

The discrete Fourier restriction phenomenon: the non-lattice case

Ciprian Demeter, Indiana University

November 2, 2013

Let S^{n-1} be the sphere of radius 1 in $\mathbb{R}^n$ and let

$$P^{n-1} = \{(x_1, \dots, x_{n-1}, x_1^2 + \dots x_{n-1}^2) : -1 \leq x_i \leq 1\}$$

be the truncated paraboloid.

For $f : \mathbb{R}^n \rightarrow \mathbb{C}$ in $L^1(\mathbb{R}^n)$ denote the Fourier transform $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{C}$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(\mathbf{x}) e(\mathbf{x} \cdot \xi) d\mathbf{x}$$

where $e(a) = e^{2\pi i a}$.

There are two extreme scenarios

Recall that if $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}$ is continuous and

$$\sup_{\xi \in \mathbb{R}^n} |\widehat{f}(\xi)| \lesssim \|f\|_1$$

In particular, if S is **any** subset of $\mathbb{R}^n$, such as S^{n-1} , P^{n-1} , we can meaningfully restrict the Fourier transform to S and

$$\|\widehat{f}\|_{L^\infty(S)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}$$

On the other hand, if $f \in L^2$ then $\hat{f}$ is not continuous in general, it is only guaranteed to be in L^2 .

As a result, $\hat{f}$ can be modified on a set of (n dimensional Lebesgue) measure zero (such as S^{n-1}, P^{n-1}) and the new function would still be the Fourier transform of f .

So it makes no sense to restrict $\hat{f}$ to S^{n-1}, P^{n-1} if $f \in L^2$.

Question: Is there any restriction theory between L^1 and L^2 , in other words can one find pairs p, q with $1 < p < 2$ and surfaces S in $\mathbb{R}^n$ such that

$$\|\widehat{f}\|_{L^q(S)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}?$$

The answer is NO, if S is a hyperplane...

...but it is YES, if S has nonzero curvature.

For S either S^{n-1} or P^{n-1} denote the statement

$$\|\widehat{f}\|_{L^q(S^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

with $R_S(p \mapsto q)$.

Simple examples show that $R_S(p \mapsto q)$ can only hold if $p < \frac{2n}{n+1}$ and $p' \geq \frac{q(n+1)}{n-1}$. It is conjectured that these conditions are also sufficient.

It is easy to see that if $p' > q$ then the restriction conjecture $R_S(p \mapsto q)$ is equivalent with the following discretized version

$$\left\| \sum_{\xi \in \Lambda} a_{\xi} e(\xi \cdot \mathbf{x}) \right\|_{L^{p'}(B(c, \delta^{-1}))} \lesssim \delta^{-\frac{n-1}{q}} \left(\sum_{\xi \in \Lambda} |a_{\xi}|^{q'} \right)^{1/q'}$$

for each center $c \in \mathbb{R}^n$ and each δ separated set $\Lambda \in S$. This shows that the continuous restriction conjecture is about oscillations at spatial scale δ^{-1} .

By summing over δ^{-1} separated centers c we get

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_{\xi} e(\xi \cdot \mathbf{x}) \right|^{p'} \right)^{1/p'} \lesssim \delta^{\frac{n}{p'} - \frac{n-1}{q}} \left(\sum_{\xi \in \Lambda} |a_{\xi}|^{q'} \right)^{1/q'}$$

for each $R \gtrsim \delta^{-1}$ and each R -ball B_R .

The case $q = 2$ is completely solved (Thomas/Stein 1970's):

$R(p \mapsto 2)$ holds in the conjectured range $1 \leq p \leq \frac{2(n+1)}{n+3}$.

This in turn implies that when $R \gtrsim \delta^{-1}$ and $p \geq \frac{2(n+1)}{n-1}$

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot \mathbf{x}) \right|^p \right)^{1/p} \lesssim \delta^{\frac{n}{p} - \frac{n-1}{2}} \|a_\xi\|_2.$$

However, it is conjectured that further cancellations occur at the larger spatial scales $\gtrsim \delta^{-2}$

Conjecture (General discrete restriction: supercritical regime)

For $R \gtrsim \delta^{-2}$ and $p \geq \frac{2(n+1)}{n-1}$

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot \mathbf{x}) \right|^p \right)^{1/p} \lesssim_\epsilon \delta^{\frac{n+1}{p} - \frac{n-1}{2} + \epsilon} \|a_\xi\|_2.$$

where $\Lambda \subset S^{n-1}$ or $\Lambda \subset P^{n-1}$ is delta-separated.

