A numerical approach for computing Euler characteristics of affine varieties*

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Abstract. We develop a numerical nonlinear algebra approach for computing the Euler characteristic of an affine variety. Our approach is to relate Euler characteristics of a smooth affine variety with the number of critical points using Morse theory. In general, we stratify a variety into the union of smooth affine varieties to obtain results on singular varieties.

Keywords: Euler characteristic · numerical algebraic geometry · homotopy continuation

1 Introduction

The Euler characteristic is one of the most fundamental topological invariants. In the past decade, a series of works appeared which relate Euler characteristics of complex algebraic varieties with the complexity of algebraic optimization problems [1, 5, 13, 16, 17]. There are several existing approaches to compute the Euler characteristics of complex algebraic varieties [3, 6, 15], each having their own benefits. Our new approach has the following advantages.

1. Our methods directly compute the Euler characteristic of an affine variety without involving any compactification. This is useful because the closure of a smooth affine variety can have bad singularities along infinity.
2. We stratify and compute the Euler characteristics of smooth affine varieties. In theory, any d-dimensional affine variety can be stratified into the union of at most $d+1$ smooth affine varieties. In contrast to the inclusion-exclusion principle, our method does not involve too many varieties.
3. We can tailor the stratification to reduce the degree of each stratum.

A standard method to compute Euler characteristics of complex algebraic varieties is to reduce to the projective hypersurface case. The drawback of this method is that to compute the Euler characteristic of a projective variety, the number of involved hypersurfaces grows exponentially in the codimension.

In contrast, we compute the Euler characteristic of a smooth equidimensional affine variety X by counting the critical points of $\dim(X)+1$ algebraic functions.

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Given a singular complex affine variety, we stratify it into smooth affine varieties to reduce to the smooth case. In theory, we can always stratify a d-dimensional affine variety into $d+1$ smooth (possibly not connected) equidimensional affine varieties of dimension $d, d-1, \dots, 1, 0$. So we need to compute the number of critical points of at most $(d+1)(d+2)/2$ algebraic functions. Our algorithms also have the practical feature of minimizing the degree of the algebraic functions at the expense of increasing the number of functions to consider.

This work is organized follows. In Section 2, we recall a theorem to determine the Euler characteristic of a smooth equidimensional variety with a general hyperplane removed by counting critical points of a function. In Section 3, we provide some key definitions from numerical algebraic geometry. In Sections 4-5 we present algorithms for computing Euler characteristics.

2 Euler characteristics and critical points

Let X be a topological space that is homotopy equivalent to a finite CW-complex. The Euler characteristic of X, denoted by $\chi(X)$, is the alternating sum of the Betti numbers of X [8, Page 146]. We are only interested in the situation where X is a complex algebraic variety [7, Corollary 6.10]. In this case, the Euler characteristic of a complex algebraic variety X is an alternating sum of cardinalities of several sets of critical points as shown in Theorem 1.

Let X be a smooth subvariety of $\mathbb{C}^n$ and let $f : \mathbb{C}^n \to \mathbb{C}$ be a regular function. If X is defined by polynomials $g_1, \dots, g_l \in \mathbb{C}[x_1, \dots, x_n]$, then the critical points of $f|_X$ for a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ are the points $P \in X$ such that the vector $(\frac{df}{dx_1}, \dots, \frac{df}{dx_n})|_P$ is contained in the span of $(\frac{dg_i}{dx_1}, \dots, \frac{dg_i}{dx_n})|_P$ for $i = 1, \dots, l$.

This theorem relates the number of critical points to the Euler characteristic.

Theorem 1 ([16]). *Let ℓ denote a general affine linear function $\ell : \mathbb{C}^n \to \mathbb{C}$ and let X denote a smooth equidimensional affine subvariety of $\mathbb{C}^n$. Then*

$$(-1)^{\dim(X)}\chi\big(X \setminus V(\ell)\big) = \#\{\textit{critical points of } \ell|_X\}.$$

As a corollary we are able to determine the Euler characteristic of X itself.

