

The discrete Fourier restriction phenomenon: the lattice case

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November 3, 2013

In 1993 Bourgain has considered the problem of discrete restriction to the paraboloid

$$\mathbb{P}_{n-1}(N) :=$$

$$\{\xi := (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n : \xi_n = \xi_1^2 + \dots + \xi_{n-1}^2, |\xi_1|, \dots, |\xi_{n-1}| \leq N\}$$

in connection to the periodic Schrödinger equation

Conjecture (Discrete restriction: the lattice case (paraboloid))

For each $a_\xi \in \mathbb{C}$ and each $\epsilon > 0$ we have

$$\left\| \sum_{\xi \in \mathbb{P}_{n-1}(N)} a_\xi e(\xi \cdot \mathbf{x}) \right\|_{L^p(\mathbb{T}^n, d\mathbf{x})} \lesssim_\epsilon N^{\frac{n-1}{2} - \frac{n+1}{p} + \epsilon} \|a_\xi\|_{\ell^2},$$

for p in the supercritical regime $p \geq p_{c,para} = \frac{2(n+1)}{n-1}$ and

$$\left\| \sum_{\xi \in \mathbb{P}_{n-1}(N)} a_\xi e(\xi \cdot \mathbf{x}) \right\|_{L^p(\mathbb{T}^n, d\mathbf{x})} \lesssim_\epsilon N^\epsilon \|a_\xi\|_{\ell^2},$$

in the subcritical regime $2 \leq p < p_{c,para}$.

Note that the critical index for the general and lattice restrictions to the paraboloid are the same. Thus the resolution of the general conjecture immediately implies the lattice case.

In the same year 1993 Bourgain has also initiated the study of the discrete restriction for the sphere

$$\mathbb{S}_{n-1}(N) = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n : |\xi_1|^2 + \dots + |\xi_n|^2 = N^2\}.$$

Conjecture (Discrete restriction: the lattice case (sphere))

For each $a_\xi \in \mathbb{C}$ and each $\epsilon > 0$ we have

$$\left\| \sum_{\xi \in \mathbb{S}_{n-1}(N)} a_\xi e(\xi \cdot \mathbf{x}) \right\|_{L^p(\mathbb{T}^n, d\mathbf{x})} \lesssim_\epsilon N^{\frac{n-2}{2} - \frac{n}{p} + \epsilon} \|a_\xi\|_{l^2},$$

for p in the supercritical regime $p > p_{C,sphere} = \frac{2n}{n-2}$ and

$$\left\| \sum_{\xi \in \mathbb{S}_{n-1}(N)} a_\xi e(\xi \cdot \mathbf{x}) \right\|_{L^p(\mathbb{T}^n, d\mathbf{x})} \lesssim_\epsilon N^\epsilon \|a_\xi\|_{l^2},$$

in the subcritical regime $2 \leq p < p_{C,sphere}$.

When $n = 2, 3, 4$ it is known that for each ϵ we have $|\mathbb{S}_{n-1}(N)| \lesssim N^{n-2+\epsilon}$, but the upper bound is only sharp for certain values of N . For example $\mathbb{S}_2(N) = \emptyset$ when $N^2 = 4^a(8m+7)$ for $a, m \in \mathbb{N}$. On the other hand, if $n \geq 5$ we have a sharp estimate $|\mathbb{S}_{n-1}(N)| \sim N^{n-2}$. Note also that trivially,

$$|\mathbb{P}_{n-1}(N)| = (2N+1)^{n-1} \sim N^{n-1}$$

The discrepancy between the critical indices $p_{c,para} = \frac{2(n+1)}{n-1}$ and $p_{c,sphere} = \frac{2n}{n-2}$ in the discrete world ($p_{c,para} < p_{c,sphere}$) is due to the lack of "uniform distribution" of lattice points on $\mathbb{S}_{n-1}(N)$ as opposed to those on $\mathbb{P}_{n-1}(N)$.

Between 1993 and 2011 Bourgain has attacked these problems with essentially three distinct methods.