Note that in particular it is anticipated that at the critical index $\frac{2(n+1)}{n-1}$ the following should hold

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot \mathbf{x}) \right|^{\frac{2(n+1)}{n-1}} \right)^{\frac{n-1}{2(n+1)}} \lesssim_\epsilon \delta^{-\epsilon} \|a_\xi\|_2.$$

By Hölder this would immediately give

Conjecture (General discrete restriction: the subcritical case)

For $R \gtrsim \delta^{-2}$, $p \leq \frac{2(n+1)}{n-1}$ and each $\epsilon > 0$

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot \mathbf{x}) \right|^p \right)^{1/p} \lesssim_\epsilon \delta^{-\epsilon} \|a_\xi\|_2.$$

where $\Lambda \subset S^{n-1}$ or $\Lambda \subset P^{n-1}$ is delta-separated.

In 1993 Bourgain has considered the problem of discrete restriction to the paraboloid

$$\mathbb{P}_{n-1}(N) :=$$

$$\{\xi := (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n : \xi_n = \xi_1^2 + \dots + \xi_{n-1}^2, |\xi_1|, \dots, |\xi_{n-1}| \leq N\}$$

in connection to the periodic Schrödinger equation

Conjecture (Discrete restriction: the lattice case (paraboloid))

For each $a_\xi \in \mathbb{C}$ and each $\epsilon > 0$ we have

$$\left\| \sum_{\xi \in \mathbb{P}_{n-1}(N)} a_\xi e(\xi \cdot \mathbf{x}) \right\|_{L^p(\mathbb{T}^n, d\mathbf{x})} \lesssim_\epsilon N^{\frac{n-1}{2} - \frac{n+1}{p} + \epsilon} \|a_\xi\|_{\ell^2},$$

for p in the supercritical regime $p \geq p_{c,para} = \frac{2(n+1)}{n-1}$ and

$$\left\| \sum_{\xi \in \mathbb{P}_{n-1}(N)} a_\xi e(\xi \cdot \mathbf{x}) \right\|_{L^p(\mathbb{T}^n, d\mathbf{x})} \lesssim_\epsilon N^\epsilon \|a_\xi\|_{\ell^2},$$

in the subcritical regime $2 \leq p < p_{c,para}$.

Bourgain (1993, GAFA) has proved this conjecture when $n = 2$ and $n = 3$.

Note that the critical index for the general and lattice restrictions to the paraboloid are the same. Thus the resolution of the general conjecture immediately implies the lattice case.

In 2011, Bourgain employed the "Bourgain-Guth scheme" relying on the Bennett-Carbery-Tao multilinear Kakeya estimate to prove the sharp subcritical estimate $p = \frac{2n}{n-1}$ for the general discrete restriction conjecture for both the sphere and the paraboloid.

For $R \gtrsim \delta^{-2}$ and $p \leq \frac{2n}{n-1}$

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot \mathbf{x}) \right|^p \right)^{1/p} \lesssim_\epsilon \delta^{-\epsilon} \|a_\xi\|_2.$$

where $\Lambda \subset S^{n-1}$ or $\Lambda \subset P^{n-1}$ is δ -separated.

On the other hand Garrigós, Schlag and Seeger have proved sharp estimates for p large in the supercritical regime.

Observation (D., 2012)

Let Λ be a collection of points ξ on the paraboloid in $\mathbb{R}^3$, that is $\xi = (\xi_1, \xi_2, \xi_1^2 + \xi_2^2)$. Then the critical estimate holds for all R large enough

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot \mathbf{x}) \right|^4 d\mathbf{x} \right)^{1/4} \lesssim_\epsilon |\Lambda|^\epsilon \|a_\xi\|_{l^2},$$

where B_R is any ball of radius R in $\mathbb{R}^3$.

If $\Lambda = \mathbb{P}_2(N)$ then one recovers Bourgain's aforementioned estimate, by simply invoking periodicity.

Interestingly, the bound $|\Lambda|^\epsilon$ does not depend on the amount of separation between points, but only on their cardinality! Curvature suggests that if the points are δ separated then one should be able to take $R \gtrsim \delta^{-2}$, but I can't prove that

The proof uses incidence theory. Recall

Szemerédi-Trotter: There are $O(m + n + (mn)^{2/3})$ incidences between points and lines in the plane.

Circle-point incidence conjecture: essentially the same should hold if lines are replaced with circles.

