Finding the Walls in Geometric Invariant Theory

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What is GIT?

A way to construct varieties (and stacks):

INPUTS	OUTPUTS
G = complex reductive group	open set $X^{ss}_\theta(G) \subseteq X$
X = representation of G	GIT quotient $X /\!\!/_\theta G := [X^{ss}_\theta(G)/G]$
$\theta : G \to \mathbb{C}^*$ homomorphism	

Examples:

$$\mathbb{P}^n = \mathbb{C}^{n+1} /\!\!/_{\theta(t)=t} \mathbb{C}^*$$

$$Gr(k,n) = M_{k\times n} /\!\!/_{\theta(g)=\det(g)} GL_k$$

many toric varieties, using $G = (\mathbb{C}^*)^k$

moduli of representations of quivers, using $G = \prod GL_{k_i}/\mathbb{C}^*$

Note: $X /\!\!/_\theta G$ can be a variety or Deligne-Mumford stack, or it can be an Artin stack.

What is VGIT?

Study dependence of $X /\!\!/_\theta G$ on θ:

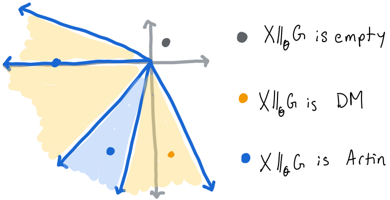
INPUTS

G = complex reductive group
X = representation of G

OUTPUTS

$\Sigma(G, X)$ = fan in $\mathrm{char}(G)_\mathbb{R}$
cones biject with possible $X^{ss}_\theta(G)$

hypothetical
$\Sigma(G, X) \subset \mathrm{char}(G)_\mathbb{R}$

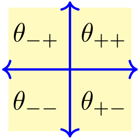


Questions:

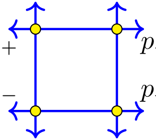
- How to compute $\Sigma(G, X)$?
- For which (G, X) does there exist non-Artin $X /\!\!/_\theta G$?

Mirror symmetry motivation for VGIT

$G = GL_2 \quad X = M_{2\times n} \times M_{n\times 2} \quad g\cdot(x,y) = (gx, yg^{-1}) \quad T = \mathrm{diag} \subseteq G$



$\Sigma(T, X)$



$\mathcal{M}_{T,X} = \mathbb{P}^1 \times \mathbb{P}^1$

Theorem (Coates-Corti-Iritani-Tseng)

A generating series $I^{X/\!\!/_\theta T}$ of Gromov-Witten invariants is an analytic function in a neighborhood of p_θ. The series $I^{X/\!\!/_\theta T}$ are related by analytic continuation and a symplectic transformation.

$\Sigma(G, X)$

$\mathcal{M}_{G,X} = \mathbb{P}^1$

Theorem (Lutz-Shafi-W, see also Priddis-Shoemaker-Wen)

The theorem holds for the Grassmannian flop by "taking Weyl invariants."

References

- Mumford '65: Introduces the study of GIT quotients of quasiprojective X
- King '93: Specializes definitions to $X = \mathbb{C}^n$; numerical criterion allows to compute small examples by hand
- Dolgachev-Hu '98: Introduces the study of variation of GIT
- Arzhantsev-Hausen '09: The VGIT fan is a fan and is determined by its walls; tools for computation

Questions:

- How to compute $\Sigma(G, X)$?
- For which (G, X) does there exist non-Artin $X /\!\!/_\theta G$?