The first one (1993, the "**number theory approach**") implements the Thomas-Stein argument via a minor/major arc decomposition. Gauss sums (paraboloid) and Kloosterman sums (sphere) are the most complicated number theoretical tools he uses. This approach by itself proves the conjectures for **large** values of p , in the supercritical range.

The second approach (he calls it the "**arithmetic approach**") is a quite immediate adaptation of the "Cordoba L^4 argument" to the discrete world. It only works for even exponents p and it only relies on the fact that circles of radius N in the plane contain $O(N^\epsilon)$ points.

For example, when $n = 3$, Bourgain completely settled the conjecture for $\mathbb{P}_2(N)$ by proving the critical estimate for $p = p_c = 4$

$$\left\| \sum_{\xi \in \mathbb{P}_2(N)} a_\xi e(\xi \cdot \mathbf{x}) \right\|_{L^4(\mathbb{T}^3, d\mathbf{x})} \lesssim_\epsilon N^\epsilon \|a_\xi\|_{l^2}.$$

This follows immediately by observing that for each fixed $\xi \neq \xi' \in \mathbb{P}_2(N)$, the set

$$A(\xi, \xi') = \{(\xi'', \xi''') \in \mathbb{P}_2(N) \times \mathbb{P}_2(N) : \xi + \xi' = \xi'' + \xi'''\}$$

contains $O(N^\epsilon)$ pairs.

Indeed, if

$$\xi'' := (n_1, m_1, n_1^2 + m_1^2)$$

$$\xi''' = (n_2, m_2, n_2^2 + m_2^2)$$

then for any given integers A, B, C the equality

$$\xi'' + \xi''' = (A, B, C)$$

implies that

$$(n_1 - \frac{A}{2})^2 + (m_1 - \frac{B}{2})^2 = \frac{2C - A^2 - B^2}{4}.$$

Thus, (n_1, m_1) (and similarly (n_2, m_2)) belong to a fixed circle of radius $O(N)$, which gives $|A(\xi, \xi')| \lesssim N^\epsilon$.

Bourgain's third and most recent approach (2011, the "**Fourier analytic approach**") is perhaps the most striking one, since it uses no number theory! It is a variant of the recent "Bourgain-Guth scheme", which in turn relies on the Bennett-Carbery-Tao multilinear estimate. The result is the sharp subcritical estimate $p = \frac{2n}{n-1}$ for both the sphere and the paraboloid. His result applies to the general non-lattice case:

For $R \gtrsim \delta^{-2}$ and $p \leq \frac{2n}{n-1}$

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot \mathbf{x}) \right|^p \right)^{1/p} \lesssim_\epsilon \delta^{-\epsilon} \|a_\xi\|_2.$$

where $\Lambda \subset S^{n-1}$ or $\Lambda \subset P^{n-1}$ is δ -separated.

Our first result was a refinement on the number theoretic approach, in the supercritical regime.

Fix $\|a_\xi\|_2 = 1$ and define

$$F(x) = \sum_{\xi \in \mathbb{S}_{n-1}(N)} a_\xi e(x \cdot \xi).$$

Theorem (Bourgain, D. 2012)

For $\alpha > N^{\frac{n-1}{4} + \epsilon}$ we have

$$|\{|F| > \alpha\}| \lesssim \alpha^{-2\frac{n-1}{n-3}} N^{\frac{2}{n-3}}.$$

Theorem (Bourgain, D. 2013)

(i) ($n=4$) For each $a_\xi \in \mathbb{C}$, $\epsilon > 0$ and $p > \frac{44}{7} > p_c = 4$ we have

$$\left\| \sum_{\xi \in \mathbb{S}_3(N)} a_\xi e(\xi \cdot \mathbf{x}) \right\|_{L^p(\mathbb{T}^4)} \lesssim_\epsilon N^{\frac{n-2}{2} - \frac{n}{p} + \epsilon} \|a_\xi\|_{l^2}$$

(ii) ($n=5$) For each $a_\xi \in \mathbb{C}$, $\epsilon > 0$ and $p > \frac{14}{3} > p_c = \frac{10}{3}$ we have

$$\left\| \sum_{\xi \in \mathbb{S}_4(N)} a_\xi e(\xi \cdot \mathbf{x}) \right\|_{L^p(\mathbb{T}^5)} \lesssim_\epsilon N^{\frac{n-2}{2} - \frac{n}{p} + \epsilon} \|a_\xi\|_{l^2}$$

We derive this by interpolating earlier results with a new L^4 estimate.