Via restricted type interpolation, the problem is equivalent with

$$\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda'} e^{i\mathbf{x} \cdot \xi} \right|^4 d\mathbf{x} \lesssim_{\epsilon} |\Lambda|^{\epsilon} |\Lambda'|^2,$$

for each subset $\Lambda' \subset \Lambda$, for R large enough. I will show that this holds true for R big enough depending only on $|\Lambda|$ and on the separation number

$$\eta := \min \{ |\xi_1 + \xi_2 - \xi_3 - \xi_4| : \xi_i \in \Lambda : \xi_1 + \xi_2 - \xi_3 - \xi_4 \neq \mathbf{0} \}.$$

Expanding the L^4 norm we need to prove

$$\sum_{\xi_i \in \Lambda'} \frac{1}{R^3} \int_{B_R} e^{i(\xi_1 + \xi_2 - \xi_3 - \xi_4) \cdot \mathbf{x}} d\mathbf{x} \lesssim_{\epsilon} |\Lambda|^{\epsilon} |\Lambda'|^2.$$

Note that if $A \neq 0$

$$\int_{-R}^R e^{iAx} dx \lesssim A^{-1}, \text{ as } R \rightarrow \infty.$$

Using this we get that whenever $\xi \neq (0, 0, 0)$ we have

$$\int_{B_R} e^{i\xi \cdot \mathbf{x}} d\mathbf{x} \lesssim_{\xi} R^2, \text{ as } R \rightarrow \infty.$$

Thus all quadruples such that $\xi_1 + \xi_2 - \xi_3 - \xi_4 \neq 0$ will contribute $O_{\eta}(R^2)$ to the integral above. Their contribution will become $O(R^3)$ for R large enough compared to $|\Lambda|$ and η .

Thus it suffices to prove the following estimate for the additive energy

$$\mathbb{E}_4(\Lambda') := |\{(\xi, \dots, \xi''') \in \Lambda'^4 : \xi + \xi' = \xi'' + \xi'''\}| \lesssim_\epsilon |\Lambda'|^{2+\epsilon} \quad (1)$$

for each subset $\Lambda' \subset \Lambda$.

Assume $\xi_1 + \xi_2 = \xi_3 + \xi_4$. Let

$$\xi_1 := (n_1, m_1, n_1^2 + m_1^2)$$

$$\xi_2 = (n_2, m_2, n_2^2 + m_2^2)$$

then for any given integers A, B, C the equality

$$\xi_1 + \xi_2 = (A, B, C)$$

implies that

$$(n_1 - \frac{A}{2})^2 + (m_1 - \frac{B}{2})^2 = \frac{2C - A^2 - B^2}{4}.$$

Thus, (n_1, m_1) (and similarly (n_i, m_i)) belong to a circle. The bound on the energy would follow if we assume the circle-point incidence conjecture.

Note also that (n_1, m_1) and (n_2, m_2) must in fact be diametrically opposite on this circle.

I needed to rely on this in order to avoid the (notoriously difficult!) circle-point incidence theory. In particular, note that each quadruple $(\xi, \xi', \xi'', \xi''')$ contributing to $\mathbb{E}_4(\Lambda')$ gives rise to a distinct right angle, the one subtended by ξ, ξ', ξ'' .

The estimate then follows from the following application of Szemerédi-Trotter

Theorem (Pach, Sharir, 1992)

The number of repetitions of a given angle among N points in the plane is $O(N^2 \log N)$.

If one considers (arbitrary) points on the sphere $\mathbb{S}^2$, the same result will follow if one assumes the conjectured bound on the circle point incidences. Indeed, it is easy to observe that if $\xi, \dots, \xi''' \in \mathbb{S}^2$ are distinct and if

$$\xi + \xi' = \xi'' + \xi'''$$

then (ξ, ξ') and (ξ'', ξ''') will form pairs of diametrically opposite points on some circle on $\mathbb{S}^2$. Via the stereographic projection, all four points will land on a circle in the plane. Unfortunately this projection will not preserve the property of being diametrically opposite and thus prevents the previous argument from working.

This seems to indicate the fact that spheres are trickier than paraboloids. It turns out that even in three dimensions, the circle-point perspective is not a fruitful one. One needs to instead observe that $\xi, \dots, \xi'''$ belong to a plane (i.e. hyperplane)!

One may try to use this method to solve the following version of the general discrete restriction conjecture in two dimensions. Here the critical index is 6. This amounts to proving

Conjecture

Let Λ be a collection of points ξ on the parabola in $\mathbb{R}^2$, that is $\xi = (z, z^2)$. Then for all R large enough

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot \mathbf{x}) \right|^6 d\mathbf{x} \right)^{1/6} \lesssim_\epsilon |\Lambda|^\epsilon \|a_\xi\|_{l^2},$$

where B_R is any ball of radius R in $\mathbb{R}^2$.

As before, it would suffice to prove the following estimate for the 6 additive energy

$$\mathbb{E}_6(\Lambda) := |\{(\xi_1, \dots, \xi_6) \in \Lambda^6 : \xi_1 + \xi_2 + \xi_3 = \xi_4 + \xi_5 + \xi_6\}| \lesssim_\epsilon |\Lambda|^{3+\epsilon} \quad (2)$$

for each subset Λ of the parabola.

