

For the examples of interest in this paper the map  $\iota$  will often be a quasi-isomorphism.

### 3. The Sen-Zwiebach DGLA and string vertices

We apply the constructions of the previous section to the dg PROP of singular chains on the moduli spaces  $C_*(M_{g,k,l}^{\text{fr}})$  of Riemann surfaces with distinguished, framed inputs and outputs. We find that the map  $\iota$  between the two types of DGLAs constructed before is a quasi-isomorphism; since in the first one the Maurer-Cartan equation has solution that is unique up to homotopy, it follows that the same is true for the second one. The components of this unique solution are the homological string vertices. The same result holds for a combinatorial version of the second type of DGLA above, constructed from ribbon graphs.

As stated in the introduction,  $C_*(-)$  will denote the normalized singular chain functor from topological spaces to dg vector spaces.

**3.1. Moduli spaces of curves with framed markings.** For  $g \geq 0$  and  $k, l \geq 0$  satisfying  $2g - 2 + k + l > 0$  we consider the coarse moduli space  $M_{g,k,l}^{\text{fr}}$  of Riemann surfaces of genus  $g$  with  $k + l$  marked and framed points,  $k$  of which are designated as inputs, and the rest as outputs. Explicitly, a point in  $M_{g,k,l}^{\text{fr}}$  is given by the data  $(\Sigma, p_1, \dots, p_k, q_1, \dots, q_l, \varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_l)$  where

- $\Sigma$  is a smooth Riemann surface of genus  $g$ ,
- the  $p$ 's and the  $q$ 's are distinct marked points on  $\Sigma$ ,
- the  $\varphi$ 's and the  $\psi$ 's are framings around the marked points, given by biholomorphic maps

$$\begin{aligned}\varphi_i &: \mathbb{D}^2 \rightarrow U_{\epsilon_i}(p_i), \quad 1 \leq i \leq k \\ \psi_j &: \mathbb{D}^2 \rightarrow U_{\epsilon_j}(q_j), \quad 1 \leq j \leq l\end{aligned}$$

where  $\mathbb{D}^2$  stands for the unit disk in  $\mathbb{C}$ , and  $U_{\epsilon}(x)$  is a radius  $\epsilon$  disk of the marked point  $x$  on  $\Sigma$ , under, say, the unique hyperbolic metric on  $\Sigma \setminus \{p_1, \dots, p_k, q_1, \dots, q_l\}$ . We require that the biholomorphic maps  $\varphi$ 's and  $\psi$ 's extend to an open neighborhood of  $\mathbb{D}^2$ . We also require that the closures of all the framed disks be disjoint, i.e. for any two distinct points  $x, y \in \{p_1, \dots, p_k, q_1, \dots, q_l\}$  we have

$$\overline{U_{\epsilon_i}(x)} \cap \overline{U_{\epsilon_j}(y)} = \emptyset.$$

We note that the fibration  $M_{g,k,l}^{\text{fr}} \rightarrow M_{g,k,l}$  which forgets the framing is a homotopy  $(S^1)^{k+l}$ -bundle.

**3.2.** There is a sewing map

$$M_{h,l,m}^{\text{fr}} \times M_{g,k,l}^{\text{fr}} \rightarrow M_{g+h+l-1,k,m}^{\text{fr}}$$

defined by first removing the  $l$  open disks from each framed surface, and then sewing the boundary circles using the identification  $zw = 1$ . Here  $z \in \partial\mathbb{D}^2$  and  $w \in \partial\mathbb{D}^2$  are coordinates on the two boundary circles to be sewed together. Applying the functor  $C_*(-)$  yields a composition map

$$\circ : C_*(M_{h,l,m}^{\text{fr}}) \otimes C_*(M_{g,k,l}^{\text{fr}}) \rightarrow C_*(M_{g+h+l-1,k,m}^{\text{fr}}).$$

**3.3. The PROP  $\mathcal{S}$ .** The collection of  $\Sigma$ -bimodules  $\{C_*(M_{g,k,l}^{\text{fr}})\}$  with composition defined above does not form a PROP because we only allow connected surfaces in  $M_{g,k,l}^{\text{fr}}$ . To overcome this we set  $\mathcal{R}$  to be the free dg PROP generated by  $\{C_*(M_{g,k,l}^{\text{fr}})\}$  modulo the relations defined by the composition maps  $\circ$  in (3.2). A geometric way to directly construct  $\mathcal{R}$  would be to allow disconnected Riemann surfaces, but we shall not need this description here.

