HODGE THEORY FOR MATROIDS

Thanks to ideas from algebraic geometry, matroid theory (also called combinatorial geometry) had some major developments in the past decade. In this note, we aim to survey two results in this development: the proof of Heron-Rota-Welsh conjecture by Adiprasito, Huh and Katz [AHK18], and the proof of Dowling-Wilson conjecture by Braden, Huh, Matherne, Proudfoot and Wang [BHM+20].

1. Some motivating results in combinatorics

We start by presenting two results which are special cases of the Heron-Rota-Welsh conjecture and the Dowling-Wilson conjecture.

1.1. **The Read conjecture.** Given a graph G and a positive integer t, a *proper coloring* of G is an assignment of one of t-colors to each vertex of G such that the two endpoints of each edge are assigned different colors. The *chromatic polynomial* of G is defined to be the total number of proper colorings as a function of t, denoted by $P_G(t)$.

Example 1.1.

- When G has a loop, $P_G(t) = 0$.
- When G is a connected tree with n-vertices, $P_G(t) = t(t-1)^{n-1}$.
- When G is the complete graph on n verticies, $P_G(t) = t(t-1)\cdots(t-n+1)$.

The chromatic polynomial satisfies the following deletion-contraction relation.

Lemma 1.2. *Given any edge e of G, we have*

$$P_G(t) = P_{G\setminus e}(t) - P_{G/e}(t),$$

where $G \setminus e$ and G/e are the graphs obtained from G by deleting and contracting the edge e, respectively.

Using inductions on the number of edges, one can easily deduce the following properties from the deletion-contraction relation.

Corollary 1.3.

(i) For any loopless graph G, the chromatic polynomial $P_G(t)$ is a monic polynomial.
(ii) The coefficients of $P_G(t)$ are alternating, that is, if $P_G(t) = t^n + a_1t^{n-1} + \cdots + a_n$, then $(-1)^i a_i \geq 0$.

The following result of Huh confirms the Read conjecture from the 60's.

Theorem 1.4 (Huh, 2012)**.** *The (absolute values of the) coefficients of $P_G(t)$ form log-concave sequence. In other words, if $P_G(t) = t^n + a_1t^{n-1} + \cdots + a_n$, then $|a_{i-1}a_{i+1}| \leq a_i^2$.*

Remark 1.5. In fact, the sequence $|a_1|, \ldots, |a_n|$ has no internal zeros (which follows from Huh's proof, but is also a previously known result). Obviously, among the nonzero terms, $|a_{i-1}a_{i+1}| \leq a_i^2$ is equivalent to $\log|a_i|$ being concave. For the rest of the note, when we say a sequence a_k is log-concave, we mean the sequence $|a_k|$ has no internal zeros, and $|a_{k-1}a_{k+1}| \leq a_k^2$ for all k.

1.2. **The realizable Dowling-Wilson conjecture.** Let V be a d-dimensional vector space over a field, and let $E \subset V$ be a finite generating set. Let

$$\mathcal{F} = \{\text{all linear subspaces of } V \text{ generated by a subset of } E\},$$

and $\mathcal{F}_k = \{F \in \mathcal{F} \mid \dim F = k\}$. Denote the cardinality of $\mathcal{F}_k$ by W_k. The sequence of numbers W_k are called *Whitney numbers of the second kind*. The following result confirms a conjecture of Dowling-Wilson in the realizable case.

Theorem 1.6 ([HW17]). *Let W_k be defined as above. For $k \leq d/2$, we have $W_k \leq W_{d-k}$.*

It turns out that for both theorems, the proofs essentially use algebraic geometry. In fact, the first theorem reduces to the Hodge index theorem, and the second reduces to the hard Lefschetz theorem for projective varieties.

Both statements can be generalized to matroids, where the corresponding variety may not exist. Then the key idea is to produce some combinatorial proofs of the analogous statements (Hodge index theorem, or more generally the Hodge-Riemann relations, and the hard Lefschetz theorems) on some combinatorial cohomology ring of matroids (or Chow rings, or intersection cohomology groups).

2. Introduction to matroids

2.1. **Definitions.** Given a finite set E in a vector space, we can consider the independent sets of E, defined as

$$\mathcal{I}_E = \{I \subset E \mid I \text{ is an independent set}\}.$$

Given a graph G with edge set $E(G)$, we can define the set of forests,

$$\mathcal{I}_G = \{I \subset E(G) \mid I \text{ does not contain a cycle}\}.$$

The set of independent sets and forests share a common combinatorial property, called the exchange lemma.

Lemma 2.1. *Let $\mathcal{I}_E$ be defined as above. If $I_1, I_2 \in \mathcal{I}_E$ satisfies $|I_2| > |I_1|$, then there exists $x \in I_2$ such that $I_1 \cup x \in \mathcal{I}$.*[1] *The same statement holds for the set of forests $\mathcal{I}_G$.*

Matroid is the combinatorial structure that captures the independence conditions from both linear algebra and graphs.

Definition 2.2. A *matroid* consists of a finite set E and a collection of subsets $\mathcal{I} \subset 2^E$, satisfying the following properties.

(i) $\emptyset \in \mathcal{I}$;
(ii) if $I \in \mathcal{I}$ and $I' \subset I$, then $I' \in \mathcal{I}$;
(iii) if $I_1, I_2 \subset \mathcal{I}$ and $|I_1| < |I_2|$, then there exists $x \in I_2$ such that $I_1 \cup x \in \mathcal{I}$.

The sets in $\mathcal{I}$ are called *independent sets*.

When E is a finite subset of a vector space, the independent sets $\mathcal{I}_E$ defines a matroid.

There are many other equivalent definitions of matroids in terms of flats, rank functions, closure operators, bases, basis polytopes, etc. Here, we mention a few.

Definition 2.3. A matroid is a pair $(E, \mathcal{B})$, where E is a finite set and $\mathcal{B} \subset 2^E$, satisfying the following properties.

(i) $\mathcal{B}$ is nonempty;
(ii) If $A, B \in \mathcal{B}$ and $x \in A \setminus B$, then there is $y \in B \setminus A$ such that $(A \setminus x) \cup y \in \mathcal{B}$.

The sets in $\mathcal{B}$ are called *bases*.

Definition 2.4. A matroid is a pair $(E, \mathcal{F})$, where E is a finite set and $\mathcal{F} \subset 2^E$, satisfying the following properties.

(i) $\mathcal{F}$ is nonempty;
(ii) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$;
(iii) given any $F \in \mathcal{F}$, every element of $E \setminus F$ is in exactly one minimal set in $\mathcal{F}$ which strictly contains F.

The sets in $\mathcal{F}$ are called *flats*.

[1] Throughout this note, we abuse notations and use the element x to also denote the singleton set $\{x\}$.

Remark 2.5. For any pair of flats F, G of a matroid M, there exists a unique minimal flat that contains both F and G, which is called the *join* of F and G, and denoted by $F \vee G$. There also exists a unique maximal flat that is contained in both F and G, which is called the *meet* of F and G, and denoted by $F \wedge G$. Clearly, $F \wedge G = F \cap G$, but $F \cup G \subset F \vee G$ and the inclusion can be strict.

Remark 2.6. When E is a finite subset of a vector space V, the set of flats

$$\mathcal{F} = \{W \cap E \mid W \subset V \text{ is a linear subspace}\}$$

defines a matroid.

Definition 2.7. A matroid is a pair (E, rk), where E is a finite set and $\mathrm{rk} : 2^E \to \mathbb{Z}_{\geq 0}$ satisfying the following properties.

(i) for any $S \subset E$, $\mathrm{rk}(S) \leq |S|$;
(ii) if $S \subset T \subset E$, then $\mathrm{rk}(S) \leq \mathrm{rk}(T)$;
(iii) the function rk is submodular, that is,

$$\mathrm{rk}(S) + \mathrm{rk}(T) \geq \mathrm{rk}(S \cup T) + \mathrm{rk}(S \cap T)$$

for any $S, T \subset E$.

The function rk is called the *rank function*.

Remark 2.8. When E is a finite subset of a vector space V, $\mathrm{rk}(S) = \dim \mathrm{span}(S)$ defines a matroid.

Exercise 2.9. Explain how to relate one definition with the other, and prove that they are all equivalent.

Definition 2.10. Let M be a matroid defined over a set E using one of the above equivalent definitions. Then E is called the *ground set* of M. The *rank* of M is defined to be $\mathrm{rk}(E)$, and also denoted by $\mathrm{rk}(M)$. An element $i \in E$ is called a *loop* if $\mathrm{rk}(i) = 0$, which is equivalent to that i is not contained in any independent set, and is further equivalent to that i is contained in every flat. Two elements $i, j \in E$ are called *parallel*, if $\mathrm{rk}(i) = \mathrm{rk}(j) = \mathrm{rk}(\{i, j\}) = 1$. If a matroid has no loops, then we say it is *loopless*. If a matroid has no loops or parallel elements, then we say it is *simple*. An element $i \in E$ is called a *coloop* if $\mathrm{rk}(E \setminus i) = \mathrm{rk}(E) - 1$, or equivalently, i is contained in very basis.

Exercise 2.11. Every matroid M has a canonical *simplification*, that is, a simple matroid M' whose poset of flats is isomorphic to the poset of flats of M.

Example 2.12. A *uniform matroid* on $E = \{1, \ldots, n\}$ with rank r, denoted by $U_{r,n}$, is a matroid whose bases are all r-element subsets of E. When $r = n$, the matroid $U_{n,n}$ is called a *Boolean matroid*.

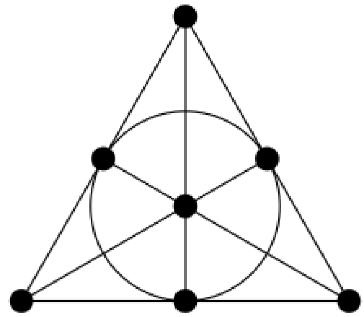
Definition 2.13. A matroid defined using trees in a graph as discussed before is called a *graphic matroid*. Given a field K, a matroid defined by a finite subset E in a K-vector space V is called *realizable over* K. A matroid realizable over some field K is called *realizable*. A finite set E in a vector space V is called a *vector configuration*.