Toric VGIT

INPUTS

$\mathcal{Q} \in M_{k\times n}(\mathbb{Z})$

matrix representing action
of $(\mathbb{C}^*)^k$ on $\mathbb{C}^n$

OUTPUTS

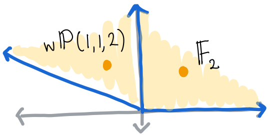
$\Sigma \subset \mathbb{R}^k$ the VGIT fan

ccr of triangulations of columns of $\mathcal{Q}$
Artin quotients $\leftrightarrow$ low-dim cones

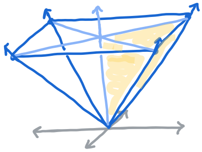
Examples:

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad X = \mathbb{C}^4,\ G = (\mathbb{C}^*)^2$$

$$(s,t)\cdot(x,y,z,w) = (sx, sy, s^{-2}tz, tw)$$



$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



Quiver VGIT

INPUTS

Q a directed graph (quiver) with vertices Q_0

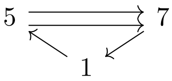
$\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$

OUTPUTS

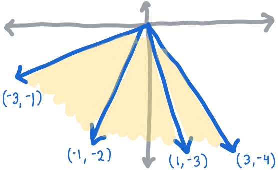
$\Sigma \subset {}^{\perp}\mathbf{d}$ the VGIT fan

ambient dim $|Q_0| - 1$

Example:



$$X = M_{5\times 1} \times M_{7\times 5} \times M_{7\times 5} \times M_{1\times 7}$$
$$G = GL_1 \times GL_5 \times GL_7/\mathbb{C}^*$$
$$(g,h,k)\cdot(x,\mathbf{y},z) = (hxg^{-1}, k\mathbf{y}h^{-1}, gzk^{-1})$$



- Schofield ‘92: recursive computation of generic subdimension vectors
- King ’93: constructs moduli of quiver representations via GIT
- Derksen-Weyman ’00: For acyclic quivers, support $\mathrm{sst}(\mathbf{d})$ of $\Sigma(Q, \mathbf{d})$ is determined by *generic subdimension vectors*
- Hille-de la Peña ’02: For acyclic quivers, study the *wall system*: minimal union of hyperplanes separating VGIT chambers
- Chindris ’08: For acyclic quivers, shows existence of the VGIT fan
- Belmans-Franzen-Petrella: Packages for computing $\mathrm{sst}(\mathbf{d})$

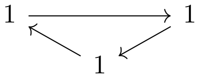
Theorem (Franzen-Petrella-W.)

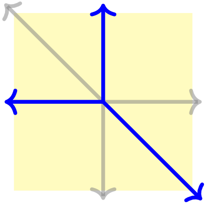
- $X /\!\!/_\theta G$ *is Artin iff* $\theta \in W_{\mathbf{e}}$ *for nonzero* $\mathbf{e} \subsetneq \mathbf{d}$*, where*

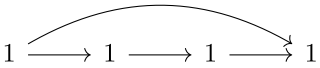
$$W_{\mathbf{e}} = \mathrm{sst}(\mathbf{e}) \cap \mathrm{sst}(\mathbf{d} - \mathbf{e}).$$

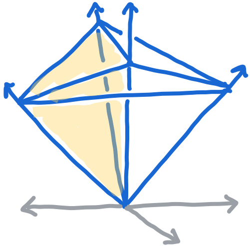
- (G, X) *has a non-Artin quotient if and only if* $\mathbf{d}$ *is an indivisible Schur root.*