Corollary 1. *Let X be a smooth subvariety of $\mathbb{C}^n$. For $i = 1, \dots, \dim(X)$, let h_i denote a general affine linear function $\mathbb{C}^n \to \mathbb{C}$. Then we have the equality*

$$\chi\big(X\big) = p + \sum_{i=1}^{\dim(X)} (-1)^{\dim(X)-i+1}\eta_i$$

where η_i is the number of critical points of $h_i|_{X\cap V(h_1,\dots,h_{i-1})}$ and p is the cardinality of $X \cap V(h_1, \dots, h_{\dim(X)})$.

Proof. The additive property of Euler characteristic implies the equality

$$\chi(X) = \sum_{i=1}^{\dim(X)} \chi\big(X \cap V(h_1, \dots h_{i-1}) \setminus V(h_i)\big) + \chi\big(X \cap V(h_1, \dots, h_{\dim(X)})\big).$$

Theorem 1 gives the equality $\eta_i = (-1)^{\dim(X)-i+1}\chi(X \cap V(h_1, \ldots h_{i-1}) \setminus V(h_i))$. It follows from Bertini's theorem that $X \cap V(h_1, \ldots, h_{\dim(X)})$ is a set of p points.

3 Numerical algebraic geometry basics

In this section we recall a witness set [4, 19], which is a fundamental concept in numerical algebraic geometry. A witness set is used to analyze algebraic varieties and is manipulated using homotopy continuation [2], as see in Sections 3.2-3.3.

3.1 Witness sets and numerical irreducible decomposition

Let X be an equidimensional subvariety of affine space $\mathbb{C}^n$. As a consequence of Bertini's Theorem, there are two invariants, dimension and degree, of X that can be understood by intersecting X with a general linear space. The dimension $\dim(X)$ of a subvariety X of $\mathbb{C}^n$ is the maximal codimension of a general affine linear space $\mathcal{L} \subseteq \mathbb{C}^n$ such that $X \cap \mathcal{L}$ is finite and nonempty. The degree $\deg(X)$ of X is the number of points in $X \cap \mathcal{L}$.

Definition 1 (Witness set). *Suppose X is an equidimensional subvariety of $\mathbb{C}^n$. A* witness set *for X is a triple (F, L, W), where F is a finite set of polynomials with each irreducible component of X being an irreducible component of $V(F)$, L is a set of* $\dim(X)$ *general*[1] *affine linear functions, and W is the set of points $X \cap V(L)$.*

In numerical algebraic geometry, W is called a *witness point set* for X. Since L consists of generic affine linear functions, the affine linear space $V(L)$ is generic and the cardinality of the set W is $\deg(X)$. Throughout, we assume the ideal generated by F in Defin. 1 defines a reduced scheme by using *deflation*.

Given a (not necessarily equidimensional) subvariety X of $\mathbb{C}^n$, we denote by X_i the union of i-dimensional irreducible components of X. We call a set of witness sets of X_i for $i = 0, 1, \ldots, k$ a *numerical equidimensional decomposition of* X. This decomposition can be refined to a *numerical irreducible decomposition of* X [4] by providing a witness set for each irreducible component of X.

Example 1 (Embedding). Suppose X is an equidimensional subvariety of $\mathbb{C}^n$ and h is a linear function $\mathbb{C}^n \to \mathbb{C}$. Let $\hat{X}$ be the image of X under the closed embedding $\mathbb{C}^n \to \mathbb{C}^{n+1}$ given by $x \mapsto (x, h(x))$. Given a witness set (F, L, W) for X, we construct a witness set for $\hat{X}$ as $(F \cup \{h - x_{n+1}\},\ L,\ \{(x, h(x)) : x \in W)\})$.

Example 2. We can also easily construct a witness set for the Cartesian product of two varieties. Suppose X_i is a subvariety of $\mathbb{C}^{n_i}$ for $i = 1, 2$. If (F_i, L_i, W_i) is a witness set for X_i, then $(F_1 \cup F_2, L_1 \cup L_2, W_1 \times W_2)$ is a witness set for $X_1 \times X_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$.