We will be most interested in the PROP  $\mathcal{S}$  obtained by adjoining the PROP  $\text{Ann}^+$  to  $\mathcal{R}$ . Conceptually this is equivalent to also allowing the surfaces (annuli) in  $M_{0,1,1}^{\text{fr}}$  and  $M_{0,2,0}^{\text{fr}}$  – in the framed setting these are stable (unlike in the unframed case). To be precise we define  $\mathcal{S}$  to be the free PROP generated by  $\{C_*(M_{g,k,l}^{\text{fr}})\}$  and  $S \in \text{Ann}^+(1, 1), M \in \text{Ann}^+(2, 0)$  modulo the relations below.



- The relations in the definition of  $\mathcal{A}nn^+$ , see (2.2).
- The relations in the definition of  $\mathcal{R}$ , see (3.2).
- For a  $d$ -chain  $\sigma \in C_d(M_{g,k,l}^{\text{fr}})$  and an index  $1 \leq i \leq k$ , define the composition  $\sigma_i \circ S$  to be (the prism decomposition of) the  $(d+1)$ -chain

$$S^1 \times \Delta^d \xrightarrow{\text{id} \times \sigma} S^1 \times M_{g,k,l}^{\text{fr}} \rightarrow M_{g,k,l}^{\text{fr}},$$

where the second arrow is the action of  $S^1$  on the framing at the  $i$ -th marked point by precomposing a framing with the rotation map  $e^{i\theta} : \mathbb{D}^2 \rightarrow \mathbb{D}^2$ . The composition on the other side  $S \circ_j \sigma$ , ( $1 \leq j \leq l$ ) is defined in the same way.

- For a  $d$ -chain  $\sigma$  as above and a pair of indices  $1 \leq i < j \leq l$ , define  $M \circ_{(i,j)} \sigma$  as the composition

$$\Delta^d \xrightarrow{\sigma} M_{g,k,l}^{\text{fr}} \rightarrow M_{g+1,k,l-2}^{\text{fr}}.$$

Here the second map performs framed sewing at the two marked points  $q_i$  and  $q_j$ , i.e., we first remove the two disks  $U_{\epsilon_i}(q_i)$ ,  $U_{\epsilon_j}(q_j)$ , and then we identify the two boundaries by  $z = w$  using the two framings  $\psi_i$  and  $\psi_j$ .

**3.4. Sen-Zwiebach's DGLA.** The PROP  $\mathcal{A}nn^+$  obviously sits inside  $\mathcal{S}$ , giving rise to an  $\mathcal{A}nn^+$ -action on  $\mathcal{S}$ . Theorem 2.10 constructs a DGLA

$$\mathfrak{g}^{\mathcal{S}} = \left( \bigoplus_n \mathcal{S}(0, n)_{\text{hS}}[1] \right) \llbracket \hbar \rrbracket.$$

Inside  $\mathcal{S}(0, n)$  we have a subspace  $C_*(M_{g,0,n}^{\text{fr}})$  consisting of connected Riemann surfaces. We put these subspaces together to form

$$\mathfrak{g} = \left( \bigoplus_{g,n} C_*(M_{g,0,n}^{\text{fr}})_{\text{hS}}[1] \right) \llbracket \hbar \rrbracket.$$

Since sewing operations with  $S$  and  $M$  preserves connectedness the subspace  $\mathfrak{g} \subset \mathfrak{g}^{\mathcal{S}}$  is a sub-DGLA.

It is convenient to adjoin to  $\mathfrak{g}$  another formal variable  $\lambda$  of degree  $-2$ . We will use the notation  $\mathfrak{g}$  for the resulting DGLA

$$\mathfrak{g} = \left( \bigoplus_{g,n} C_*(M_{g,0,n}^{\text{fr}})_{\text{hS}}[1] \right) \llbracket \hbar, \lambda \rrbracket.$$

**3.5. Definition.** A degree  $-1$  element  $\mathcal{V} \in \mathfrak{g}$  of the form

$$\mathcal{V} = \sum_{g,n} \mathcal{V}_{g,n} \hbar^g \lambda^{2g-2+n}, \quad \mathcal{V}_{g,n} \in C_*(M_{g,0,n}^{\text{fr}})_{\text{hS}}$$

is called a homological string vertex if it satisfies the following properties:

- (1)  $\mathcal{V}$  is a Maurer-Cartan element of the DGLA  $\mathfrak{g}$ , i.e., in terms of the components  $\{\mathcal{V}_{g,n}\}$  we have

$$(\partial + uB)\mathcal{V}_{g,n} + \Delta\mathcal{V}_{g-1,n+2} + \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+2}} \{\mathcal{V}_{g_1,n_1}, \mathcal{V}_{g_2,n_2}\} = 0.$$

- (2)  $\mathcal{V}_{0,3} = \frac{1}{6} \cdot \text{pt}$  with  $\text{pt} \in C_0(M_{0,0,3}^{\text{fr}})_{\text{hS}}$  representing a point class.