A gallery of walls: cones not hyperplanes

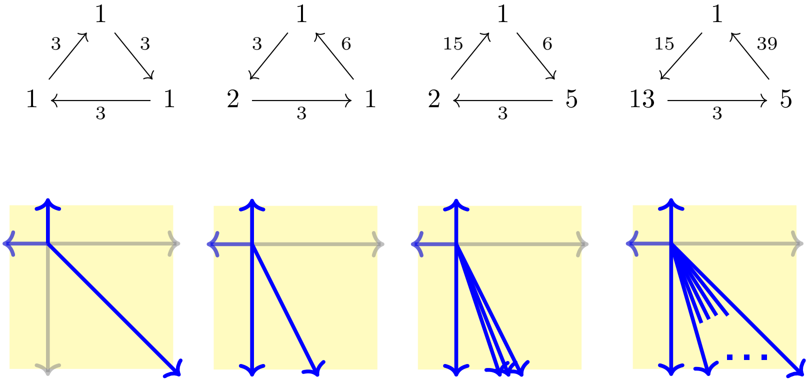






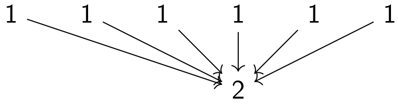


A gallery of walls: mutation



Seiberg duality: Zhang, He-Zhang

A gallery of walls: Segre cubic



$\theta = (2, 2, 2, 2, 2, 2, -6)$

- good moduli space of $X /\!\!/_\theta G$ is the (singular) Segre cubic
- VGIT fan has 1678 full-dimensional cones

General reductive groups

Choose G a connected complex reductive group. Then

$$G \approx H_1 \times H_2 \times \ldots \times H_M \times (\mathbb{C}^*)^k$$

with possible H_i indexed by connected Dynkin diagrams.

Possible H_i:

- $A_n = SL_{n+1}$
- $B_n =$ s.c. cover of SO_{2n+1}
- $C_n = Sp_{2n+1}$
- $D_n =$ s.c. cover of SO_{2n}
- E_6, E_7, E_8, F_4, G_2

Notice: For toric, quiver VGIT the group G is type A—all H_i's are A_{n_i}

Me: Is there a non-Artin $X /\!\!/_\theta G$ for some G not of type A?

Quasimap theory: Work with $X /\!\!/_\theta G$, no restrictions on the type of G, but assume $X /\!\!/_\theta G$ is non-Artin.

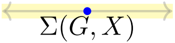
Lie theorists: Type A is very special.

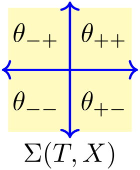
Arzhantsev-Hausen: Here's an example with $G = Sp_{2n} \times (\mathbb{C}^*)^k$. All $X /\!\!/_\theta G$ are Artin.

Me: How to compute the VGIT fan of (X, G)?

Arzhantsev-Hausen: Like this!

Me: I still don't know how to answer the first question . . .

Recall: 

is the Weyl-invariant part of 

In general:

Set $T =$ maximal torus in G,

$W =$ Weyl group

Then $\mathrm{char}(G)_{\mathbb{R}} = \mathrm{char}(T)_{\mathbb{R}}^{W}$, and

$\Sigma(T, X) \cap \mathrm{char}(T)_{\mathbb{R}}^{W}$ is a fan

Theorem (Kurama-Li-Talbott-W.)

(1) *For $r > \dim(G)$, have $\Sigma(G, X^{\oplus r}) = \Sigma(T, X^{\oplus r}) \cap \mathrm{char}(T)_{\mathbb{R}}^{W}$.*

(2) *If $X /\!\!/_{\theta} G$ is nonempty/Artin, then $X /\!\!/_{\theta} T$ is nonempty/Artin.*

(3) *The converse of (2) is false.*

Theorem (Kurama-Li-Talbott-W.)

(1) *For $r > \dim(G)$, have $\Sigma(G, X^{\oplus r}) = \Sigma(T, X^{\oplus r}) \cap \mathrm{char}(T)^W_{\mathbb{R}}$.*

(2) *If $X /\!\!/_\theta G$ is nonempty/Artin, then $X /\!\!/_\theta T$ is nonempty/Artin.*

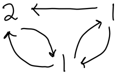
(3) *The converse of (2) is false.*

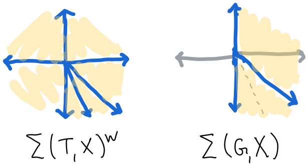
Examples:

$G = GL_k$

$X = M_{k \times n}$

$k > n$





Theorem (Kurama-Li-Talbott-W.)

