

L^p norms of eigenfunctions and Keakeya-Nikodym averages

Matthew D. Blair

University of New Mexico

November 2013

Joint work with Christopher Sogge (Johns Hopkins)

Stein-Tomas restriction theorem

Theorem (Stein, Tomas)

Let $\mathcal{R}f = \widehat{f}|_{\mathbb{S}^{n-1}}$, then for every $1 \leq r \leq \frac{2(n+1)}{n+3}$

$$\|\mathcal{R}f\|_{L^2(\mathbb{S}^{n-1})} \leq C_r \|f\|_{L^r(\mathbb{R}^n)}$$

and hence for every $\frac{2(n+1)}{n-1} \leq q \leq \infty$

$$\|\mathcal{R}^*g\|_{L^q(\mathbb{R}^n)} \leq C_q \|g\|_{L^2(\mathbb{S}^{n-1})}.$$

Adjoint of the restriction operator

- For $\frac{2(n+1)}{n-1} \leq q \leq \infty$

$$\|\mathcal{R}^*g\|_{L^q(\mathbb{R}^n)} \leq C_q \|g\|_{L^2(\mathbb{S}^{n-1})}.$$

- Can compute

$$\mathcal{R}^*g(x) = \int_{\mathbb{S}^{n-1}} e^{ix \cdot \xi} g(\xi) d\sigma(\xi)$$

- Since integral is over $|\xi|^2 = 1$, we have

$$\Delta(\mathcal{R}^*g) = \mathcal{R}^*g \quad \text{where} \quad \Delta = - \sum_{j=1}^n \partial_{x_j}^2$$

(rescale for solutions to $\Delta u = \lambda^2 u$)

- Formally, $\mathcal{R}^* \circ \mathcal{F}$ is a “singular” projection onto $|\xi|^2 = 1$

Adjoint of the restriction operator

- For $\frac{2(n+1)}{n-1} \leq q \leq \infty$

$$\|\mathcal{R}^*g\|_{L^q(\mathbb{R}^n)} \leq C_q \|g\|_{L^2(\mathbb{S}^{n-1})}.$$

- Can compute

$$\mathcal{R}^*g(x) = \int_{\mathbb{S}^{n-1}} e^{ix \cdot \xi} g(\xi) d\sigma(\xi)$$

- Since integral is over $|\xi|^2 = 1$, we have

$$\Delta(\mathcal{R}^*g) = \mathcal{R}^*g \quad \text{where} \quad \Delta = -\sum_{j=1}^n \partial_{x_j}^2$$

(rescale for solutions to $\Delta u = \lambda^2 u$)

- Formally, $\mathcal{R}^* \circ \mathcal{F}$ is a “singular” projection onto $|\xi|^2 = 1$

Eigenfunctions on a compact manifold

- Let (M, g) be a C^∞ , compact Riemannian manifold, $\dim(M) \geq 2$, $\partial M = \emptyset$
- Let Δ_g be the positive Laplace-Beltrami operator,

$$\Delta_g u = \operatorname{div}(\nabla_g u),$$

in coordinates,

$$\Delta_g u = - \sum_{ij} \frac{1}{\sqrt{\det g_{kl}}} \partial_{x_i} \left(g^{ij} \sqrt{\det g_{kl}} \partial_{x_j} u \right)$$

- Spectrum of Δ_g is discrete, there exists a sequence $\{e_{\lambda_j}\}_{j=1}^\infty$ of eigenfunctions forming an O.N. basis for $L^2(M)$

$$\Delta_g e_{\lambda_j} = \lambda_j^2 e_{\lambda_j}, \quad \lambda_j \geq 0, \lambda_j \nearrow \infty$$

Eigenfunctions on $\mathbb{S}^n$

- For example, on $\mathbb{S}^2$, with spherical coordinates (θ, ϕ)

$$\Delta_g u = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta u) + \frac{1}{\sin^2 \theta} \partial_\phi^2 u$$