One may consider the analog problem for the circle. The trivial upper bound is $O(|\Lambda|^4)$ (for both circle and parabola)

Bourgain has initially used additive combinatorics to prove the bound $o(|\Lambda|^4)$, in order to answer a question related to the nodal length of eigenfunctions raised by KRISHNAPUR, KURLBERG, AND WIGMAN (2012)

In a recent Bombieri-Bourgain paper, they improve this bound for the circle to get

$$\mathbb{E}_6(\Lambda) \lesssim_\epsilon |\Lambda|^{\frac{7}{2}+\epsilon}.$$

The proof relies on Szemerédi-Trotter.

They also consider the case where Λ are the lattice points on the circle

$$x^2 + y^2 = m, \quad m \in \mathbb{N}$$

and show that for some values of m

$$\mathbb{E}_6(\Lambda) \lesssim |\Lambda|^3$$

The same exponent $7/2$ can be shown for the parabola:
Indeed, Bourgain has observed (1993, GAFA) that if

$$(x_1, x_1^2) + (x_2, x_2^2) + (x_3, x_3^2) = (n, j),$$

then the point $(3(x_1 + x_2), \sqrt{3}(x_1 - x_2))$ belongs to the circle centered at $(2n, 0)$ and of radius squared equal to $6j - 2n^2$. Note that there are $|\Lambda|^2$ such points, call this set T .

Assume we have M_n such circles containing roughly 2^n points in T . Then clearly

$$\mathbb{E}_6(\Lambda) \lesssim \sum_{2^n \leq |\Lambda|} M_n 2^{2n}.$$

It is easy to see that

$$M_n 2^n \lesssim |\Lambda|^3,$$

as each point in T can belong to at most $|\Lambda|$ circles.

The nontrivial estimate is

$$M_n 2^{3n} \lesssim |\Lambda|^4,$$

which is an immediate consequence of the Szemerédi-Trotter Theorem for Szemerédi-Trotter curves (i.e. two curves intersect in $O(1)$ points, and there are $O(1)$ curves passing through any two given points from T).

Note that since our circles have centers on the x axis, any two points in T determine a unique circle.

Thus

$$M_n 2^n \lesssim M_n + |\Lambda| + (M_n |\Lambda|)^{2/3}.$$

Taking the geometric mean of the two inequalities we get

$$M_n 2^{2n} \lesssim |\Lambda|^{\frac{7}{2}}.$$

Crucial to this approach is the fact that our circles form a Szemerédi-Trotter family.

The Bombieri-Bourgain proof for the 6-energy of the circle is very similar. One ends up with yet another Szemerédi-Trotter family of circles: the unit circles of arbitrary center. In this case, the conjectured bound on $\mathbb{E}_6$ would follow from the (notoriously hard) **unit distance conjecture**

It seems natural to conjecture the following:

Given N^2 points in the plane and a Szemerédi-Trotter family of circles such that

(1) each point is incident to (at least) $\sim N$ circles

(2) each circle contains at most N points

then

$$M_n 2^{2n} \lesssim N^{3+\epsilon},$$

where M_n is the number of circles containing $\sim 2^n$ points.

I will construct an S-T family of $\sim N^{5/2}$ circles and a family of $\sim N^2$ points such that each point is incident to $\sim N$ circles. In this example a big fraction of the circles will contain roughly $N^{1/2}$ points so

$$M_{N^{1/2}} N \sim N^{7/2}$$

Start with the points

$$\{(n, m) \in \mathbb{Z}^2 : 1 \leq n, m \leq N\}.$$

Through each such point consider the line with slope a/b where $a, b \sim N^{1/2}$ where $(a, b) = 1$. Note that there are $\sim N$ such slopes (logs ignored) and that a big fraction of the lines contain $\sim N^{1/2}$ points. There will thus be $\sim N^{5/2}$ lines, since each line will be repeated $N^{1/2}$ times. It suffices now to map the points and lines from the complex plane using $z \mapsto z^{-1}$. The circles will all pass through the origin.

One can also look at higher order energies. The conjectured bound for the 8-energy is

$$\mathbb{E}_8(\Lambda) \lesssim |\Lambda|^{5+\epsilon} \quad (*)$$

This is immediately implied by the conjectured bound on $\mathbb{E}_6(\Lambda)$. It leads to similar point sphere-incidence problems in $\mathbb{R}^3$, which I don't know how to solve.

Interestingly, the bound $(*)$ follows from the work of Garrigós-Seeger (2009) in the special case when Λ is a saturated set on the unit circle (or parabola). This means that the points are δ -separated and $|\Lambda| \sim \delta^{-1}$.

The proof of $(*)$ in this case relies at its core on three ingredients: the "thick" version of the axiom that two lines intersect in one point, rescaling (these two are enough to prove the right bound for $\mathbb{E}_{10}(\Lambda)$) and the easy Cordoba-type estimate $\mathbb{E}_4(\Lambda) \lesssim |\Lambda|^2$.