[1] Here by general, we mean the intersection $X \cap V(L)$ is transverse and has cardinality $\deg(X)$.

3.2 Witness collections of subvarieties in $\mathbb{C}^n \times \mathbb{C}^n$

A witness collection is a generalization of a witness set and is used to study varieties that are defined by polynomials with a natural multi-variable group structure. For a complete description of witness collections see [14, 10, 9]. For our purposes, it suffices to study the following special case.

A *witness collection* for a d-dimensional irreducible subvariety Z of $\mathbb{C}^n \times \mathbb{C}^n$ is the following collection of triples, which we call *multi-affine witness sets*:

$$\left(F, L^i \cup M^{d-i}, Z \cap (\mathcal{L}^i \times \mathcal{M}^{d-i})\right) \text{ for } i = 0, 1, \ldots, d,$$

where F is a set of polynomials such that $V(F)$ contains Z as an irreducible component; $\mathcal{L}^i$ and $\mathcal{M}^i$ are general codimension i affine linear spaces in $\mathbb{C}^n$ defined the sets of general linear functions L^i and M^i respectively.

Witness collections are used to understand the intersection of Z with a Cartesian product of linear spaces. Let $\mathcal{A}^i$ and $\mathcal{B}^i$ denote general codimension i affine linear spaces in $\mathbb{C}^n$. With homotopy continuation, we determine the isolated points in the intersection $Z \cap (\mathcal{A}^i \times \mathcal{B}^{d-i})$. These points are contained in the set of endpoints of the homotopy $H^i : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^{N+d}$ with

$$(x, y, t) \quad \mapsto \quad \left(F(x, y), tL^i(x) + (1-t)A^i(x), tM^{d-i}(y) + (1 - tB^{d-i}(y)\right) \quad (1)$$

where $L^i, A^i, M^{d-i}, B^{d-i}$ are sets of affine linear functions defining $\mathcal{L}^i, \mathcal{A}^i, \mathcal{M}^{d-i}$ and $\mathcal{B}^{d-i}$ respectively. For more details see [9, Remark 1.3]. To conveniently denote linear functions, for $a, x \in \mathbb{C}^n$, we take $a \circ x$ to be the usual inner product.

Example 3 (Conormal variety). Let $X = V(f_1, \ldots, f_k)$ be a smooth equidimensional variety in $\mathbb{C}^n$ with $(f_1, \ldots, f_k)$ generating a radical ideal. The *(affine) conormal variety* of X is a subvariety $\mathcal{C}(X)$ in $\mathbb{C}^n \times \mathbb{C}^n$ with an ideal

$$\langle f_1, \ldots, f_k \rangle + \langle (1 + \operatorname{codim}(X))\text{-minors of } \operatorname{Jac}_x(x \circ y, f_1, \ldots, f_k) \rangle \subset \mathbb{C}[x, y],$$

where $\operatorname{Jac}_x(x \circ y, f_1, \ldots, f_k)$ is a $(k+1) \times n$ matrix of partial derivatives with respect to x. The dimension of $\mathcal{C}(X)$ is n. For a projective formulation, see [18].

A witness collection for the conormal variety of X is given by

$$\left(F,\ L^i \cup M^{n-i},\ W_i\right) \ \text{ for } i = 0, \ldots, n, \quad (2)$$

where $V(F)$ contains $\mathcal{C}(X)$ as an irreducible component and

$$W_i := \mathcal{C}(X) \cap (\mathcal{L}^i \times \mathcal{M}^{n-i}).$$

Each multi-affine witness set $\left(F,\ L^i \cup M^{n-i},\ W_i\right)$ has information about the variety X. For example, the dimension of X is the maximal i such that $W_i \neq \emptyset$ and the degree of X is the cardinality of $W_{\dim(X)}$. Moreover, for $i = 0$, the linear space $V(M^n) \subset \mathbb{C}^n$ contains a unique point, say c. The set of points (x, y) in W_0 such that $x \circ c \neq 0$ is the set of critical points of the general linear function $x \circ c$ on $X \setminus V(x \circ c)$. The cardinality of the set $W_0 \setminus V(x \circ c)$ is the Euler characteristic of $X \setminus V(x \circ c)$ up to a sign. In general, the number of points in W_i is an upperbound to the η_i appearing in Corollary 1.