The proof of the L^4 estimate relies on two new ingredients: hyperplane-point incidence theory and the Siegel mass formula. The sharp L^4 estimate was known in dimensions $n \geq 6$ from the earlier work of Bourgain.

We use a sharp hyperplane-point incidence theorem, due to Elekes and Szabó (2012), that relies on the standard "divide and conquer" technique via a cell decomposition. Thus, the main contribution of this method to the discrete restriction problem is the use of the topology of $\mathbb{R}^n$.

To make this approach work we use multiple times the fact that ellipses with axes of length $O(N)$ in $\mathbb{R}^n$ contain $O(N^\epsilon)$ lattice points. This will imply that not "too many" hyperplanes intersect in "too many" points.

Here is how hyperplanes appear into the picture. As before it suffices to estimate $\mathbb{E}_4$. If

$$\mathbb{P}_3(N) = \{\xi := (\xi_1, \xi_2, \xi_3, \xi_1^2 + \xi_2^2 + \xi_3^2) : |\xi_i| \leq N\},$$

and if $\xi + \eta = \nu := (\nu_1, \nu_2, \nu_3, \nu_4)$, then

$$\sum_{i=1}^3 [\xi_i^2 + (\xi_i - \nu_i)^2] = \sum_{i=1}^3 [\eta_i^2 + (\eta_i - \nu_i)^2] = \nu_4,$$

and thus ξ, η belong to the hyperplane

$$H_\nu := \{\theta \in \mathbb{R}^4 : 2\nu_1\theta_1 + 2\nu_2\theta_2 + 2\nu_3\theta_3 - 2\theta_4 = \nu_1^2 + \nu_2^2 + \nu_3^2 - \nu_4\}.$$

It is immediate that

Incidence Axiom in 4d: $|H_\nu \cap H_{\nu'} \cap P_N^3| \lesssim_\epsilon N^\epsilon$.

This is false in five dimensions; the correct substitute is

Incidence Axiom in 5d There are $O(N^\epsilon)$ hyperplanes H_ν in $\mathbb{R}^5$ containing a given three dimensional affine subspace W of $\mathbb{R}^5$.

Subject to these "axioms" only, we obtain the "sharp" energy estimates

$$\mathbb{E}(\Lambda') \lesssim_{\epsilon} N^{\epsilon} |\Lambda'|^{7/3}.$$

for each $\Lambda' \subset \mathbb{S}_3(N)$, and

$$\mathbb{E}(\Lambda') \lesssim_{\epsilon} N^{\epsilon} |\Lambda'|^{5/2}.$$

for each $\Lambda' \subset \mathbb{S}_4(N)$.

Let's focus on $n = 4$. In reality, the estimate

$$\mathbb{E}(\Lambda') \lesssim_{\epsilon} N^{\epsilon} |\Lambda'|^{7/3}$$

for $\Lambda' \subset \mathbb{S}_3(N)$ is not sharp. It is conjectured that

$$\mathbb{E}(\Lambda') \lesssim_{\epsilon} N^{\epsilon} |\Lambda'|^2. \quad (1)$$

In order to achieve this via incidence theory, one would need to replace the earlier "axiom" with a stronger one, that reflects better the distribution of lattice points on spheres.

On the other hand, the Siegel mass formula is a purely number theoretical tool that proves the desired (1) estimate, but only for the whole sphere $\Lambda' = \mathbb{S}_3(N)$

$$\mathbb{E}(\mathbb{S}_3(N)) \lesssim_{\epsilon} N^{\epsilon} |\mathbb{S}_3(N)|^2. \quad (2)$$

The Bourgain-D. theorem follows by "interpolating" (1) and (2).