Proposition 2.14. *A graphic matroid is realizable over any field.*

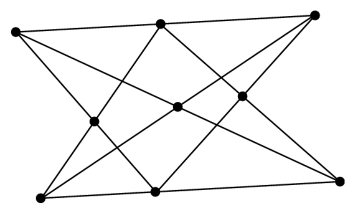
Proof. Given a finite graph G, we label its vertices by $1, 2, \ldots, n$. Fixing any field K, we let V be the K-vector space with basis $v_1, \ldots, v_n$. Let the set of edges be $E = \{e_1, \ldots, e_m\}$, and assume that the two ends of each e_i be $a_i, b_i \in \{1, \ldots, n\}$. Now, we choose the subset $E_V \subset V$ to be $\{v_{a_i} - v_{b_i} \mid i = 1, \ldots, m\}$. Then it is easy to check that a subset of E form a forest if and only if the corresponding set of vectors in V are independent. □

Remark 2.15. As in the above proof, the subset E does not generate V. In fact, the codimension of $\mathrm{span}(E)$ is equal to the number of connected components of G.

Example 2.16. The fano matroid, defined as in the following picture, is realizable over any field of characteristic 2, and not over any other characteristics.



The non-Pappus matroid, defined as in the following, is not realizable over any field.



2.2. **Matroid operations.** Given a matroid M and an element e in the ground set E, we can form two new matroids $M \setminus e$ and M/e, the deletion and contraction of e. They corresponds to the deletion and contraction of an edge in a graph.

Definition 2.17. Let M be a matroid with ground set E. For any $e \in E$, we define the *deletion matroid* $M\setminus e$ to be the matroid on $E\setminus\{e\}$ with rank functions $\mathrm{rk}_{M\setminus e}(S) = \mathrm{rk}_M(S)$ for any $S \subset E$. We define the *contraction matroid* M/e to be the matroid on $E\setminus\{e\}$ with rank function $\mathrm{rk}_{M/e}(S) = \mathrm{rk}(S\cup\{e\})-r(e)$.

Remark 2.18. Equivalently, the deletion and contraction can be defined using independent sets:

$$\mathcal{I}_{M\setminus e} = 2^{E\setminus\{e\}} \cap \mathcal{I}_M$$

and

$$\mathcal{I}_{M/e} = \{I \subset E \setminus e \mid I \cup \{e\} \in \mathcal{I}_M\}.$$

Remark 2.19. If e is a loop, then $M \setminus e = M/e$, whose flats are naturally bijective the the flats of M. If e is not a loop, then $\mathrm{rk}(M/e) = \mathrm{rk}(M) - 1$. On the other hand, if e appears in every basis (in this case, e is called a *coloop*), then $\mathrm{rk}(M \setminus e) = \mathrm{rk}(M) - 1$; otherwise, $\mathrm{rk}(M \setminus e) = \mathrm{rk}(M)$.

Definition 2.20. Given a flat F of a matroid M, we define the matroid M^F to be the matroid deleting every element of $E \setminus F$ from M. We also define the matroid M_F to be the matroid contracting every element of F from M.

Exercise 2.21. Prove that the poset of flats of M^F is naturally isomorphic to the poset of the flats $\{G \in \mathcal{F} \mid G \leq F\}$. Similarly, prove that the poset of flats of M_F is naturally isomorphic to the poset of the flats $\{G \in \mathcal{F} \mid G \geq F\}$.

Remark 2.22. If M is a simple matroid, then M^F is also simple for any flat F. However, M_F may fail to be simple. If M is a loopless matroid, then M^F and M_F are loopless for any flat F.

Definition 2.23. Another way to construct new matroid is taking direct sum. Let M_1 and M_2 be matroids with ground sets E_1 and E_2 respectively. Then their *direct sum* $M_1 \oplus M_2$ is defined to be the matroid with ground set $E_1 \sqcup E_2$ and

$$\mathcal{I}_{M_1\oplus M_2} = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_{M_1}, I_2 \in \mathcal{I}_{M_2}\}.$$

Remark 2.24. Equivalently, the direct sum matroid can be defined using basis

$$\mathcal{B}_{M_1\oplus M_2} = \{B_1 \cup B_2 \mid B_1 \in \mathcal{B}_{M_1}, B_2 \in \mathcal{B}_{M_2}\},$$

or rank function

$$\mathrm{rk}_{M_1\oplus M_2}(S_1\cup S_2)=\mathrm{rk}_{M_1}(S_1)+\mathrm{rk}_{M_2}(S_2),$$

or flats

$$\mathcal{F}_{M_1\oplus M_2}=\{F_1\cup F_2\mid F_1\in\mathcal{F}_{M_1},F_2\in\mathcal{F}_{M_2}\}.$$

2.3. **Characteristic polynomials.**

Definition 2.25. Let M be a matroid with ground set E. Its characteristic polynomial is defined as

$$\chi_M(t)=\sum_{S\subset E}(-1)^{|S|}t^{r(M)-r(S)}.$$

Proposition 2.26. *The characteristic polynomial $\chi_M(t)$ satisfies the following properties:*

(i) (loop property) if M has a loop, then $\chi_M(t)=0$;

(ii) (normalization) the characteristic polynomial of the uniform matroid $U_{1,1}$ satisfies

$$\chi_{U_{1,1}}(t)=t-1;$$

(iii) (direct sum) if $M=M_1\oplus M_2$, then

$$\chi_{M_1\oplus M_2}(t)=\chi_{M_1}(t)\cdot\chi_{M_2}(t);$$

(iv) (deletion/contraction) if e is not a coloop of M, then

$$\chi_M(t)=\chi_{M\setminus e}(t)-\chi_{M/e}(t).$$

Moreover, the characteristic polynomial is the unique way to associate each matroid a polynomial such that all the above properties are satisfied.

Proposition 2.27. *Let G be a graph, and let M be the associated matroid. Then,*

$$P_G(t)=t^l\cdot\chi_M(t)$$

where l is the number of connected components of G.

2.4. **Hyperplane arrangement.** As we have discussed, a realizable matroid corresponds to a vector configuration. Dually, a (loopless) realizable matroid also corresponds to a hyperplane arrangement.

Definition 2.28. Fixing a d-dimensional vector space V, a *central hyperplane arrangement* is a collection $\mathcal{A}=\{H_1,\dots,H_n\}$ of $(d-1)$-dimensional linear subspaces (also called hyperplanes). Here we allow two hyperplanes to be the same. The hyperplane arrangement $\mathcal{A}$ is called *essential*, if the intersection of all the hyperplanes H_i is equal to zero. Given a hyperplane arrangement $\mathcal{A}=\{H_1,\dots,H_n\}$ in V, we define the *hyperplane arrangement complement* to be the open subvariety $U=V\setminus(H_1\cup\cdots\cup H_n)$ of V.

Proposition 2.29. *Given such a central hyperplane arrangement $\mathcal{A}$, it defines a matroid M on $E=\{1,\dots,n\}$ whose rank function is given by*

$$r(S)=d-\dim\bigcap_{i\in S}H_i.$$

In the realizable case, the characteristic polynomial of a matroid is closely related to the geometry of the corresponding hyperplane arrangement complement.

Proposition 2.30. *Let M be the loopless matroid associated to an essential hyperplane arrangement $\mathcal{A}$ in a K-vector space V, and let U be the hyperplane arrangement complement. If $\chi_M(t)=t^d+a_1t^{d-1}+\cdots+a_d$, then in $K_0(\mathrm{Var}_K)$, the Grothendieck ring of algebraic varieties over K,*

$$[U]=\mathbb{L}^d+a_1\mathbb{L}^{d-1}+\cdots+a_d,$$

where $\mathbb{L}=[\mathbb{A}^1]$.

As consequences of the above proposition, we have the following.

Corollary 2.31. *Let $\mathcal{A}$ be an essential hyperplane arrangement in a vector space V over a finite field $\mathbb{F}_q$ with q elements, and let U be the complement. Denote the associated matroid to be M. Then the number of $\mathbb{F}_{q^r}$ points on U is qual to $\chi_M(q^r)$.*

Corollary 2.32. *Let $\mathcal{A}$ be an essential hyperplane arrangement in a complex vector space V, and let U be the complement. Denote the associated matroid to be M. Then,*

$$\begin{aligned}\chi_M(t) &= \sum_{0\le k\le d}(-1)^k \dim_{\mathbb{Q}} \mathrm{H}^k(U,\mathbb{Q})t^{d-k}\\ &= \sum_{0\le k\le d}(-1)^{d-k}\dim_{\mathbb{Q}} \mathrm{H}_c^{d+k}(U,\mathbb{Q})t^k.\end{aligned}$$

Sketch of proof. The two summations are equal to each other by Poincaré duality. So it is enough to show $\chi_M(t)$ is equal to the second summation.

To relate the Betti numbers and the class in the Grothendieck ring, we need a fact that the mixed Hodge structure on $\mathrm{H}^k(U,\mathbb{Q})$ is of (k,k)-type. In fact, when $k=1$, this follows from the fact that U admits a good compactification $U\subset\mathbb{P}^{d-1}$ with $\mathrm{H}^1(\mathbb{P}^{d-1},\mathbb{Q})=0$. In general, we know that the cohomology ring $\mathrm{H}^\bullet(U,\mathbb{Q})$ is generated in degree one (e.g., by the theorem of Orlik-Solomon). Thus, the mixed Hodge structure on $\mathrm{H}^k(U,\mathbb{Q})$ is of (k,k)-type. By Poincaré duality, the Hodge structure on $\mathrm{H}_c^k(U,\mathbb{Q})$ is of $(k-d,k-d)$-type.