(1) *For* $r > \dim(G)$*, have* $\Sigma(G, X^{\oplus r}) = \Sigma(T, X^{\oplus r}) \cap \mathrm{char}(T)^W_{\mathbb{R}}$*.*

(2) *If* $X /\!\!/_\theta G$ *is nonempty/Artin, then* $X /\!\!/_\theta T$ *is nonempty/Artin.*

(3) *The converse of (2) is false.*

Corollary

An example with $\Sigma(T, X) \cap \mathrm{char}(T)^W_{\mathbb{R}}$ *not in the walls of* $\Sigma(T, X)$ *gives an example of non-Artin* $X /\!\!/_\theta G$ *(after replacing* X *with* $X^{\oplus r}$*).*

Theorem (Kurama-Li-Talbott-W.)

(1) *If* $\Sigma(T, X) \cap \mathrm{char}(T)_{\mathbb{R}}^W$ *is not in the walls, then* G *has type* A.

(2) *In this case, if* $G = \prod_{i=1}^M SL_{n_i} \times \mathbb{C}^*$, *then*

- *the* n_i *are pairwise coprime*
- $X = \oplus_{a \in \mathbb{Z}} X_a \otimes \mathbb{C}_a$, *where*

$$\text{for } a > 0 \text{ we have } X_a = \bigotimes_{i=1}^M \left\{ \begin{array}{c} \text{(dual of) } Sym^1(\mathbb{C}^{n_i}) \\ \text{(dual of) } Sym^{2\ell_a+1}(\mathbb{C}^2) \end{array} \right\}$$

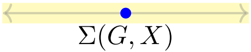
(3) *If* $G = \prod SL_{n_i} \times (\mathbb{C}^*)^k$ *and* $X = X_+ \otimes (\mathbb{C}_{a_1} \oplus \ldots \oplus \mathbb{C}_{a_m})$ *for some nondegenerate* X_+ *and* a_i *spanning* $\mathrm{char}((\mathbb{C}^*)^k)$, *then* $\Sigma(T, X) \cap \mathrm{char}(T)_{\mathbb{R}}^W$ *is not in the walls.*

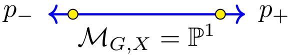
Corollary

There are no "easy" examples of non-Artin $X /\!\!/_\theta G$, *for* G *not of type* A.

Theorem (many authors)

For toric quotients or the Grassmannian flop, a generating series $I^{X /\!\!/_\theta G}$ of Gromov-Witten invariants is an analytic function in a neighborhood of p_θ. The series $I^{X /\!\!/_\theta G}$ are related by analytic continuation and a symplectic transformation.





Cautious takeaway: This mirror symmetry picture is only a story about type-A GIT quotients.

Observation (Ben Elias)

If W preserves a vector space basis of $\mathrm{char}(T)_\mathbb{R}$, then G has type A.

Definite takeaway: Type A is very special.

For non-type A, W-invariants $\subseteq$ walls

$$\mathrm{degen}(X_\lambda) = \text{min } \ell \text{ s.t. } 0 = \sum_{i=1}^{\ell+1} a_i \xi_i, a_i > 0,\ \xi_i \in \mathrm{weights}(X_\lambda)$$

H	dominant weight λ	bound on $\mathrm{degen}(X_\lambda)$
A_1	$(k, 0)$ k odd	1
	$(k, 0)$ k even	0
$A_n,\ n \geq 2$	(1,0 . . . , 0)	n
	(1,1 . . . ,1, 0)	n
	all other λ	$n-1$
$B_n,\ n \geq 2$	ω_n	1
$C_n,\ n \geq 3$	ω_1	1
$D_n,\ n \geq 4$ even	ω_1	1
	ω_{n-1}	1
	ω_n	1
$D_n,\ n \geq 5$ odd	ω_1	1
	ω_{n-1}	3
	ω_n	3
E_6	ω_1	2
	ω_6	2
E_7	ω_7	1
E_8, F_4, G_2	0	0

Thanks for listening!