- Spectrum of Δ_g on $\mathbb{S}^n$ is well understood:

$$\text{spec}(\sqrt{\Delta_g}) = \{ \sqrt{k(k+n-1)} \approx k : k = 0, 1, 2, 3, \dots \}$$

and dimension of each eigenspace is

$$(2k+n-1) \frac{(k+n-2)!}{k!(n-1)!} \approx k^{n-1}$$

Eigenfunctions on $\mathbb{S}^n$

- For example, on $\mathbb{S}^2$, with spherical coordinates (θ, ϕ)

$$\Delta_g u = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta u) + \frac{1}{\sin^2 \theta} \partial_\phi^2 u$$

- Spectrum of Δ_g on $\mathbb{S}^n$ is well understood:

$$\text{spec}(\sqrt{\Delta_g}) = \{ \sqrt{k(k+n-1)} \approx k : k = 0, 1, 2, 3, \dots \}$$

and dimension of each eigenspace is

$$(2k + n - 1) \frac{(k + n - 2)!}{k!(n - 1)!} \approx k^{n-1}$$

Theorem (Sogge '86)

Let P_k be the orthogonal projection on the eigenspace corresponding to $k(k + n - 1)$, then

$$\|P_k f\|_{L^q(\mathbb{S}^n)} \leq C_q k^{\delta(q)} \|f\|_{L^2(\mathbb{S}^n)}$$

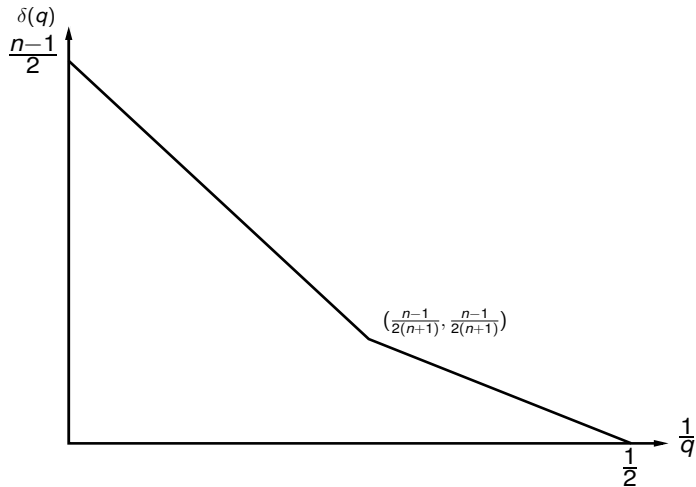
$$\delta(q) = \begin{cases} \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{q} \right), & 2 \leq q \leq \frac{2(n+1)}{n-1} \\ \frac{n-1}{2} - \frac{n}{q}, & \frac{2(n+1)}{n-1} \leq q \leq \infty \end{cases}$$

Moreover, $\delta(q)$ is sharp in that it cannot be replaced by any smaller exponent

Hence if e_λ is any L^2 normalized eigenfunction

$$\|e_\lambda\|_{L^q(M)} \leq C_q \lambda^{\delta(q)} \quad \lambda = \sqrt{k(k + n - 1)}$$

L^q bounds



- Main problem: given (M, g) and an eigenspace

$$V_\lambda = \{e_\lambda : \Delta_g e_\lambda = \lambda^2 e_\lambda\}$$

find (the best possible) upper bounds on

$$\sup_{e_\lambda \in V_\lambda : \|e_\lambda\|_{L^2(M)}=1} \|e_\lambda\|_{L^q(M)}, \quad 2 < q \leq \infty$$

- Main difficulty: for most (M, g) , projection onto V_λ is difficult to understand!

- Main problem: given (M, g) and an eigenspace

$$V_\lambda = \{e_\lambda : \Delta_g e_\lambda = \lambda^2 e_\lambda\}$$

find (the best possible) upper bounds on

$$\sup_{e_\lambda \in V_\lambda : \|e_\lambda\|_{L^2(M)}=1} \|e_\lambda\|_{L^q(M)}, \quad 2 < q \leq \infty$$

- Main difficulty: for most (M, g) , projection onto V_λ is difficult to understand!