A fundamental result about string vertices is the following theorem from [Cos09].

**3.6. Theorem.** *The string vertex exists and is unique up to homotopy (i.e., gauge equivalence between Maurer-Cartan elements).*

**Proof.** The proof is a standard argument in deformation theory. The obstruction to existence lies in  $H_{6g-7+2n}(M_{g,0,n}^{\text{fr}})_{\text{hS}}$  while the deformation space is given by  $H_{6g-6+2n}(M_{g,0,n}^{\text{fr}})_{\text{hS}}$ . Both homology groups are known to vanish. We refer to [Cos09, Section 9] for a detailed proof.  $\square$

**3.7. The Koszul resolution of  $\mathfrak{g}$ .** Let  $\mathcal{S}^+$  denote the sub-PROP of  $\mathcal{S}$  generated by chains on moduli spaces of Riemann surfaces with positive number of inputs. The construction of Theorem 2.20, applied to the morphism  $i : \text{Ann}^+ \rightarrow \mathcal{S}^+$ , yields a second DGLA

$$\widehat{\mathfrak{g}}^{\mathcal{S}^+} = \left( \bigoplus_{k \geq 1, l} \mathcal{S}^+(k, l)_{\text{hS}}[2-k] \right) \llbracket \hbar \rrbracket.$$

Again, the subspace of connected stable surfaces forms a sub-DGLA to which we adjoin the formal variable  $\lambda$  to get

$$\widehat{\mathfrak{g}} = \left( \bigoplus_{g \geq 0, k \geq 1, l} C_*(M_{g,k,l}^{\text{fr}})_{\text{hS}}[2-k] \right) \llbracket \hbar, \lambda \rrbracket.$$

The canonical map  $\iota : (\mathfrak{g}^{\mathcal{S}})^+ \rightarrow \widehat{\mathfrak{g}}^{\mathcal{S}^+}$  of (2.22) also restricts to give a morphism of DGLAs  $\iota : \mathfrak{g}^+ \rightarrow \widehat{\mathfrak{g}}$ .

**3.8. Proposition.** *The map  $\iota : \mathfrak{g}^+ \rightarrow \widehat{\mathfrak{g}}$  is a quasi-isomorphism of DGLAs.*

**Proof.** Observe that for fixed  $g \geq 0$  and  $n > 0$  the various moduli spaces  $M_{g,k,l}^{\text{fr}}$  with  $k + l = n$  are all isomorphic. Then the result follows from the algebraic fact that if  $W$  is any  $\Sigma_n$ -representation, then the associated Koszul complex

$$0 \rightarrow W_{\Sigma_n} \rightarrow W_{\Sigma_{1,n-1}} \rightarrow W_{\Sigma_{2,n-2}} \rightarrow \cdots \rightarrow W_{\Sigma_{n,0}} \rightarrow 0$$

is exact. Here  $W_{\Sigma_{k,l}}$  denotes the space of coinvariants of  $W$  under the action of  $\Sigma_k \times \Sigma_l$ , where the first group acts via the sign representation  $\text{sgn}_k$  and the second one acts via the trivial representation.  $\square$

**3.9. The combinatorial version of  $\widehat{\mathfrak{g}}$ .** There exists a version  $\mathcal{S}^{\text{comb},+}$  of  $\mathcal{S}^+$  constructed in terms of ribbon graphs with framed inputs and outputs (also sometimes called fat graphs with black and white vertices). We will not give the details of the construction of the PROP  $\mathcal{S}^{\text{comb},+}$  here since a complete description of it will appear in the upcoming work [CC20], following ideas of Kontsevich-Soibelman [KS09] and Wahl-Westerland [WW16]. It suffices to say that the chain complexes  $\mathcal{S}^{\text{comb},+}(k, l)$  split up as direct sums of complexes indexed by a genus  $g \geq 0$ ,

$$\mathcal{S}^{\text{comb},+}(k, l) = \bigoplus_{g \geq 0} \mathcal{S}_g^{\text{comb},+}(k, l).$$

Each summand  $\mathcal{S}_g^{\text{comb},+}(k, l)$  is quasi-isomorphic to  $C_*(M_{g,k,l}^{\text{fr}})$ , and will be denoted by  $C_*^{\text{comb}}(M_{g,k,l}^{\text{fr}})$ . A basis for  $C_*^{\text{comb}}(M_{g,k,l}^{\text{fr}})$  consists of isomorphism classes of framed ribbon graphs of genus  $g$  with  $k$  faces and  $l$  white vertices.