The class $[U]\in K_0(\mathrm{Var}_K)$ determines the (compactly supported) Hodge-Deligne polynomial of U:

$$E(U)=\sum_k\sum_{p,q}(-1)^k h^{p,q}\left(\mathrm{H}_c^k(U,\mathbb{Q})\right)x^py^q$$

where $h^{p,q}(\mathrm{H}_c^k(U,\mathbb{Q}))$ denotes the dimension of the (p,q)-component of the mixed Hodge structure on $\mathrm{H}_c^k(U,\mathbb{Q})$. Since $\mathrm{H}_c^k(U,\mathbb{Q})$ is of $(k-d,k-d)$-type, we know that

$$\begin{aligned}E(U) &= \sum_{d\le k\le 2d}(-1)^k\dim \mathrm{H}_c^k(U,\mathbb{Q})(xy)^{k-d}\\ &= \sum_{0\le k\le d}(-1)^k\dim \mathrm{H}_c^{2d-k}(U,\mathbb{Q})(xy)^{d-k}.\end{aligned}$$

On the other hand, since $E(\mathbb{C}^k)=(xy)^k$, by Proposition 2.30,

$$E(U)=\chi_M(xy).$$

Combining the above two equalities and substitute k by $d-k$, we have the desired equality

$$\chi_M(t)=\sum_{0\le k\le d}(-1)^{d-k}\dim \mathrm{H}_c^{d+k}(U,\mathbb{Q})t^k.$$

□

Remark 2.33. The above two corollaries imply that the Betti numbers of a hyperplane arrangement complement (over $\mathbb{C}$) and the number of $\mathbb{F}_{q^r}$ points on the complement (over a finite field) are combinatorial invariants. A deeper theorem of Orlik-Solomon says that the cohomology ring of a hyperplane arrangement complement (over $\mathbb{C}$) is also a combinatorial invariant. In this note, we will not get into details along this direction.

Remark 2.34. In [Huh12], Huh proved the log-concavity of the coefficients of the characteristic polynomial for matroids realizable over a field of characteristic 0. He used the above corollary (more precisely the projective analog) and a theorem of Dimca-Papadima to realize the coefficients of the characteristic polynomial as intersections numbers on a projective variety. In the later paper of Huh and Katz [HK12], they used combinatorial arguments to show the coefficients of the (reduced) characteristic polynomial are the desired intersection numbers.

3. Chow ring of a matroid

3.1. **Log-concavity in the realizable case.** Before we talk about the general theory of Chow ring of a matroid, let us give a sketch of the proof of the following theorem of Huh and Katz. Throughout this section, when we talk about a matroid M, we assume it has positive rank.

Theorem 3.1 ([HK12])**.** *Let M be a realizable matroid. Then the (absolute values of the) coefficients of $\chi_M(t)$ form log-concave sequence. In other words, if $\chi_M(t) = t^d + a_1 t^{d-1} + \cdots + a_d$, then $|a_{i-1}a_{i+1}| \le a_i^2$.*

First of all, it is a simple fact that if M has positive rank, then $\chi_M(t)$ is always divisible by $t-1$ (for graphic matroid, this is saying that proper 1-coloring does not exist, and for realizable matroid, this follows from Proposition 2.30 and the fact that the hyperplane arrangement complement U admits a free G_m-action). For a matroid M, let $\overline{\chi}_M(t) = \chi_M(t)/(t-1)$, which is called the reduced characteristic polynomial. It is also an elementary fact that the coefficients of $\overline{\chi}_M(t)$ form an alternating sequence.

Theorem 3.2 ([HK12])**.** *Let M be a rank d matroid realizable over a field K, and assume that $\overline{\chi}_M(t) = t^{d-1} + b_1 t^{d-2} + \cdots + b_{d-1}$ $(b_0 = 1)$. Then there exists a $(d-1)$-dimensional smooth projective variety X defined over K, and two nef divisor classes $\underline{\alpha}, \underline{\beta}$ such that*

$$\deg_X(\underline{\alpha}^k \underline{\beta}^{d-k-1}) = |b_{d-k-1}|.$$

This variety X is the wonderful model of a hyperplane arrangement realizing M. It will be defined in the next subsection.

Theorem 3.3 (Khovanskii–Teissier inequality)**.** *Given a smooth projective variety X of dimension d, and two nef divisor classes A and B on X, then the sequence*

$$\deg_X(A^d),\ \deg_X(A^{d-1}B),\ \ldots,\ \deg_X(B^d)$$

is log-concave.

Corollary 3.4. *The reduced characteristic polynomial $\overline{\chi}_M(t)$ has log-concave coefficients.*

It is a simple fact that if two polynomials both have alternating and log-concave coefficients, then so does their product. Since $\chi_M(t) = (t-1)\overline{\chi}_M(t)$, $\chi_M(t)$ also has log-concave coefficients.

3.2. **The Chow ring of a matroid.**

Definition 3.5 ([AHK18])**.** Let M be a loopless matroid with ground set $E = \{1, \ldots, n\}$. The *Chow ring* of M, denoted by $\underline{\mathrm{CH}}^\bullet(M)$,[2] is defined to be the graded $\mathbb{Q}$-algebra generated by x_F (with degree 1) for all nonempty proper flats F (i.e., $0 \subsetneq F \subsetneq E$), with relations

$$x_F x_G = 0 \quad \text{for all pairs of incomparable flats } F \text{ and } G,$$

and

$$\sum_{i \in F} x_F = \sum_{j \in G} x_G \quad \text{for any } i, j \in E.$$

Let us give the geometric object behind this definition. Assume that M is realizable over a field K. Then there exists a central and essential hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in a K-vector space $V \cong K^d$, where $d = \mathrm{rk}(M)$. Let $\overline{H}_k \subset \mathbb{P}^{d-1}$ be the projectivization of H_k, then $\overline{\mathcal{A}} = \{\overline{H}_1, \ldots, \overline{H}_n\}$ is a projective hyperplane arrangement in $\mathbb{P}^{d-1}$. Given any proper flat F of M, we let $Z_F = \cap_{i \in F} \overline{H}_i$, which is a linear subspace in $\mathbb{P}^{d-1}$. The codimension of Z_F is equal to $\mathrm{rk}(F)$, or equivalently, the dimension of Z_F is equal to $\mathrm{crk}(F) - 1$.[3]

Remark 3.6. If $\mathrm{rk}(M) = 0$ or 1, then we let $\underline{\mathrm{CH}}^\bullet(M) = \mathbb{Q}$.

Remark 3.7. In the paper of [AHK18], they work with the real coefficient Chow ring so that it is easier to take limits. In [BHM+22], taking limits is less essential. So we used the rational coefficient Chow ring. In this note, we will follow the latter one and use rational coefficients.

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[2]The Chow ring of M is denoted by $A^\bullet(M)$ in [AHK18].

[3]Here crk denotes the *corank* of a flat. For a flat F in a matroid of rank d, its corank is defined by $\mathrm{crk}(F) = d - \mathrm{rk}(F)$.

Starting with $\mathbb{P}^{d-1}$, we blow up all Z_F for corank one flats F, i.e., all Z_F that are points. Then we blow up the strict transform of all Z_F for corank two flats F, i.e., all Z_F that are lines. Notice that after we blow up the points, all lines are separated. In other words, the strict transforms of two different lines Z_F and Z_G do not intersect any more. Continue this process dimension by dimension, until we blow up all Z_F for rank two flats F.

Definition 3.8. The smooth projective variety we obtain from the above process is called the wonderful model of the hyperplane arrangement $\mathcal{A}$, and denoted by $\underline{X}_{\mathcal{A}}$ or simply $\underline{X}$.

Proposition 3.9. *Let $\mathcal{A}$ be a central essential hyperplane arrangement in $V = K^d$, and let M be the associated matroid. Then there is a natural graded ring isomorphism*

$$A^\bullet_{\mathbb{Q}}(\underline{X}_{\mathcal{A}}) \cong \underline{\mathrm{CH}}^\bullet(M), \tag{1}$$

where $A^\bullet_{\mathbb{Q}}$ denotes the rational Chow ring of a smooth variety.

Remark 3.10. When $\mathcal{A}$ is defined over $\mathbb{C}$, we also have that

$$\mathrm{H}^{2\bullet}(\underline{X}_{\mathcal{A}}, \mathbb{Q}) \cong \underline{\mathrm{CH}}^\bullet(M).$$

Instead of proving the proposition, we explain how the isomorphism is constructed, and how we can see the corresponding relations in $A^\bullet_{\mathbb{Q}}(\underline{X}_{\mathcal{A}})$. For any nonempty proper flat F, let $\widetilde{Z}_F \subset X$ be the strict transform of Z_F, which is a divisor by our construction. In the isomorphism (1), the class $[\widetilde{Z}_F]$ maps to x_F. It is easy to see that if F and G are incomparable flats, then $\widetilde{Z}_F$ and $\widetilde{Z}_G$ do not intersect each other, which corresponds to the relation that $x_F x_G = 0$.

Let $\pi : X \to \mathbb{P}^{d-1}$ be the composition of all the blowup maps, and let $[H] \in A^1(\mathbb{P}^{d-1})$ be the hyperplane class. Then $[H] = [Z_{F_1}]$ all any rank one flat F_1. Given any rank one flat F_1 and $i \in F_1$, it is easy to see that

$$\pi^*([Z_{F_1}]) = \sum_{i \in F} [\widetilde{Z}_F]$$

in $A^1[X]$, where the sum is over all proper flats F containing i. Therefore, $\sum_{i\in F}[\widetilde{Z}_F] = \pi^*([H])$, which explains the relation $\sum_{i\in F} x_F = \sum_{j\in G} x_G$ for all $i, j \in E$.

Remark 3.11. Whether M is realizable or not, there is always a simplicial fan called the Bergman fan Π_M, such that the Chow ring of the toric variety $X(\Pi_M)$ is isomorphic to $\underline{\mathrm{CH}}^\bullet(M)$. This perspective is useful to define pullback, pushforward maps, and to show the Poincaré duality of $\underline{\mathrm{CH}}^\bullet(M)$. We will skip the details in this direction.

There are two distinguished divisor classes in $\underline{\mathrm{CH}}^\bullet(M)$, which are essential in the proof of the Heron-Rota-Welsh conjecture. The class $\underline{\alpha}_M = \underline{\alpha} \in \underline{\mathrm{CH}}^1(M)$ is defined as

$$\underline{\alpha}_M = \sum_{i \in F} x_F$$

where i is fixed and the sum is over all proper flats F containing i. By the relations in $\underline{\mathrm{CH}}^\bullet(M)$, the definition of $\underline{\alpha}_M$ does not depend on the choice of i. Similarly, the class $\underline{\beta}_M = \underline{\beta}$ is defined as

$$\underline{\beta}_M = \sum_{i \notin F} x_F$$

where i is fixed and the sum is over all nonempty flats F that does not contain i.