Fig. 1. We illustrate W_0 (left) and W_1 (right) for the circle $X = V(x_1^2 + x_2^2 - 1)$ in $\mathbb{C}^2$.

3.3 Regeneration and removing a hypersurface

Given a witness collection for an irreducible variety X and a polynomial g, regeneration determines a witness set for $X \cap V(g)$. One of two situations can occur. First, if X is contained in $V(g)$, then $X = X \cap V(g)$ and we are done. Second, if X is not contained in $V(g)$, then for $i = 1, \ldots, \deg(g)$ compute a witness set $(F \cup \{\ell_i\}, L, W_i)$ for $X \cap V(\ell_i)$ where $\ell_i : \mathbb{C}^n \to \mathbb{C}$ is a general affine linear function. This is easy to do using standard homotopy continuation methods when given the witness set $(F, L \cup \{\ell\}, W)$ for X. This produces a witness set $\left(F \cup \{\ell_1 \cdots \ell_{\deg(g)}\},\, L,\, \cup_{i=1}^{\deg(g)} W_i\right)$ for $X \cap V(\ell_1 \cdots \ell_{\deg(g)})$. Finally, the homotopy $H(x,t) = (F(x), t\ell_1(x) \cdots \ell_{\deg(g)}(x) + (1-t)g(x), L(x))$ provides a witness set for $X \cap V(g)$ when $t = 0$. This procedure for computing a witness set for $X \cap V(g)$ from a witness set for X is called *regeneration* [12, 11, 10].

Example 4. For $X \subset \mathbb{C}^n$, consider the embedding $\hat{X} \subset \mathbb{C}^{n+1}$ as in Ex. 1. Fix a polynomial $g \in \mathbb{C}[x_1, \ldots, x_n]$. The affine variety $\hat{X} \cap V(gx_{n+1} - 1)$ is isomorphic to $X \setminus V(g)$. A useful application of regeneration to compute a witness set for $\hat{X} \cap V(gx_{n+1} - 1)$. From a witness set for $\hat{X} \subset \mathbb{C}^{n+1}$ and a polynomial $gx_{n+1} - 1$, regeneration produces a witness set for $\hat{X} \cap V(gx_{n+1} - 1)$.

4 Euler characteristics of smooth varieties

Theorem 1 leads to an algorithm that outputs the the Euler characteristic of an algebraic variety by computing critical points. The proof of correctness of the following algorithm is easily derived from Corollary 1.

Algorithm 2 (Smooth X) Input: *A smooth equidimensional affine variety* $X \subset \mathbb{C}^n$. **Output:** $\chi(X)$. **Procedure:**

1. *Let* $h_1, h_2, \ldots, h_{\dim(X)} : \mathbb{C}^n \to \mathbb{C}$ *denote general affine linear functions.*
2. **For** $i \in \{1, \ldots, \dim(X)\}$ **do***: Compute the number of critical points of* h_i *on* $X \cap V(h_1, \ldots, h_{i-1}) \setminus V(h_i)$*, which we denote by* η_i*.*
3. *Calculate* p *the number of points in* $X \cap V(h_1, \ldots, h_{\dim(X)})$*.*
4. *Return* $\sum_{i=1}^{\dim(X)} (-1)^{\dim(X)-i+1} \eta_i + p$*.*

We can compute these critical points from a witness set as follows.