The main results that we will need about  $\mathcal{S}^{\text{comb},+}$  are summarized in the following theorems.

**3.10. Theorem.** *There is a sub-PROP of  $\mathcal{S}^{\text{comb},+}$  equivalent to  $\text{Ann}^+$ .*

**Proof.** The operations  $S \in \text{Ann}^+(1, 1)$  and  $M \in \text{Ann}^+(2, 0)$  correspond to the following ribbon graphs

$$S = \text{---} \circ \text{---}, \quad M = \text{---} \circ \text{---}$$

The thick leaf in the graph  $S$  is a marked leaf of the white vertex (see [WW16]) which indicates a generic framing for the marked point corresponding to the white vertex.  $\square$

**3.11. Theorem.** *The PROP  $\mathcal{S}^{\text{comb},+}$  is quasi-equivalent to  $\mathcal{S}$ .*

**Proof.** This follows from work of Egas [Ega15].  $\square$

**3.12. Theorem.** *Let  $A$  be a cyclic  $A_\infty$ -algebra of Calabi-Yau dimension  $d$ . Then  $A$  gives rise to a 2d TFT structure on the dg-vector space  $L = CC_*(A)[d]$  of shifted Hochschild chains of  $A$ , i.e., a PROP map*

$$\rho^A : \mathcal{S}^{\text{comb},+} \longrightarrow \text{End}(L).$$

**Proof.** This construction was sketched in [KS09, Section 11.6]. Complete details of this result will appear in [CC20].  $\square$

**3.13. Theorem.** *The ribbon graphs that form a basis of  $C_*^{\text{comb}}(M_{g,k,l}^{\text{fr}})$  and the operators  $b, B, \iota, \Delta, \{-, -\}_i$  can be algorithmically computed in explicit form.*

**Proof.** This follows from the explicit description of  $\mathcal{S}^{\text{comb},+}$  in [CC20].  $\square$

**3.14.** To summarize, we have the following diagram:

$$\begin{array}{ccc} & \text{Ann}^+ & \\ & \swarrow \quad \downarrow \quad \searrow & \\ \mathcal{S} & \longleftarrow \mathcal{S}^+ \xrightarrow{\cong} \mathcal{S}^{+, \text{comb}} & \end{array}$$

This yields a roof diagram of quasi-isomorphisms of DGLAs:

$$\begin{array}{ccc} & \mathfrak{g} = \left( \bigoplus_{g,n>0} C_*(M_{g,0,n}^{\text{fr}})_{\text{HS}}[1] \right) \llbracket \hbar \rrbracket & \\ & \swarrow \cong \text{ (by Proposition 3.8) } & \\ \widehat{\mathfrak{g}} = \left( \bigoplus_{g,k \geq 1, l} C_*(M_{g,k,l}^{\text{fr}})_{\text{HS}}[2-k] \right) \llbracket \hbar \rrbracket & & \\ & \swarrow \cong \text{ (since } \mathcal{S}^{+, \text{comb}} \cong \mathcal{S}^+ \text{)} & \\ & \widehat{\mathfrak{g}}^{\text{comb}} = \left( \bigoplus_{g,k \geq 1, l} C_*^{\text{comb}}(M_{g,k,l}^{\text{fr}})_{\text{HS}}[2-k] \right) \llbracket \hbar \rrbracket & \end{array}$$

By the homotopy invariance of Maurer-Cartan moduli space, Theorem 3.6 implies the following result.

**3.15. Theorem.** *In the DGLA*

$$\widehat{\mathfrak{g}}^{\text{comb}} = \left( \bigoplus_{g,k \geq 1, l} C_*^{\text{comb}}(M_{g,k,l}^{\text{fr}})_{\text{hS}}[2-k] \right) \llbracket \hbar, \lambda \rrbracket$$

there exists a degree  $-1$  element  $\widehat{\mathcal{V}}^{\text{comb}}$ , unique up to homotopy, of the form

$$\widehat{\mathcal{V}}^{\text{comb}} = \sum_{g,k \geq 1, l} \widehat{\mathcal{V}}_{g,k,l}^{\text{comb}} \hbar^g \lambda^{2g-2+k+l},$$

such that the following conditions are satisfied:

- (1)  $\widehat{\mathcal{V}}^{\text{comb}}$  is a Maurer-Cartan element of  $\widehat{\mathfrak{g}}^{\text{comb}}$ , i.e.,  $\widehat{\mathcal{V}}^{\text{comb}}$  satisfies the following equations for each triple  $(g, k \geq 1, l)$ :

$$(\partial' + uB')\widehat{\mathcal{V}}_{g,k,l}^{\text{comb}} + \iota\widehat{\mathcal{V}}_{g,k-1,l+1}^{\text{comb}} + \Delta\widehat{\mathcal{V}}_{g-1,k,l+2}^{\text{comb}} + \frac{1}{2} \sum_{r!} \frac{1}{r!} \{ \widehat{\mathcal{V}}_{g_1,k_1,l_1}^{\text{comb}}, \widehat{\mathcal{V}}_{g_2,k_2,l_2}^{\text{comb}} \}_r = 0.$$

The last sum is over all  $r \geq 1$  and all  $(g_1, g_2, k_1, k_2, l_1, l_2)$  such that

$$\begin{aligned} g_1 + g_2 + r - 1 &= g \\ k_1 + k_2 - r &= k \\ l_1 + l_2 - r &= l \end{aligned}$$

- (2)  $\widehat{\mathcal{V}}_{0,1,2}^{\text{comb}} = \frac{1}{2} \cdot \text{pt}$  with pt representing a point class in  $C_0^{\text{comb}}(M_{0,1,2}^{\text{fr}})_{\text{hS}}$ , i.e., the ribbon graph expression

$$\widehat{\mathcal{V}}_{0,1,2}^{\text{comb}} = \frac{1}{2} \text{---} \circ \begin{array}{c} \times \\ | \\ \circ \end{array}$$

**3.16. Definition.** We shall refer to the  $\mathbb{Q}$ -linear combinations of ribbon graphs  $\widehat{\mathcal{V}}_{g,k,l}^{\text{comb}}$  as combinatorial string vertices.

**3.17. Explicit formulas for some combinatorial string vertices.** We present explicit formulas for the first few combinatorial string vertices, ignoring orientations of ribbon graphs and associated signs. Our conventions are that a factor of  $u^{-k}$  associated to a boundary component of a diagram is related to the class  $\psi^{k-1}$ . If we do not indicate a power of  $u$ , then we mean  $u^{-1}$ .

$$\widehat{\mathcal{V}}_{0,1,2}^{\text{comb}} = \frac{1}{2} \text{---} \circ \begin{array}{c} \times \\ | \\ \circ \end{array}$$

$$\widehat{\mathcal{V}}_{1,1,0}^{\text{comb}} = \frac{1}{24} \text{---} \circ \begin{array}{c} \times u^{-2} \\ | \\ \circ \end{array} + \frac{1}{4} \text{---} \circ \begin{array}{c} \times \\ | \\ \circ \end{array}$$

$$\begin{aligned}
\widehat{\mathcal{V}}_{0,2,1}^{\text{comb}} &= \frac{1}{2} \times \text{---} \circ \text{---} \bigcirc \text{---} \times + \frac{1}{2} \times \text{---} \bigcirc \text{---} \times + \times \text{---} \bigcirc \text{---} \times \\
\widehat{\mathcal{V}}_{0,3,0}^{\text{comb}} &= \frac{1}{2} \times \text{---} \bigcirc \text{---} \times + \frac{1}{2} \times \text{---} \bigcirc \text{---} \times \text{---} \bigcirc \text{---} \times + \frac{1}{2} \times \text{---} \bigcirc \text{---} \times \text{---} \bigcirc \text{---} \times \\
\widehat{\mathcal{V}}_{0,1,3}^{\text{comb}} &= \frac{1}{2} \circ \text{---} \times \text{---} \circ + \frac{1}{2} \circ \text{---} \times \text{---} \circ + \frac{1}{2} \circ \text{---} \times \text{---} \circ + \frac{1}{6} \circ \text{---} \times \text{---} \circ.
\end{aligned}$$

Note, for example, that the coefficients  $1/24$  and  $1/6$  in  $\widehat{\mathcal{V}}_{1,1,0}^{\text{comb}}$  and  $\widehat{\mathcal{V}}_{0,1,4}^{\text{comb}}$ , respectively, correspond to the integrals of a  $\psi$ -class on  $\overline{M}_{1,1}$  and  $\overline{M}_{0,4}$ , respectively. This will be explained in [CT20].

## 4. String functionals from topological field theories

In this section we consider the action of the combinatorial string vertices constructed in the previous section on the Hochschild chains of a cyclic  $\mathcal{A}_\infty$ -algebra. The resulting functionals  $\widehat{\beta}_{g,k,l}$  are natural homological analogues of the string functionals introduced by Sen-Zwiebach in [SZ94].