When M is realizable, we have constructed a smooth projective variety X such that $A^\bullet_{\mathbb{Q}}(X) \cong \underline{\mathrm{CH}}^\bullet(M)$. The composition of all blowup maps is denoted by $\pi : X \to \mathbb{P}^{d-1}$.

Let $l_1, \ldots, l_n$ be the linear forms defining the hyperplanes $H_1, \ldots, H_n$ in the hyperplane arrangement $\mathcal{A}$. Then we can define two maps from $\mathbb{P}^{d-1}$ to $\mathbb{P}^{n-1}$. The first is a linear embedding,

$$\mathbb{P}^{d-1} \to \mathbb{P}^{n-1}, \quad x \mapsto [l_1(x), \ldots, l_n(x)],$$

and the second is the linear embedding composing with the Cremona transformation

$$\mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^{n-1}, \quad x \mapsto \left[\frac{1}{l_1(x)}, \dots, \frac{1}{l_n(x)}\right].$$

Composing both maps with $\pi : X \to \mathbb{P}^{d-1}$, then we have two regular maps

$$\Phi_1 : X \to \mathbb{P}^{n-1} \quad \text{and} \quad \Phi_2 : X \to \mathbb{P}^{n-1}.$$

and a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\Phi_1} & & \searrow^{\Phi_2} & \\ \mathbb{P}^{n-1} & & \overset{\text{Crem}}{\dashrightarrow} & & \mathbb{P}^{n-1}. \end{array} \tag{2}$$

Identifying $A^\bullet(X)$ and $\underline{\mathrm{CH}}^\bullet(M)$ with the natural isomorphism, then $\Phi_1^*([H]) = \underline{\alpha}_M$ and $\Phi_2^*([H]) = \underline{\beta}_M$, where $[H]$ is the hyperplane class in $\mathbb{P}^{n-1}$.

3.3. **Pullback and pushforward maps.**

Definition 3.12. Given a proper nonempty flat F of M, we can define a *pullback* graded ring homomorphism

$$\underline{\varphi}^F : \underline{\mathrm{CH}}^\bullet(M) \to \underline{\mathrm{CH}}^\bullet(M_F) \otimes \underline{\mathrm{CH}}^\bullet(M^F),$$

such that for a nonempty proper flat G of M,

- if G is incomparable to F, then $\underline{\varphi}^F(x_G) = 0$;
- if $G < F$, then $\underline{\varphi}^F(x_G) = 1 \otimes x_G$;
- if $G > F$, then $\underline{\varphi}^F(x_G) = x_{G \setminus F} \otimes 1$;
- $\underline{\varphi}^F(x_F) = -1 \otimes \underline{\alpha}_{M^F} - \underline{\beta}_{M_F} \otimes 1$.

Exercise 3.13. Prove the following properties of $\underline{\varphi}^F$:

(i) $\underline{\varphi}^F$ is surjective;
(ii) $\underline{\varphi}^F(\underline{\alpha}_M) = \underline{\alpha}_{M_F} \otimes 1$;
(iii) $\underline{\varphi}^F(\underline{\beta}_M) = 1 \otimes \underline{\beta}_{M^F}$.

Definition 3.14. Given a proper nonempty flat F of M, there is also a natural *pushforward* map $\underline{\psi}^F : \underline{\mathrm{CH}}^\bullet(M_F) \otimes \underline{\mathrm{CH}}^\bullet(M^F) \to \underline{\mathrm{CH}}^\bullet(M)$, which is a graded $\underline{\mathrm{CH}}^\bullet(M)$-module homomorphism (the $\underline{\mathrm{CH}}^\bullet(M)$-module structure on $\underline{\mathrm{CH}}^\bullet(M_F) \otimes \underline{\mathrm{CH}}^\bullet(M^F)$ is induced by $\underline{\varphi}^F$). The map $\underline{\psi}^F$ is defined by

$$\underline{\psi}^F\left(\prod_{i \in I} x_{F_i \setminus F}^{m_i} \otimes \prod_{j \in J} x_{F_j}^{n_j}\right) = x_F \cdot \left(\prod_{i \in I} x_{F_i}^{m_i}\right) \cdot \left(\prod_{j \in J} x_{F_j}^{n_j}\right)$$

where $F_i, i \in I$ is a collection of proper flats of M containing F, $F_j, j \in J$ is a collection of nonempty flats of M contained in F.

Proposition 3.15. *For any matroid M and a nonempty proper flat F, the composition $\underline{\psi}^F \circ \underline{\varphi}^F$: $\underline{\mathrm{CH}}^\bullet(M) \to \underline{\mathrm{CH}}^\bullet(M)$ is equal to the multiplication by x_F. In other words, the following diagram commutes.*

$$\begin{array}{ccc} \underline{\mathrm{CH}}^\bullet(M) & \xrightarrow{\quad \cdot x_F \quad} & \underline{\mathrm{CH}}^\bullet(M) \\ & {\scriptstyle \underline{\varphi}^F} \searrow \quad \nearrow {\scriptstyle \underline{\psi}^F} & \\ & \underline{\mathrm{CH}}^\bullet(M_F) \otimes \underline{\mathrm{CH}}^\bullet(M^F) & \end{array}$$

Let us now explain the geometric meaning of the pullback and pushforward maps. When M is realizable, as we have discussed before, there is a smooth projective variety X such that $A^\bullet(X) \cong \underline{\mathrm{CH}}^\bullet(M)$. Given a nonempty proper flat F, there is a divisor $\widetilde{Z}_F$, which is the strict transform of the linear subspace $Z_F \subset \mathbb{P}^{d-1}$.

The divisor $\widetilde{Z}_F$ is naturally isomorphic to a product of two smooth projective varieties $X_{M_F} \times X_{M^F}$. By Künneth formula,

$$A^\bullet(X_M) \cong A^\bullet(X_{M_F}) \otimes A^\bullet(X_{M^F}).$$

Moreover, the Chow rings of X_{M_F} and X_{M^F} are isomorphic to the Chow rings of M_F and M^F respectively. Hence,

$$\underline{\mathrm{CH}}^\bullet(M) \cong \underline{\mathrm{CH}}^\bullet(M_F) \otimes \underline{\mathrm{CH}}^\bullet(M^F).$$

The pullback map $\underline{\varphi}^F$ just corresponds to the pullback map on Chow ring induced by the inclusion map $\widetilde{Z}_F \hookrightarrow X$, and the pushforward map is the induced Gysin pushforward map. It is a simple fact that the composition of pullback and pushforward maps is equal to the multiplication of the divisor class of $\widetilde{Z}_F$, which is equal to x_F in $\underline{\mathrm{CH}}^\bullet(M)$.

Finally, let us describe the two varieties X_{M_F} and X_{M^F}. They are constructed using the same blowup procedure as X out of different hyperplane arrangements. Recall that X is defined from a central essential hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in $V \cong K^d$. Define $V_F = \bigcap_{i\in F} H_F$. Then V_F is a linear subspace of V, and we let $V^F = V/V_F$. The hyperplanes H_i with $i \in F$ all contain V_F. Therefore, they induce a central hyperplane arrangement in V^F, which we denote by $\mathcal{A}^F$. It is easy to see that $\mathcal{A}^F$ is essential. On the other hand, each hyperplane H_i with $i \notin F$ intersects V_F along a codimension one linear subspace. These subspaces form a central and essential hyperplane arrangement in V_F, which we denote by $\mathcal{A}_F$. The varieties X_{M_F} and X_{M^F} are constructed by the same procedure as X out of hyperplane arrangements $\mathcal{A}_F$ and $\mathcal{A}^F$ respectively.

Remark 3.16. The pullback and pushforward maps can also be explained in terms of fans and toric varieties, which we will not get into to details here.

3.4. **Hodge theory of the Chow ring.** Given an matroid M, the key result of [AHK18] says that $\underline{\mathrm{CH}}^\bullet(M)$ satisfies the Kähler package. We will explain the precise statement in this subsection. The Kähler package has three components: Poincaré duality, hard Lefschetz theorem and the Hodge-Riemann relations. The last two involve the notion of ample classes. So we start with describing the ample cone.

Definition 3.17. Let E be a finite set. A *strictly submodular* function on E is a function $c : 2^E \to \mathbb{Q}$ satisfying $c(\emptyset) = c(E) = 0$ and for any two incomparable subsets $S_1, S_2 \subset E$,

$$c(S_1) + c(S_2) > c(S_1 \cap S_2) + c(S_1 \cup S_2).$$

Example 3.18. Let $c' : [0, |E|] \to \mathbb{Q}$ be a strictly concave function with $c'(0) = c'(|E|) = 0$. Then $c(S) = c'(|S|)$ defines a strictly submodular function.

Definition 3.19. Let M be a loopless matroid with ground set E. A divisor class $\ell \in \underline{\mathrm{CH}}^1(M)$ is called *ample*, if there exists a strictly submodular function c on E, such that

$$\ell = \sum_F c(F) x_F$$

where the sum is over all nonempty proper flats of M.

Exercise 3.20. The divisor classes form an open strongly convex polyhedral cone in $\underline{\mathrm{CH}}^1(M)$.

Definition 3.21. A divisor class $\ell \in \underline{\mathrm{CH}}^1(M)$ is called *nef* if it is the limit of ample classes.

Proposition 3.22. *Let M be a loopless matroid of rank d with ground set E. For any complete flag of flats $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = E$, $\underline{\alpha}_M^{d-1} = x_{F_1} \cdots x_{F_{d-1}} \in \underline{\mathrm{CH}}^{d-1}(X)$. Moreover, this element is nonzero and spans $\underline{\mathrm{CH}}^{d-1}(X)$.*

The proof of this proposition involves arguments about the Bergman fan, which we skip. Using this proposition, we can make the following definition.

Definition 3.23. Given any loopless matroid M of rank $d \geq 1$, we define a *degree map* $\deg : \underline{\mathrm{CH}}^{d-1}(M) \to \mathbb{Q}$ by $\deg(\underline{\alpha}_M^{d-1}) = 1$.