Algorithm 3 (Numerical Smooth X) Input: *A witness set for a smooth equidimensional affine variety* $X \subset \mathbb{C}^n$. **Output:** $\chi(X)$. **Procedure:**

1. *Compute a witness set for* $X \times \mathbb{C}^n$*, as in Example 2. Use regeneration to compute a witness collection* (2) *for the conormal variety* $\mathcal{C}(X) \subset \mathbb{C}^n \times \mathbb{C}^n$*.*
2. **For** $i = 0, \ldots, \dim(X)$ **do***:*
   (a) *Consider a set of* i *generic points* $p_1, \ldots, p_i$ *in* $\mathbb{C}^n$*.*
   (b) *Let* $q_1, \ldots, q_{n-i} \in \mathbb{C}^n$ *denote a basis for the perpendicular complement of the span of the origin and* $p_1, \ldots, p_i$*.*
   (c) *Let* a, b *denote generic points in* $\mathbb{C}^n$ *and fix the following sets of affine linear functions:* $A^i := \{p_1 \circ (x-a), \ldots, p_i \circ (x-a)\}$*, and* $B^{n-i} := \{q_1 \circ (y-b), \ldots, q_{n-i} \circ (y-b)\}$*.*
   (d) *Recall the multi-affine witness set* $(F, L^i \cup M^{n-i}, W_i)$ *for* $\mathcal{C}(X)$ *in* (2)*, and perform the homotopy* (1) *with* L^i, M^{n-i} *and* A^i, B^{n-i} *as above and start points* W_i*. Denote the set of endpoints by* S_i*.*
   (e) *Denote by* ζ_i*, the number of nonsingular isolated points* (x, y) *in* S_i *such that* $x \circ b \neq 0$*.*
3. *Return* $\sum_{i=0}^{\dim(X)} (-1)^{\dim(X)-i} \zeta_i$*.*

Proof (Correctness sketch). Example 3 explains how $\zeta_{\dim(X)} = p$ and $\zeta_0 = \eta_{\dim(X)}$. With some substitutions and algebra one shows $\zeta_i = \eta_{\dim(X)-i}$ for $i = 1, \ldots, \dim(X)$, and then the result follows.

Example 5. Consider the smooth curve $X = V(x_1^2 + x_2^2 - 1) \subset \mathbb{C}^2$ and its conormal variety $\mathcal{C}(X) = V(x_1^2 + x_2^2 - 1, x_2 y_1 - x_1 y_2) \subset \mathbb{C}^2 \times \mathbb{C}^2$. The algorithms find $(p, \eta_1, \eta_2) = (\zeta_2, \zeta_1, \zeta_0) = (2, 2, 0)$ and both output $\chi(X) = 0$. The points corresponding to p and η are illustrated in Figure 3 by plotting points on X with the respective normal vectors. The homotopy used to determine ζ_i is given by $H^i : \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^4$. Concretely, for H^1, with generic affine linear functions $\ell, m : \mathbb{C}^2 \to \mathbb{C}$, we have $H^1(x, y, t)$ is

$$\left(x_1^2 + x_2^2 - 1, x_2 y_1 - x_1 y_2, (1-t)\ell(x) + t p_1 \circ (x-a), (1-t)m(y) + t q_1 \circ (y-b)\right).$$

5 Euler characteristics of singular varieties

Algorithm 4 (Excision-restriction method) Input: *An affine variety* $X \subset \mathbb{C}^n$ *with* X_i *denoting the union of* i*-dimensional irreducible components of* X*.* **Output:** *The Euler characteristic* $\chi(X)$*.* **Procedure:**

1. *Initialize* S *to be* $|X_0|$ *and let* $Y = \cup_{i=1}^{\dim(X)} X_i$*.*
2. **If** Y *is smooth* **then** *For* $i = 1, \ldots, \dim(X)$*, if* $X_i \neq \emptyset$*, then replace* S *by* S *plus the output of Algorithm 2 on* X_i*.* **else**
   (a) *Find a polynomial function* g *such that* $X_{\dim(X)} \subsetneq V(g)$ *and* $V(g)$ *contains a union of irreducible components of* $\mathrm{Sing}(Y) \cup \bigcup_{i=1}^{\dim(X)-1} X_i$*.*
   (b) *(Restriction) Set* S_1 *to be the output of Algorithm 4 on* $Y \cap V(g)$*.*
   (c) *(Excision) Set* S_2 *to be the output of Algorithm 4 on* $Y \setminus V(g)$*, considered as a subvariety of* $\mathbb{C}^{n+1}$ *with coordinate functions* $x_1, \ldots, x_n, 1/g$*, as in Example 4.*
3. *Return* $S + S_1 + S_2$

The freedom in choosing the g is a feature of this algorithm. To minimize the number of recursions, choose a g vanishing on $\mathrm{Sing}(Y) \cup \bigcup_{i=1}^{\dim(X)-1} X_i$. Alternatively, we could choose g so that the degree is small as possible. This heuristic works well for a numerical approach. Next, we tailor the previous algorithm for regeneration, which was described in Section 3.3.