In this section we assume that all  $A_\infty$ -algebras are defined over a field  $\mathbb{K}$  of characteristic zero.

**4.1. Two dimensional TFTs of dimension  $d$ .** For a non-negative integer  $d \geq 0$  define a shifted version  $\mathcal{S}_{[d]}$  of the PROP  $\mathcal{S}$  by performing the following degree shifts on the generators:

$$\begin{aligned}
C_*(M_{g,k,l}^{\text{fr}}) &\mapsto C_*(M_{g,k,l}^{\text{fr}})[d(2 - 2g - 2k)], \\
S &\mapsto S, \\
M &\mapsto M[-2d].
\end{aligned}$$

In other words, after the shift  $M$  has degree  $-2d$ , while  $S$  still has degree 1. The composition map  $\circ$  from (3.2) still has degree zero. We have similar shifted versions of  $\mathcal{S}^+$ ,  $\mathcal{S}^{+, \text{comb}}$  which will be denoted by  $\mathcal{S}_{[d]}^+$  and  $\mathcal{S}_{[d]}^{+, \text{comb}}$ , respectively.

Following [Cos09] we define:

- A two dimensional TFT of dimension  $d$  is a morphism of dg PROPs

$$F : \mathcal{S}_{[d]} \rightarrow \text{End}(V).$$

- A positive boundary two dimensional TFT is a morphism of dg PROPs

$$F : \mathcal{S}_{[d]}^+ \rightarrow \text{End}(V).$$

- A combinatorial positive boundary two dimensional TFT is a morphism of dg PROPs

$$F : \mathcal{S}_{[d]}^{+, \text{comb}} \rightarrow \text{End}(V).$$

The following result of Kontsevich-Soibelman [KS09] and [Cos07] provides us with non-trivial examples of combinatorial positive boundary two dimensional TFTs. Explicit formulas for the PROP action will be given in [CC20] following the ideas in [KS09] and [WW16].

**4.2. Theorem.** *Let  $A$  be a cyclic unital  $A_\infty$ -algebra of Calabi-Yau dimension  $d$ . Then there exists a combinatorial positive boundary two dimensional TFT of dimension  $d$  given by a dg PROP morphism*

$$\rho^A : \mathcal{S}_{[d]}^{+, \text{comb}} \rightarrow \text{End}(L),$$

whose underlying dg vector space is the shifted reduced Hochschild chain complex  $L = CC_*(A)[d]$  of the  $A_\infty$ -algebra  $A$ .

**4.3. The  $\text{Ann}^+$ -algebra structure.** An immediate consequence of Theorem 4.2 is the existence of an  $\text{Ann}^+$ -algebra structure on the dg vector space  $L$ . The degree one operator  $B = \rho^A(S) : CC_*(A)[d] \rightarrow CC_*(A)[d]$  is the Connes cyclic differential. The degree  $-2d$  operator

$$\langle -, - \rangle_{\text{Muk}} = \rho^A(M) : CC_*(A)[d] \otimes CC_*(A)[d] \rightarrow \mathbb{K}$$

is known as the Mukai pairing. It is a symmetric pairing of degree  $-2d$ . Applying the constructions of Theorems 2.10 and 2.20 we obtain two DGLAs naturally associated with the  $\text{Ann}^+$ -algebra  $L$ . Denote them by

$$\begin{aligned} \mathfrak{h} &= ((\text{Sym } L_-)[1]) \llbracket \hbar, \lambda \rrbracket \\ \widehat{\mathfrak{h}} &= \left( \bigoplus_{k \geq 1, l} \text{Hom}^c(\wedge^k L_+, \text{Sym}^l L_-)[2 - k] \right) \llbracket \hbar, \lambda \rrbracket \end{aligned}$$

Here we continue to use the notations of (2.11):  $L_+ = uL \llbracket u \rrbracket$  and  $L_- = L[u^{-1}]$ . We also denote by  $\text{Hom}^c$  the space of continuous homomorphisms in the  $u$ -adic topology ( $\widehat{\mathfrak{h}}$  is a Lie subalgebra of the one defined in Theorem 2.20).

**4.4. Lemma.** *Let  $A$  be a smooth, cyclic, and unital  $A_\infty$ -algebra which is assumed to satisfy the Hodge-de Rham degeneration property <sup>1</sup>. Then the morphism defined in (2.22)*

$$\iota : \mathfrak{h}^+ = ((\mathrm{Sym}^{\geq 1} L_-)[1])[[\hbar, \lambda]] \longrightarrow \widehat{\mathfrak{h}}$$

is a quasi-isomorphism of DGLAs.