Theorem 3.24. [AHK18] *Given any loopless matroid M of rank $d > 0$. Its Chow ring satisfies Kähler package, that is, the following three properties.*

(i) (Poincaré duality) The composition

$$\underline{\mathrm{CH}}^k(M) \times \underline{\mathrm{CH}}^{d-k-1}(M) \xrightarrow{\cdot} \underline{\mathrm{CH}}^{d-1}(M) \xrightarrow{\deg} \mathbb{Q}$$

defines a perfect pairing between $\underline{\mathrm{CH}}^k(M)$ and $\underline{\mathrm{CH}}^{d-k-1}(M)$.

(ii) (Hard Lefschetz theorem) Given any ample class $\ell \in \underline{\mathrm{CH}}^1(M)$, the hard Lefschetz map

$$\ell^{d-2k-1} : \underline{\mathrm{CH}}^k(M) \to \underline{\mathrm{CH}}^{d-k-1}(M)$$

is an isomorphism for any $k \le \frac{d-1}{2}$.

(iii) (The Hodge-Riemann relations) Given any ample class $\ell \in \underline{\mathrm{CH}}^1(M)$ and $k \le \frac{d-1}{2}$, the Hodge-Riemann bilinear form *on $\underline{\mathrm{CH}}^k(M)$ defined by*

$$(\eta, \xi) = \deg(\eta \cdot \xi \cdot \ell^{d-2k-1})$$

is $(-1)^k$-definite on the kernel of $\ell^{d-2k} : \underline{\mathrm{CH}}^k(M) \to \underline{\mathrm{CH}}^{d-k}(M)$.

Corollary 3.25. *Let $A, B \in \underline{\mathrm{CH}}^1(M)$ be two nef classes. Then the sequence*

$$\deg(A^{d-1}),\ \deg(A^{d-2}B),\ \ldots,\ \deg(B^{d-1})$$

is log-concave.

Proposition 3.26. *The classes $\underline{\alpha}_M$ and $\underline{\beta}_M$ are both nef classes. Moreover, if the reduced characteristic polynomial of M is of the form $\overline{\chi}_M(t) = t^{d-1} - b_1 t^{d-2} + \cdots + (-1)^{d-1} b_{d-1}$, then*

$$b_k = \deg(\underline{\alpha}_M^{d-1-k} \underline{\beta}_M^k).$$

Corollary 3.27. *The reduced characteristic polynomial (and hence, the characteristic polynomial) of any matroid M has log-concave coefficients.*

Now, let us briefly explain how Theorem 3.24 is proved in [AHK18]. In the realizable case, the theorem says that $A^\bullet(X)$ satisfies Kähler package, where X is the wonderful model discussed before. We need to prove this result without referring to any classical Hodge theory, so that the arguments can be generalized to a pure combinatorial setting.

Recall that X is obtained from $\mathbb{P}^{d-1} = \mathbb{P}(V)$ by blowing up the subspaces Z_F corresponding to flats of M in certain order. Denote the sequence of blowup by

$$X = X_l \to X_{l-1} \to \cdots \to X_0 = \mathbb{P}^{d-1}$$

To show that $A^\bullet(X)$ satisfies Kähler package, the idea is to use induction and prove that $A^\bullet(X_i)$ satisfies Kähler package for every i. Starting from $\mathbb{P}^{d-1}$, its Chow ring obviously satisfies Kähler package. Every step, we blow up X_{i-1} along a smooth center which is the strict transform of some Z_F in X_{i-1} (here the strict transform of Z_F is birational to Z_F before the blowup). It turns out that the strict transform of Z_F is of the same type as those X_i's, that is, obtained from some smaller $\mathbb{P}^m$ by blowing up the strict transform of a sequence of linear subspaces.

More precisely, given a central essential hyperplane arrangement $\mathcal{A}$ in a vector space V, it defines a matroid $M = M_{\mathcal{A}}$. Given any order filter (an upper closed subset) $\mathcal{P}$ of the poset of flats $\mathcal{F}$, we can obtain a smooth projective variety by blowing up $\mathbb{P}(V)$ along all Z_F for all proper flats F in $\mathcal{P}$ from smaller dimension to bigger, and we denote the resulting variety by $X_{\mathcal{A},\mathcal{P}}$. Essentially, the following theorem is proved in [AHK18].

Theorem 3.28. *The Chow ring $A^\bullet(X_{\mathcal{A},\mathcal{P}})$ satisfies Kähler package.*

It turns out that when $\mathcal{P}_1 \subset \mathcal{P}_2$ are two order filters with $\mathcal{P}_2$ containing one more flat than $\mathcal{P}_1$. Then $X_{\mathcal{A},\mathcal{P}_2}$ is equal to the blowup of $X_{\mathcal{A},\mathcal{P}_1}$ along a smooth locus Z. It turns out that $Z = X_{\mathcal{A}',\mathcal{P}'}$ for some hyperplane arrangement $\mathcal{A}'$ on a smaller dimensional vector space and an order filter $\mathcal{P}'$ in the flats of $M_{\mathcal{A}'}$. So using inductions on dimension and the size of $\mathcal{P}$ to prove the above theorem, it is enough to prove the following (without referring to classical Hodge theory).

Theorem 3.29. *Let X be a smooth projective variety and let Z be a smooth subvariety. If both $A^\bullet(X)$ and $A^\bullet(Z)$ satisfy Kähler package, then $A^\bullet(\mathrm{Bl}_Z X)$ also satisfies Kähler package.*

This is a technical theorem in [AHK18]. The difficulty here is that an ample divisor of $A^\bullet(\mathrm{Bl}_Z X)$ is the pullback of an ample divisor of X minus a small multiple of the exceptional divisor. To deduce the Hodge-Riemann relations of $A^\bullet(\mathrm{Bl}_Z X)$ from the ones of $A^\bullet(X)$ and $A^\bullet(Z)$ involves some highly nontrivial limiting arguments.

Once we have a proof in the realizable case without referring to the classical Hodge theory, the arguments can be generalized to arbitrary matroids. In fact, given any matroid M and any order filter $\mathcal{P}$ of the flats, there is a graded ring $\underline{\mathrm{CH}}^\bullet(M, \mathcal{P})$ generalizing $A^\bullet(X_{\mathcal{A},\mathcal{P}})$ in the realizable case. Then we can apply the same proof strategy to $\underline{\mathrm{CH}}^\bullet(M, \mathcal{P})$, and prove that they satisfy Kähler package.

Remark 3.30. In [BHM+22], we give a simpler proof, where we realize the wonderful model $X_{\mathcal{A}}$ as a sequence of blowups from $(\mathbb{P}^1)^{d-1}$ such that each blowup is along a codimension two smooth subvariety. Such blowup satisfies the nice property that the pullback of an ample class by the blowup map still behaves like an ample class (called a *lef* class by de Cataldo-Migliorini). Hence we can avoid the more complicated limiting arguments in [AHK18].

4. Intersection cohomology of a matroid

In this section, we give a sketch of the proof of the Dowling-Wilson conjecture in the realizable case first, and then in the general case.

4.1. The Dowling-Wilson conjecture in the realizable case. Let us recall the statement of Dowling-Wilson conjecture in the realizable case (for simplicity we assume the field is $\mathbb{C}$). Let $V = \mathbb{C}^d$ and $E \subset V$ be a finite generating set. It defines a matroid M, whose poset of flats is $\mathcal{F}$. Let $\mathcal{F}_k \subset \mathcal{F}$ be the set of rank k flats. The Dowling-Wilson conjecture says that

$$|\mathcal{F}_k| \leq |\mathcal{F}_{d-k}| \quad \text{for any } k \leq d/2.$$

The proof in the realizable case uses a singular projective variety Y, which is now called a *matroid Schubert variety* (or *Schubert variety of hyperplane arrangement*).

Assume that $E = \{v_1, \ldots, v_n\}$. Then as an element in V, each v_i determines a linear function $V^* \to \mathbb{C}$. Putting all the v_i's together, we have a linear map

$$F = (v_1, \ldots, v_n) : V^* \to \mathbb{C}^n.$$

Consider the coordinate-wise compactification $\mathbb{C}^n \subset (\mathbb{P}^1)^n$. Let Y be the closure of $F(V^*)$ in $(\mathbb{P}^1)^n$.

Proposition 4.1. *The variety Y admits a stratification into affine spaces, such that the graded poset of the strata is naturally isomorphic to the graded poset of flats of M. Here the ordering on the strata is defined by $S_1 \leq S_2$ if $S_1 \subset \overline{S_2}$, and the grading is given by dimension.*

Remark 4.2. The above stratification of Y is also induced by the stratification of $(\mathbb{P}^1)^n$ defined by $(\mathbb{P}^1)^n = (\mathbb{C} \cup \{\infty\})^n$. In other words, each stratum of Y is equal to the intersection of Y and some stratum of the stratification $(\mathbb{C} \cup \{\infty\})^n$.

There are two proofs of this proposition. The first is by the work of Ardila-Boocher, the defining ideal of Y is computed using Gröbner bases. The assertion can be derived immediately from the defining ideals of Y. The second is to use the equivariant structure of Y. In fact, the inclusion map $F(V^*) \to V$ is a map of algebraic groups, and hence an $F(V^*)$-equivariant map. Since $V \subset (\mathbb{P}^1)^n$ is a V-equivariant compactification, the compactification Y of $F(V^*)$ is clearly an $F(V^*)$-equivariant compactification. Using this equivariant structure and induction on the cardinality of the ground set E, there is a geometric proof of the proposition. This idea is explained an further explored in [Cro22].

Using the stratification into affine spaces, we can essentially realize Y as a CW-complex with only even dimensional cells. In particular, the boundary maps in the chain complex are all zero, and the Betti numbers are determined by the number of cells in each dimension.

Corollary 4.3. *The Betti numbers (over any field coefficients) of Y satisfy*

$$b_{2k}(Y) = |\mathcal{F}_k| \quad \text{and} \quad b_{2k+1}(Y) = 0$$

for any $k \in \mathbb{Z}_{\geq 0}$, where $\mathcal{F}_k$ is the set of rank k flats in the matroid M associated to the vector configuration $E \subset V$.