Algorithm 5 (Numerical Excision-Restriction) **Input:** *A numerical equidimensional decomposition* $\cup_i(F_i, L_i, W_i)$ *for an affine variety* $X = \cup_i X_i \subset \mathbb{C}^n$ *where* (F_i, L_i, W_i) *is a witness set for* X_i *the union of i-dimensional irreducible components of* X. **Output:** *The Euler characteristic* $\chi(X)$. **Procedure:**

1. *Initialize* S *to be the cardinality of* W_0*, and let* $Y = \bigcup_{i=1}^{\dim(X)} X_i$.
2. **If** Y *is smooth* **then** *For* $i = 1, \ldots, \dim(X)$*, replace* S *by* S *plus the output of Algorithm* 3 *on the witness set* (F_i, L_i, W_i) *for* X_i*,* **else**
   (a) *Find a polynomial* g *such that* $X_{\dim(X)} \not\subseteq V(g)$ *and* $V(g)$ *contains a union of irreducible a components of* $\mathrm{Sing}(Y) \cup \bigcup_{i=1}^{\dim(X)-1} X_i$
   (b) *Use regeneration as in Section 3.3 to compute a numerical decomposition* $\mathcal{D}_1$ *for* $Y \cap V(g)$*. Set* S_1 *to be the output of Algorithm 5 on* $\mathcal{D}_1$.
   (c) *Compute a witness for each equidimensional component of* $Y \setminus V(g)$*, like in Example 4, to get a numerical decomposition* $\mathcal{D}_2$ *of* $Y \setminus V(g)$.
   Set S_2 *to be output of Algorithm 5 on* $\mathcal{D}_2$.
3. *Return* $S + S_1 + S_2$

Example 6. Consider the Whitney umbrella $X = V(x^2y - z^2) \subset \mathbb{C}^3$. The variety X is two dimensional with a singular locus given by the y-axis. The output of the algorithm to compute $\chi(X)$ is summarized by the following equations,

$$\begin{aligned}\chi(X) &= \chi(X \cap V(g)) + \chi(X \setminus V(g)) \\ &= \chi(X \cap V(g) \cap V(g')) + \chi(X \cap V(g) \setminus V(g')) + \chi(X \setminus V(g)) \\ &= 1 + 0 + 0,\end{aligned}$$

where we have made the following choices: $g = 2x + 5z$ and $g' = y - x - 4/25$ such that $V(g) \supset \mathrm{Sing}(X)$ and $V(g') \supset \mathrm{Sing}(X \cap V(g))$.

We input a numerical equidimensional decomposition for X. Since X is itself irreducible, we have $Y = X$ and $S = |W_0| = 0$. Using regeneration we compute a numerical decomposition $\mathcal{D}_1$ for $X \cap V(g)$. Note that $X \cap V(g)$ is a union of two lines that intersect at the point $(0, 4/25, 0)$. Following the recursion, we apply Algorithm 5 to $\mathcal{D}_1$. In this case we let g' play the role of g. The variety $X \cap V(g) \cap V(g')$ is the point $(0, 4/25, 0)$ so we have $\chi(X \cap V(g) \cap V(g')) = 1$. On the other hand, $X \cap V(g) \setminus V(g')$ is smooth, so by using Algorithm 3 we get $\chi(X \cap V(g) \setminus V(g')) = 0$.

For the next part, we find a numerical decomposition $\mathcal{D}_2$ for $X \setminus V(g)$. Since $X \setminus V(g)$ is smooth, we use Algorithm 3 to find $\chi(X \setminus V(g)) = 0$.

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