**Proof.** It suffices to prove that for each  $n \geq 1$  the following sequence of dg vector spaces is exact:

$$0 \rightarrow \mathrm{Sym}^n L_- \rightarrow \mathrm{Hom}^c(L_+, \mathrm{Sym}^{n-1} L_-) \rightarrow \cdots \rightarrow \mathrm{Hom}^c(\wedge^n L_+, \mathbb{C}) \rightarrow 0.$$

We argue that the total complex of the sequence has zero homology, by considering the spectral sequence associated with the  $u$ -filtration. Since  $A$  satisfies the degeneration property the  $E^1$  page is given by

$$0 \rightarrow \mathrm{Sym}^n H_- \rightarrow \mathrm{Hom}^c(H_+, \mathrm{Sym}^{n-1} H_-) \rightarrow \cdots \rightarrow \mathrm{Hom}^c(\wedge^n H_+, \mathbb{C}) \rightarrow 0$$

where  $H = HH_*(A)[d]$  is the shifted Hochschild homology of  $A$ . Because  $A$  is assumed to be smooth,  $H$  is finite dimensional. From this finiteness condition we can deduce the exactness of the  $E^1$ -page as follows. Observe that there is an isomorphism for each  $k + l = n$

$$I_{k,l} : \wedge^k H_- \otimes \mathrm{Sym}^l H_- \rightarrow \mathrm{Hom}^c(\wedge^k H_+, \mathrm{Sym}^l H_-)$$

defined using the Mukai pairing on  $H$ . In the case of  $k = 1, l = 0$  this map is explicitly given by

$$I_{1,0}(x)(y) = \mathrm{res}_{u=0} \langle x, y \rangle.$$

For general  $k$  and  $l$  it is defined similarly. The fact that  $I_{k,l}$  is an isomorphism follows from a result of Shklyarov [Shk13] proving that the categorical Mukai pairing is non-degenerate when  $A$  is smooth (and finite dimensional, which follows from the cyclic property of  $A$ ). Putting the isomorphisms  $I_{k,l}$  together we conclude that the  $E^1$  page is isomorphic to

$$0 \rightarrow \mathrm{Sym}^n H_- \rightarrow H_- \otimes \mathrm{Sym}^{n-1} H_- \rightarrow \wedge^2 H_- \otimes \mathrm{Sym}^{n-2} H_- \rightarrow \cdots \rightarrow \wedge^n H_- \rightarrow 0,$$

which is endowed with the usual Koszul differential. The exactness follows.  $\square$

<sup>1</sup>If  $A$  is  $\mathbb{Z}$ -graded the degeneration property is a consequence of  $A$  being smooth by a result of Kaledin [Kal08].



**4.5. Pushing forward the string vertex.** By Theorem 4.2 there is a combinatorial positive boundary two dimensional TFT given by a morphism of dg PROPs

$$\rho^A : \mathcal{S}_{[d]}^{+, \text{comb}} \rightarrow \text{End}(L).$$

The PROP  $\text{Ann}_{[d]}^+$  sits inside  $\mathcal{S}_{[d]}^{+, \text{comb}}$  which gives us a commutative diagram of PROPs

$$\begin{array}{ccc} & \text{Ann}_{[d]}^+ & \\ & \swarrow \quad \searrow & \\ \mathcal{S}_{[d]}^{+, \text{comb}} & \xrightarrow{\quad} & \text{End}(L). \end{array}$$

Since the degree shifts in the definition of the shifted PROPs above are all even, the action  $\rho^A$  induces a morphism (which we still denote by  $\rho^A$ ) of  $\mathbb{Z}/2\mathbb{Z}$ -graded DGLAs

$$\rho^A : \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{h}}.$$

The combinatorial string vertex constructed in Theorem 3.15,

$$\widehat{\mathcal{V}}^{\text{comb}} = \sum_{g, k \geq 1, l} \widehat{\mathcal{V}}_{g, k, l}^{\text{comb}} \hbar^g \lambda^{2g-2+k+l}$$

yields a Maurer-Cartan element of  $\widehat{\mathfrak{h}}$  of the form

$$\widehat{\alpha}^A = \rho^A(\widehat{\mathcal{V}}^{\text{comb}}) = \sum_{g, k \geq 1, l} \widehat{\alpha}_{g, k, l}^A \hbar^g \lambda^{2g-2+k+l}$$

with

$$\widehat{\alpha}_{g, k, l}^A = \rho^A(\widehat{\mathcal{V}}_{g, k, l}^{\text{comb}}) \in \text{Hom}^c(\wedge^k L_+, \text{Sym}^l L_-).$$

Note that by the uniqueness of  $\widehat{\mathcal{V}}^{\text{comb}}$  the Maurer-Cartan element  $\widehat{\alpha}^A \in \widehat{\mathfrak{h}}$  is also uniquely defined up to gauge equivalence (though the Maurer-Cartan equation in  $\widehat{\mathfrak{h}}$  may have many non-equivalent solutions!).