The proof of the Dowling-Wilson conjecture in the realizable case uses the hard Lefschetz theorem of intersection cohomology groups of Y. Intersection cohomology is a (stratified) cohomology theory developed by Goresky-MacPherson for singular spaces so that the intersection cohomology groups of singular complex projective varieties also have Kähler package (the hard Lefschetz theorem and the Hodge-Riemann relations of the Kähler package is proved in [BBD82]). We will not give the precise definition of intersection cohomology groups. Instead, we list some properties besides the Käher package.

Proposition 4.4. *Let Y be an irreducible complex projective variety of dimension d. Then the following statements hold.*

(i) $\mathrm{IH}^\bullet(Y, \mathbb{Q})$ *is in general not a ring, but a graded* $\mathrm{H}^\bullet(Y, \mathbb{Q})$*-module.*

(ii) There is a natural graded $\mathrm{H}^\bullet(Y, \mathbb{Q})$*-module morphis* $\mathrm{H}^\bullet(Y, \mathbb{Q}) \to \mathrm{IH}^\bullet(Y, \mathbb{Q})$ *such that the kernel on* $\mathrm{H}^k(Y, \mathbb{Q})$ *is equal to* $W_{k-1}\mathrm{H}^k(Y, \mathbb{Q})$*, i.e., the subspace of weight strictly less than k. Here, we are using the weight filtration of Deligne's mixed Hodge structure on* $\mathrm{H}^\bullet(Y, \mathbb{Q})$*.*

Since our variety admits a stratification into affine spaces, we know that its cohomology groups all have the expected weight, i.e., $W_{2k-1}\mathrm{H}^{2k}(Y, \mathbb{Q}) = 0$ and $W_{2k}\mathrm{H}^{2k}(Y, \mathbb{Q}) = \mathrm{H}^{2k}(Y, \mathbb{Q})$. Therefore, we have the following corollary.

Corollary 4.5. *The natural map*

$$\mathrm{H}^\bullet(Y, \mathbb{Q}) \to \mathrm{IH}^\bullet(Y, \mathbb{Q})$$

is injective.

Remark 4.6. There is another proof of this corollary, which does not involve the theory of weights and can be generalized to the combinatorial setting. In fact, there is a natural resolution of singularity $f : X' \to Y$, called the *augmented wonderful model*, whose cohomology ring has a combinatorial description (it is the augmented Chow ring to be introduced later). Moreover, we can check that the pullback map $f^* : \mathrm{H}^\bullet(Y, \mathbb{Q}) \to \mathrm{H}^\bullet(X', \mathbb{Q})$ is injective. The decomposition theorem in [BBD82] implies that there exists an injective graded $H^\bullet(Y, \mathbb{Q})$-module homomorphism $\mathrm{IH}^\bullet(Y, \mathbb{Q}) \to \mathrm{H}^\bullet(X', \mathbb{Q})$ such that the composition

$$\mathrm{H}^\bullet(Y, \mathbb{Q}) \to \mathrm{IH}^\bullet(Y, \mathbb{Q}) \to \mathrm{H}^\bullet(X', \mathbb{Q})$$

is equal to the pullback map f^*. Since the composition is injective, the first map $\mathrm{H}^\bullet(Y, \mathbb{Q}) \to \mathrm{IH}^\bullet(Y, \mathbb{Q})$ must be injective.

For any ample class $A \in \mathrm{H}^2(Y, \mathbb{Q})$, consider the following commutative diagram,

$$\begin{array}{ccc} \mathrm{H}^{2k}(Y, \mathbb{Q}) & \longrightarrow & \mathrm{IH}^{2k}(Y, \mathbb{Q}) \\ \downarrow A^{d-2k} & & \cong \downarrow A^{d-2k} \\ \mathrm{H}^{2d-2k}(Y, \mathbb{Q}) & \longrightarrow & \mathrm{IH}^{2d-2k}(Y, \mathbb{Q}). \end{array}$$

The two horizontal maps are injective by the previous corollary, and the second vertical map is an isomorphism by the hard Lefschetz theorem for intersection cohomology groups. Therefore, the first vertical arrow must be injective, which implies that

$$\dim \mathrm{H}^{2k}(Y, \mathbb{Q}) \leq \dim \mathrm{H}^{2d-2k}(Y, \mathbb{Q}),$$

and hence by Corollary 4.3,

$$|\mathcal{F}_k| \leq |\mathcal{F}_{d-2k}|.$$

When M is realizable by a more general field, the same arguments work except that we need to replace singular cohomology by ℓ-adic cohomology.

4.2. **The graded Möbius algebra.** We have seen that the Betti numbers of the matroid Schubert variety is an invariant of the associated matroid. In fact, the cohomology ring is also a matroid invariant.

Definition 4.7. Let M be a matroid of rank d. Let $\mathrm{H}^k(M)$ be the $\mathbb{Q}$-vector space with bases y_F for all rank k flats F of M. Let

$$\mathrm{H}^\bullet(M) = \bigoplus_{0\le k\le d} \mathrm{H}^k(M)$$

be a graded vector space such that the degree k component is $\mathrm{H}^k(M)$. We define a graded ring structure on $\mathrm{H}^\bullet(M)$ by

$$y_F \cdot y_G = \begin{cases} y_{F\vee G} & \text{if } \mathrm{rk}(F\vee G) = \mathrm{rk}(F) + \mathrm{rk}(G) \\ 0 & \text{otherwise.} \end{cases}$$

The graded ring $\mathrm{H}^\bullet(M)$ is called the *(graded) Möbius algebra*.

Remark 4.8. Given a vector configuration $E \subset V$, let M be the associated matroid and let Y be the associated matroid Schubert variety constructed above. Then $\mathrm{H}^{2\bullet}(Y,\mathbb{Q}) \cong \mathrm{H}^\bullet(M)$ as graded algebras.

4.3. **Augmented wonderful model.** The original definition of intersection cohomology by Goresky-MacPherson uses chain complexes defined with certain transversality conditions. An equivalent sheaf-theoretic definition was introduced in [BBD82], which could be generalized in our setting (in a forthcoming work). However, to show that the intersection cohomology groups satisfy the Kähler package, we have to use the definition from a resolution of singularity of Y.

As we have discussed before, the variety Y admits a natural stratification into affine spaces. It turns out that if we blow up the 0-dimensional stratum, then blow up (the strict transforms of) all the 1-dimensional strata, and so on. Then the resulting variety, denoted by X', is smooth and called the *augmented wonderful model* of the vector configuration.

There is another way to construct the augmented wonderful model, which is more comparable to the construction of the wonderful model discussed before. Recall that a vector configuration is naturally dual to a hyperplane arrangement, and they define the same matroid. So let us start with a hyperplane arrangement $\mathcal{A} = \{H_1,\dots,H_n\}$ in $V = \mathbb{C}^d$, and denote its associated matroid by M. Consider the projective compactification $\mathbb{P}^d = \mathbb{P}(V\oplus\mathbb{C})$, which naturally contains V as an open subset, and the hyperplane at infinity. The hyperplane at infinity H_∞ is naturally isomorphic to $\mathbb{P}(V)$. As in the construction of the wonderful model, given any nonempty proper flat F of M, there is a linear subspace $Z_F \subset H_\infty = \mathbb{P}(V)$.

We first blow up $\mathbb{P}^d = V \cup H_\infty$ along Z_F for corank one flats F, then blow up the strict transforms of Z_F for corank two flats F, and so on. Finally, we blow up the strict transforms of Z_F for rank one flats F. Notice that the ambient space is one-dimensional larger than the one in the construction of wonderful models. We need to stop the blowup process at codimension two subvarieties, which correspond to rank one flats now.

Proposition 4.9. *The resulting variety is isomorphic to the augmented wonderful model X' defined above.*

Composing the resolution of singularity $X' \to Y$ and the inclusion map $Y \to (\mathbb{P}^1)^n$, we have a natural map $X' \to (\mathbb{P}^1)^n$. This map is in some sense analogous to the second map from the wonderful modal $X \to \mathbb{P}^{n-1}$ (as in (2)), which defines the nef class $\underline{\beta}$. From the second definition of X', we also have a map $X' \to \mathbb{P}^d$, which is analogous to the first map $X \to \mathbb{P}^{n-1}$ defining the nef class $\underline{\alpha}$.

By the decomposition theorem, $\mathrm{IH}^\bullet(Y,\mathbb{Q})$ is a direct summand of $\mathrm{H}^\bullet(X',\mathbb{Q})$. For a general matroid M, the substitute of $\mathrm{H}^\bullet(X',\mathbb{Q})$ is the augmented Chow ring $\mathrm{CH}^\bullet(M)$, and we will define the intersection cohomology groups of M as a submodule of $\mathrm{CH}^\bullet(M)$.

4.4. **Augmented Chow ring.**

Definition 4.10 ([BHM+20]). Let M be a rank d matroid with ground set $E = \{1,\dots,n\}$. The *augmented Chow ring* of M, denoted by $\mathrm{CH}^\bullet(M)$, is defined to be the graded $\mathbb{Q}$-algebra with degree one generators: y_i for $i \in E$ and x_F for proper flats F of M, and relations

$$y_i = \sum_{i \notin F} x_F, \quad \text{for every } i \in E,$$

and

$$x_F x_G = 0, \quad \text{for any incomparable flats } F \text{ and } G,$$

and

$$y_i x_F = 0, \quad \text{for any } i \in E \text{ and } F \text{ not containing } i.$$

Let us explain the geometric meanings of the generators of $\mathrm{CH}^\bullet(M)$. Recall that in the realizable case, we have an augmented wonderful model X', which is obtained either as a resolution of singularity of the matroid Schubert variety Y, or as a sequence of blowups of $\mathbb{P}^d$ along (the strict transforms of) Z_F's in the hyperplane at infinity H_∞. For a nonempty proper flat F, let $\widetilde{Z}_F$ be the strict transform of Z_F in X', and for the empty flat we let $\widetilde{Z}_\varnothing$ be the strict transform of H_∞, then x_F corresponds to the divisor class of $\widetilde{Z}_F$.