Here is a glimpse into the Siegel mass formula for the case that we need ($m = n + 1$). Let $\Lambda \in M_{n,n}(\mathbb{Z})$ be a positive definite matrix. Denote by $A(\Lambda)$ the number of solutions $\mathcal{L} \in M_{m,n}(\mathbb{Z})$ for

$$\mathcal{L}^* \mathcal{L} = \Lambda. \quad (3)$$

Define the densities (p prime)

$$\nu_p(\Lambda) = \lim_{r \rightarrow \infty} \frac{1}{p^{r(mn - \frac{n(n+1)}{2})}} |\{\mathcal{L} \in M_{m,n}(\mathbb{Z}_{p^r}) : \mathcal{L}^* \mathcal{L} \equiv \Lambda \pmod{p^r}\}|.$$

Then

Theorem (Siegel, 1935)

$$A(\Lambda) \lesssim_n \prod_{p \text{ prime}} \nu_p(\Lambda). \quad (4)$$

Using the Siegel mass formula we prove ($n = 3$)

Theorem (Bourgain, D.)

Let S be a symmetric 3×3 matrix with integer entries $O(N)$.
Then the number of solutions $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathbb{Z}^4)^3$ of the system of equations

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} = S$$

is in average $\lesssim_{\epsilon} N^{\epsilon}$ for each $\epsilon > 0$.

Recall that we need to count the number of quadruples $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) \in (\mathbb{Z}^4)^4$ that satisfy

$$\begin{cases} \mathbf{x} + \mathbf{y} = \mathbf{z} + \mathbf{w} \\ |\mathbf{x}|^2 = |\mathbf{y}|^2 = |\mathbf{z}|^2 = |\mathbf{w}|^2 = N^2. \end{cases}$$

Since

$$N^2 = |\mathbf{x} + \mathbf{y} - \mathbf{z}|^2 = 3N^2 + 2\mathbf{x} \cdot \mathbf{y} - 2\mathbf{x} \cdot \mathbf{z} - 2\mathbf{y} \cdot \mathbf{z},$$

it turns out that

$$\begin{aligned} \mathbb{E}(\mathbb{S}_3(N)) &= \\ |\{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathbb{Z}^4)^3 : |\mathbf{x}|^2 = |\mathbf{y}|^2 = |\mathbf{z}|^2 = N^2, -\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z} = N^2\}| &= \\ \sum_{\substack{a, b \in \mathbb{Z} \\ |a|, |b| \leq N^2}} |\{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathbb{Z}^4)^3 : |\mathbf{x}|^2 = |\mathbf{y}|^2 = |\mathbf{z}|^2 = N^2, \\ \mathbf{x} \cdot \mathbf{y} = a, \mathbf{y} \cdot \mathbf{z} = b, \mathbf{x} \cdot \mathbf{z} = N^2 + a - b\}|. \end{aligned}$$

Note that a triple $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ as above satisfies

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} = \begin{bmatrix} N^2 & a & N^2 + a - b \\ a & N^2 & b \\ N^2 + a - b & b & N^2 \end{bmatrix}.$$

We show further that the average over a, b of the number of solutions is $O(N^\epsilon)$. In five dimensions the average is over four parameters, and involves delicate estimates using the divisor function.

If the **general** discrete restriction conjecture is proved, we could further improve the range (in the lattice case) to $p \geq 6$ when $n = 4$ and $p \geq 4$ when $n = 5$. It is very likely that to get the full range for the lattice points on the sphere, some involved number theory will be needed.

Our theorem derived from the Siegel mass formula is general enough to allow a full range of applications.

For example, it has been recently asked by A. Greenleaf, A. Iosevich, B. Liu and E. Palsson what is the number of noncongruent four point configurations that exist in $\{0, 1, 2, \dots, N\}^3$, given that one point is fixed at the origin.

An application of our theorem proves the number is $\gtrsim N^3$.