Applying to the tensors  $\widehat{\alpha}_{g, k, l}^A$  the sign corrections in (1.13) we obtain tensors

$$\widehat{\beta}_{g, k, l}^A \in \text{Hom}^c(\text{Sym}^k(L_+[1]), \text{Sym}^l L_-)$$

such that

$$\widehat{\beta} = \sum_{g, k \geq 1, l} \widehat{\beta}_{g, k, l}^A \hbar^g \lambda^{2g-2+k+l}$$

satisfies the Maurer-Cartan equation in an appropriately modified version of  $\widehat{\mathfrak{h}}$ .

Furthermore, by Lemma 4.4 there exists a unique (up to gauge equivalence) Maurer-Cartan element  $\beta^A \in \mathfrak{h}^+$  such that

- $\beta^A = \sum_{g,n \geq 1} \beta_{g,n}^A \hbar^g \lambda^{2g+n-2}$  with  $\beta_{g,n}^A \in \text{Sym}^n L_-$ .
- $\iota(\beta^A)$  is gauge equivalent to  $\widehat{\beta}^A = \rho^A(\widehat{\mathcal{V}}^{\text{comb}})$ .

The elements  $\beta^A$  and  $\widehat{\beta}^A$  are homological analogues of the string functionals defined by Sen-Zwiebach [SZ94]. They will be used in [CT20] to define the categorical enumerative invariants.

## References

- [BCOV94] Bershadsky, M., Cecotti, S., Ooguri H., Vafa, C., *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Comm. Math. Phys. 165(2), 311-427 (1994).
- [CC20] Căldăraru, A., Cheung, W.-H., *Explicit formulas for the Kontsevich-Soibelman PROP*, in preparation
- [CT20] Căldăraru, A., Tu, J., *Categorical Enumerative Invariants, II: Givental formula*, arXiv:20???.?????
- [Cos07] Costello, K., *Topological conformal field theories and Calabi-Yau categories*, Advances in Mathematics, Volume 210, Issue 1, 20 March 2007, 165-214.
- [Cos09] Costello, K., *The partition function of a topological field theory*, J. Topol. 2 (2009), no. 4, 779-822.
- [CZ19] Costello, K., Zwiebach, B., *Hyperbolic String Vertices*, preprint, arXiv:1909.00033.
- [Ega15] Egas Santander, D., *Comparing fat graph models of moduli space*, arXiv:1508.03433.
- [GPS15] Ganatra, S., Perutz T., Sheridan N., *Mirror symmetry: from categories to curve counts*, arXiv:1510.03839.
- [Kal08] Kaledin, D. *Non-commutative Hodge-to-de-Rham degeneration via the method of Deligne-Illusie*. Pure Appl. Math. Q. 4 (2008), no. 3, Special Issue: In honor of Fedor Bogomolov. Part 2, 785-875.
- [KS09] Kontsevich, M., Soibelman, Y., *Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I*, Homological mirror symmetry, 153-219, Lecture Notes in Phys., 757, Springer, Berlin, 2009.
- [Lur09] Lurie, J., *On the classification of topological field theories*, Current Developments in Mathematics Volume 2008 (2009), 129-280.
- [Mar06] Markl, M., *Operads and PROPs*, Handbook of Algebra 5: 87-140, January 2006.
- [SZ94] Sen, A., Zwiebach, B., *Quantum background independence of closed-string field theory*, Nuclear Phys. B 423(2-3), (1994), 580-630.
- [Shk13] Shklyarov, D., *Hirzebruch-Riemann-Roch-type formula for DG algebras*, Proceedings of the London Mathematical Society (2013) 106 (1): 1-32.

[WW16] Wahl, N.; Westerland, C., *Hochschild homology of structured algebras*. Adv. Math. 288 (2016), 240-307.

*Andrei Căldăraru, Mathematics Department, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706–1388, USA.*

*Kevin Costello, Perimeter Institute, Perimeter Institute of Theoretical Physics, 31 Caroline St. N, Waterloo, ON N2L 2Y5, Canada*

*Junwu Tu, Institute of Mathematical Sciences, ShanghaiTech University, Shanghai, China, 201210.*