Consider the composition $X' \to Y \to (\mathbb{P}^1)^n \to \mathbb{P}^1$ of the resolution of singularity $X' \to Y$, the closed embedding $Y \to (\mathbb{P}^1)^n$ and the projection to the i-th factor $(\mathbb{P}^1)^n \to \mathbb{P}^1$. The divisor class y_i corresponds to the pullback of the cohomology class of a point in $\mathbb{P}^1$ under this composition map. One can check that the preimage of ∞ in X' is equal to the union of $\widetilde{Z}_F$ for all proper flats F not containing i. This explains the first relation. If we realize y_i by the preimage of a point different from ∞, then the divisor does not intersect any $\widetilde{Z}_F$ for F not containing i. This explain the last relation. The second relation is because $\widetilde{Z}_F$ and $\widetilde{Z}_G$ does not intersection for incomparable flats F and G.

Definition 4.11. There is a distinguished element α_M defined by $\alpha_M = \sum_F x_F$, where the sum is over all proper flats F of M. In the realizable case, α_M corresponds to the pullback of the hyperplane class by (the composition of) the blowup maps $X' \to \mathbb{P}^d$.

Similar to the Chow ring of a matroid, $\mathrm{CH}^\bullet(M)$ also satisfies the Kähler package.

Theorem 4.12 ([BHM+20])**.** *There is an nonempty strictly convex polyhedral cone in* $\mathrm{CH}^1(M)$ *called the* ample cone *of* $\mathrm{CH}^\bullet(M)$*, such that for any element* ℓ *in the ample cone, the analogous statements as in Theorem 3.24 hold for* $\mathrm{CH}^\bullet(M)$*, except that the Poincaré duality, hard Lefschetz theorem and Hodge-Riemann relations now are of degree* d *instead of degree* $d-1$*.*

Remark 4.13. The ample cone in $\mathrm{CH}^1(M)$ corresponds to a strictly convex function on a fan called augmented Bergman fan, and the cone is a bit involved to describe. Examples of ample classes are of the form $\sum_F c_F x_F + \sum_i \lambda_i y_i$, where c_F comes from a strictly submodular function on 2^E and each $\lambda_i > 0$.

4.5. **Pullback and pushforward maps.** Similar to the Chow rings, we can also define natural pullback and pushforward maps. In the realizable case, they correspond to pullback and pushforward maps induced by smooth subvarieties of X'.

Definition 4.14. Let M be a rank d matroid on the ground set E. Given a proper flat F of M, we define the *pullback* graded ring homomorphism

$$\varphi^F : \mathrm{CH}^\bullet(M) \to \underline{\mathrm{CH}}^\bullet(M_F) \otimes \mathrm{CH}^\bullet(M^F)$$

by the following.

- If G is incomparable to F, then $\varphi^F(x_G) = 0$.
- If G is properly contained in F, then $\varphi^F(x_G) = 1 \otimes x_G$.
- If G properly contains F, then $\varphi^F(x_G) = x_{G\setminus F} \otimes 1$.
- $\varphi^F(x_F) = -1 \otimes \alpha_{M^F} - \underline{\beta}_{M_F} \otimes 1$.
- If $i \in F$, then $\varphi^F(y_i) = 1 \otimes y_i$.
- If $i \in E \setminus F$, then $\varphi^F(y_i) = 0$.

Exercise 4.15. The pullback map satisfies $\varphi^F(\alpha_M) = \underline{\alpha}_{M_F} \otimes 1$.

Definition 4.16. There is also a natural *pushforward map*, which is a morphism of graded $\mathrm{CH}^\bullet(M)$-modules,

$$\psi^F : \underline{\mathrm{CH}}^\bullet(M_F) \otimes \mathrm{CH}^\bullet(M^F) \to \mathrm{CH}^\bullet(M)$$

defined by

$$\psi^F\left(\prod_{i\in I} x_{F_i\setminus F}^{m_i} \otimes \prod_{j\in J} x_{F_j}^{n_j}\right) = x_F \cdot \left(\prod_{i\in I} x_{F_i}^{m_i}\right) \cdot \left(\prod_{j\in J} x_{F_j}^{n_j}\right)$$

where $F_i, i \in I$ is a collection of proper flats of M containing F, $F_j, j \in J$ is a collection of nonempty flats of M contained in F.

Proposition 4.17. *Similar to the Chow ring case, the composition $\psi^F \circ \varphi^F$ is equal to the multiplication of x_F.*

Proposition 4.18. *Given any flat F and any two spanning independent subsets $I_1, I_2 \subset F$, i.e., $|I_1| = |I_2| = \mathrm{rk}(I_1) = \mathrm{rk}(I_2) = \mathrm{rk}(F)$, then $\prod_{i\in I_1} y_i = \prod_{i\in I_2} y_i$.*

Definition 4.19. Given any flat F, we define y_F to be $\prod_{i\in I} y_i$, where I is a spanning independent subset of F. By the previous proposition, the definition of y_F does not depend on the choice of I.

Proposition 4.20. *There exist a pullback ring homomorphism $\varphi_F : \mathrm{CH}^\bullet(M) \to \mathrm{CH}^\bullet(M_F)$ and a pushforward $\mathrm{CH}^\bullet(M)$-module map $\psi_F : \mathrm{CH}^\bullet(M_F) \to \mathrm{CH}^\bullet(M)$, such that the composition $\psi_F \circ \varphi_F$ is equal to the multiplication by y_F.*

Remark 4.21. The maps φ_F and ψ_F played less important role in the singular Hodge theory of matroids. So we will not discuss them further in this note.

4.6. **Intersection cohomology of a matroid.** Given a matroid M, there are different ways to define its intersection cohomology groups $\mathrm{IH}^\bullet(M)$, which is a graded $\mathrm{H}^\bullet(M)$-module. The easier, but more abstract way is using the Krull-Schmidt theorem.

Definition 4.22. Given a matroid M, let

$$\mathrm{CH}^\bullet(M) = N_1^\bullet \oplus N_2^\bullet \oplus \cdots \oplus N_l^\bullet$$

be a decomposition of $\mathrm{CH}^\bullet(M)$ as indecomposable graded $\mathrm{H}^\bullet(M)$-modules. Since $\mathrm{CH}^0(M) \cong \mathbb{Q}$, there exists a unique $N_i^\bullet$ such that the degree zero component of $N_i^0 \neq 0$. We define the *intersection cohomology groups* $\mathrm{IH}^\bullet(M)$ to be this graded $\mathrm{H}^\bullet(M)$-module $N_i^\bullet$.

Remark 4.23. By the Krull-Schmidt theorem, the isomorphism class of N_i does not depend on the choice of the decomposition. Therefore, this definition gives the isomorphism class of $\mathrm{IH}^\bullet(M)$.

Even though in the decomposition of [BBD82], the decomposition is not canonical (depend on a choice of a relative ample class), there is a canonical way to write $\mathrm{IH}^\bullet(M)$ as an $\mathrm{H}^\bullet(M)$-submodule of $\mathrm{CH}^\bullet(M)$. Before defining $\mathrm{IH}^\bullet(M)$, we need to define $\underline{\mathrm{IH}}^\bullet(M)$. When M is realized by a hyperplane arrangement $\{H_1, \ldots, H_n\}$ in $V = \mathbb{C}^d$, $\underline{\mathrm{IH}}^\bullet(M)$ is isomorphic to the intersection cohomology groups of the projective reciprocal plane, i.e., the closure of the image of the rational map

$$\mathbb{P}(V) = \mathbb{P}^{d-1} \hookrightarrow \mathbb{P}^{n-1} \quad x \mapsto \left[\frac{1}{l_1(x)} : \ldots : \frac{1}{l_n(x)}\right]$$

where l_i are the linear defining equations of H_i.

Definition 4.24. We recursively define the graded subspaces $\underline{\mathrm{K}}_F^\bullet(M)$, $\underline{\mathrm{IH}}^\bullet(M)$ and $\underline{\mathrm{J}}^\bullet(M)$ of $\underline{\mathrm{CH}}^\bullet(M)$ as follows.

(i) For any proper nonempty flat F of M, we define

$$\underline{\mathrm{K}}_F^\bullet(M) = \underline{\psi}^F\left(\underline{\mathrm{J}}^\bullet(F) \otimes \underline{\mathrm{CH}}^\bullet(M^F)\right).$$

(ii) We define the graded subspace $\underline{\mathrm{IH}}^\bullet(M)$ of $\underline{\mathrm{CH}}^\bullet(M)$ as the orthogonal complement

$$\underline{\mathrm{IH}}^\bullet(M) = \left(\sum_{\emptyset<F<E} \underline{\mathrm{K}}^\bullet_F(M)\right)^\perp$$

where the orthogonality is with respect to the Poincaré pairing on $\underline{\mathrm{CH}}^\bullet(M^F)$.

(iii) The graded subspace $\underline{\mathrm{J}}^\bullet(M)$ of $\underline{\mathrm{CH}}^\bullet(M)$ is defined by setting

$$\underline{\mathrm{J}}^k(M) = \begin{cases} \underline{\mathrm{IH}}^k(M) & \text{if } k \le (d-2)/2, \\ \underline{\beta}_M^{2k-d+2}\underline{\mathrm{IH}}^{d-k-2}(M) & \text{if } k \ge (d-2)/2. \end{cases}$$

It turns out that each $\underline{\mathrm{K}}^\bullet_F(M)$ and $\underline{\mathrm{IH}}^\bullet(M)$ is closed under multiplication by $\underline{\beta}_M$. Moreover, the sum $\sum_{\emptyset<F<E}\underline{\mathrm{K}}^\bullet_F(M)$ is a direct sum, and the restriction of the Poincaré pairing on each $\underline{\mathrm{K}}^\bullet_F(M)$, and hence on $\underline{\mathrm{IH}}^\bullet(M)$, is non-degenerate.

Theorem 4.25 ([BHM+20])**.** *The graded subspace $\underline{\mathrm{IH}}^\bullet(M)$ satisfies the Kähler package with respect to the restriction of the Poincaré pairing of $\underline{\mathrm{CH}}^\bullet(M)$ and the ample class $\underline{\beta}_M$. Moreover, we have an orthogonal direct sum decomposition*

$$\underline{\mathrm{CH}}^\bullet(M) = \underline{\mathrm{IH}}^\bullet(M) \oplus \bigoplus_{\emptyset<F<E} \underline{\psi}^F\left(\underline{\mathrm{J}}^\bullet(M_F)\otimes \underline{\mathrm{CH}}^\bullet(M^F)\right). \tag{3}$$

Remark 4.26. Even though the above theorem only involves underlined objects, we do not know how to prove it within the world of underlined objects. The current proof uses a large inductive scheme which also involves augmented objects.

Then we define the graded $\mathrm{H}^\bullet(M)$-submodules $K^\bullet_F(M)$ and $\mathrm{IH}^\bullet(M)$ of $\mathrm{CH}^\bullet(M)$.

Definition 4.27. For a nonempty proper flat F of M, we define the $\mathrm{H}^\bullet(M)$-submodule $\mathrm{K}^\bullet_F(M)$ of $\mathrm{CH}^\bullet(M)$ by

$$\mathrm{K}^\bullet_F(M) = \psi^F\left(\underline{\mathrm{J}}^\bullet(M_F)\otimes \mathrm{CH}^\bullet(M^F)\right).$$

We define $\mathrm{H}^\bullet(M)$-submodule $\mathrm{IH}^\bullet(M)$ of $\mathrm{CH}^\bullet(M)$ by

$$\mathrm{IH}^\bullet(M) = \left(\bigoplus_{\emptyset<F<E} \mathrm{K}^\bullet_F(M)\right)^\perp.$$

Remark 4.28. By definition, if $\mathrm{rk}(M)=1$, then $\underline{\mathrm{IH}}^\bullet(M)=\underline{\mathrm{IH}}^0(M)=\mathbb{Q}$ and $\underline{\mathrm{J}}^\bullet(M)=0$. Therefore, if F is a corank one flat, then $\underline{\mathrm{K}}^\bullet_F(M)=\mathrm{K}^\bullet_F(M)=0$.

Theorem 4.29 ([BHM+20])**.** *The graded subspace $\mathrm{IH}^\bullet(M)$ satisfies the Kähler package with respect to the restriction of the Poincaré pairing of $\underline{\mathrm{CH}}^\bullet(M)$ and any ample class of the form $\sum_{i\in E} c_i y_i$ with each $c_i>0$. Moreover, we have an orthogonal direct sum decomposition*

$$\mathrm{CH}^\bullet(M) = \mathrm{IH}^\bullet(M) \oplus \bigoplus_{\emptyset<F<E} \underline{\psi}^F\left(\underline{\mathrm{J}}^\bullet(M_F)\otimes \mathrm{CH}^\bullet(M^F)\right). \tag{4}$$

The proof of the above two theorems involve extremely complicated inductive arguments. We will not get into any details. Let us first explain the definition of $\underline{\mathrm{J}}^\bullet(M)$, and then the geometric meanings of the two direct sum decompositions.

4.7. **The example of a cone.** We start with a simple example of intersection cohomology and the decomposition theorem in action.

Let N be a smooth projective variety of dimension $d-1$, and let C be its affine cone with cone point 0. Then,

$$\mathrm{IH}^k(C,\mathbb{Q}) = \begin{cases} \mathrm{H}^k(C\setminus\{0\},\mathbb{Q}) & \text{if } k\le d-1, \\ 0 & \text{if } k\ge d. \end{cases}$$

Since $C \setminus \{0\}$ is the (restriction of) Hopf bundle over N, the Leray spectral sequence implies that there is a noncanonical isomophism

$$\mathrm{H}^k(C \setminus \{0\}, \mathbb{Q}) \cong \ker\left(A : \mathrm{H}^{k-1}(N) \to \mathrm{H}^{k+1}(N)\right) \oplus \operatorname{coker}\left(A : \mathrm{H}^{k-2}(M) \to \mathrm{H}^k(N)\right)$$

where $A \in H^2(N)$ is the first chern class of $\mathcal{O}(1)$ (this is the very ample line bundle on N induced by the cone C). By the hard Lefschetz theorem,

$$\ker\left(A : \mathrm{H}^{k-1}(N) \to \mathrm{H}^{k+1}(N)\right) = \begin{cases} 0 & \text{if } k \le d-1, \\ \mathrm{PH}^{2d-k-1}(N) & \text{if } k \ge d, \end{cases}$$

and

$$\operatorname{coker}\left(A : \mathrm{H}^{k-2}(N) \to \mathrm{H}^k(N)\right) = \begin{cases} \mathrm{PH}^k(N) & \text{if } k \le d-1, \\ 0 & \text{if } k \ge d, \end{cases}$$

where $\mathrm{PH}^k(N)$ is the primitive classes defined by

$$\mathrm{PH}^k(N) = \begin{cases} \ker\left(A^{2d-2k} : \mathrm{H}^k(N) \to \mathrm{H}^{2d-k}(N)\right) & \text{if } k \le d-1 \\ 0 & \text{if } k \ge d. \end{cases}$$

Therefore,

$$\mathrm{H}^k(C \setminus \{0\}, \mathbb{Q}) \cong \begin{cases} \mathrm{PH}^k(N) & \text{if } k \le d-1, \\ \mathrm{PH}^{2d-k-1}(N) & \text{if } k \ge d, \end{cases}$$

and

$$\mathrm{IH}^k(C, \mathbb{Q}) = \begin{cases} \mathrm{PH}^k(N) & \text{if } k \le d-1, \\ 0 & \text{if } k \ge d. \end{cases}$$

Let $p : \widetilde{C} \to C$ be the blowup of the cone point. Then $\widetilde{C}$ is a line bundle of M. The decomposition theorem implies that $Rp_*(\mathbb{Q}_{\widetilde{C}})$ is the direct sum of the intersection complex IC_C and some (shifted) skyscraper sheaves at the cone point. Taking cohomology, we have the following decomposition,

$$\begin{aligned} \mathrm{H}^\bullet(N) &\cong \mathrm{H}^\bullet(\widetilde{C}) \\ &\cong \mathrm{IH}^\bullet(C) \oplus \bigoplus_{2 \le k \le d-1} \frac{\mathrm{H}^k(N)}{\mathrm{PH}^k(N)}[-k] \oplus \bigoplus_{d \le k \le 2d-2} \mathrm{H}^k(N)[-k] \\ &\cong \mathrm{IH}^\bullet(C) \oplus \bigoplus_{2 \le k \le d-1} \mathrm{H}^{k-2}(N)[-k] \oplus \bigoplus_{d \le k \le 2d-2} A^{k-d-1} \cdot \mathrm{H}^{2d-k-2}(N)[-k], \end{aligned}$$

where $[-k]$ means we put the vector space in degree k, and the last isomorphism follows from the hard Lefschetz theorem. The smaller summands

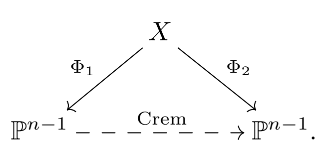
$$\bigoplus_{2 \le k \le d-1} \mathrm{H}^{k-2}(N)[-k] \oplus \bigoplus_{d \le k \le 2d-2} A^{k-d-1} \cdot \mathrm{H}^{2d-k-2}(N)[-k]$$

is consistent with the definition of $\underline{\mathrm{J}}(N)$ if we replace k by $2k$ and divide the degree of $H^\bullet$ by 2.

Remark 4.30. If the projective variety N is singular, then the same computations hold if we replace $\mathrm{H}^\bullet(N)$ and $\mathrm{H}^\bullet(\widetilde{C})$ by $\mathrm{IH}^\bullet(N)$ and $\mathrm{IH}^\bullet(\widetilde{C})$ respectively.

4.8. **The geometry of the canonical decompositions.** Assume that $\mathcal{A}$ is a central and essential hyperplane arrangement in $V = \mathbb{C}^d$, and let M be the associated matroid. We explain for geometric meanings of the two canonical decompositions (3) and (4).

For the underlined canonical decomposition (3), we start with the reciprocal plane, that is the image $\Phi_2(X)$ from the commutative diagram we have seen before,



The reciprocal plane $\Phi_2(X)$ admits a natural stratification induced by the coordinate stratifications of $\mathbb{P}^{n-1}$, and poset of the strata is isomorphic to the poset of flats with positive rank. When we blow up the closure of a stratum corresponding to F, since all smaller strata have already been blown up, the closure of the stratum corresponding to F is isomorphic to the wonderful model $X_{\mathcal{A}^F}$. Moreover, right before blowing up the closure of the stratum, the closure is equi-singular, and a transversal slice is isomorphic to the cone of the reciprocal plane $\Phi_2(X_{\mathcal{A}_F})$ of the smaller dimensional hyperplane arrangement $\mathcal{A}_F$. After the blowup, the new exceptional divisor is isomorphic to $\Phi_2(X_{\mathcal{A}_F}) \times X_{\mathcal{A}^F}$ and its intersection cohomology group is equal to

$$\underline{\mathrm{IH}}^\bullet(M_F) \otimes \underline{\mathrm{CH}}^\bullet(M^F).$$

Now our discussion in the previous subsection explains geometric meaning of the smaller summand

$$\underline{\mathrm{IH}}^\bullet(M_F) \otimes \underline{\mathrm{CH}}^\bullet(M^F),$$

which is the summand produced when the stratum corresponding to F is blown up.

Similarly, the matroid Schubert variety Y admits a stratification into affine spaces Y_F parametrized by all flats F of M, which we have discussed before. The poset of the strata is isomorphic to the poset of flats of M. In other words, the closure

$$\overline{Y_F} = \bigcup_{G \leq F} Y_G.$$

The augmented wonderful model X' can be obtained from Y by blowing up the most singular point $Y_\emptyset$, then blowing up the closure of the 1-dimensional strata $Y_F, \mathrm{rk}(F) = 1$, and so on.

Before the stratum Y_F is blown up, the closure of Y_F is isomorphic to equal to the augmented wonderful model $X'_{\mathcal{A}^F}$, and a transversal slice of the stratum is isomorphic the cone of the reciprocal plane $\Phi_2(X_{\mathcal{A}_F})$. Then we can see the geometric meaning of the smaller summands as in the canonical decomposition (3).

The proofs of Theorems 4.25 and 4.29 depend on each other. In fact, there is a big inductive scheme involving many statements including the two theorems. We will not get into any details of the proof.

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