Spectral clusters

- Given $\lambda \geq 1$, define

$$\Pi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1]} \langle f, e_{\lambda_j} \rangle e_{\lambda_j}$$

- Functions in the range of Π_λ are called “spectral clusters”, using that $\lambda_j^2 - \lambda^2 = (\lambda_j - \lambda)(\lambda_j + \lambda)$ we have

$$\|(\Delta_g - \lambda^2)\Pi_\lambda f\|_{L^2(M)} \leq (2\lambda + 1)\|f\|_{L^2(M)}$$

i.e. $(\Delta_g - \lambda^2)\Pi_\lambda f = \mathcal{O}(\lambda)$, yields an approx. eigenfunction

- But $\Delta_g e_\lambda = \lambda^2 e_\lambda$ implies

$$\Pi_\lambda e_\lambda = e_\lambda$$

- On $\mathbb{S}^n$, $\Pi_\lambda = P_k$ when $\lambda = \sqrt{k(k+n-1)}$ given clustering of eigenvalues

Spectral clusters

- Given $\lambda \geq 1$, define

$$\Pi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1]} \langle f, e_{\lambda_j} \rangle e_{\lambda_j}$$

- Functions in the range of Π_λ are called “spectral clusters”, using that $\lambda_j^2 - \lambda^2 = (\lambda_j - \lambda)(\lambda_j + \lambda)$ we have

$$\|(\Delta_g - \lambda^2)\Pi_\lambda f\|_{L^2(M)} \leq (2\lambda + 1)\|f\|_{L^2(M)}$$

i.e. $(\Delta_g - \lambda^2)\Pi_\lambda f = \mathcal{O}(\lambda)$, yields an approx. eigenfunction

- But $\Delta_g e_\lambda = \lambda^2 e_\lambda$ implies

$$\Pi_\lambda e_\lambda = e_\lambda$$

- On $\mathbb{S}^n$, $\Pi_\lambda = P_k$ when $\lambda = \sqrt{k(k+n-1)}$ given clustering of eigenvalues

L^q bounds on spectral clusters

Theorem (Sogge '88)

If $\dim(M) = n \geq 2$ and $g \in C^\infty$, then

$$\|\Pi_\lambda f\|_{L^q(M)} \leq C \lambda^{\delta(q)} \|f\|_{L^2(M)}$$

$$\delta(q) = \begin{cases} \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{q} \right), & 2 \leq q \leq \frac{2(n+1)}{n-1} \\ \frac{n-1}{2} - \frac{n}{q}, & \frac{2(n+1)}{n-1} \leq q \leq \infty \end{cases}$$

Moreover, on any manifold there are families of spectral clusters which show the exponent $\delta(q)$ is sharp.

Consequently, if $e_\lambda \in V_\lambda$, $\|e_\lambda\|_{L^2(M)} = 1$

$$\|e_\lambda\|_{L^q(M)} = \mathcal{O}(\lambda^{\delta(q)})$$

L^q bounds on spectral clusters

Theorem (Sogge '88)

If $\dim(M) = n \geq 2$ and $g \in C^\infty$, then

$$\|\Pi_\lambda f\|_{L^q(M)} \leq C \lambda^{\delta(q)} \|f\|_{L^2(M)}$$

$$\delta(q) = \begin{cases} \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{q} \right), & 2 \leq q \leq \frac{2(n+1)}{n-1} \\ \frac{n-1}{2} - \frac{n}{q}, & \frac{2(n+1)}{n-1} \leq q \leq \infty \end{cases}$$

Moreover, on any manifold there are families of spectral clusters which show the exponent $\delta(q)$ is sharp.

Consequently, if $e_\lambda \in V_\lambda$, $\|e_\lambda\|_{L^2(M)} = 1$

$$\|e_\lambda\|_{L^q(M)} = \mathcal{O}(\lambda^{\delta(q)})$$

L^q bounds on spectral clusters

Theorem (Sogge '88)

If $\dim(M) = n \geq 2$ and $g \in C^\infty$, then

$$\|\Pi_\lambda f\|_{L^q(M)} \leq C \lambda^{\delta(q)} \|f\|_{L^2(M)}$$

$$\delta(q) = \begin{cases} \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{q} \right), & 2 \leq q \leq \frac{2(n+1)}{n-1} \\ \frac{n-1}{2} - \frac{n}{q}, & \frac{2(n+1)}{n-1} \leq q \leq \infty \end{cases}$$

Moreover, on any manifold there are families of spectral clusters which show the exponent $\delta(q)$ is sharp.

Consequently, if $e_\lambda \in V_\lambda$, $\|e_\lambda\|_{L^2(M)} = 1$

$$\|e_\lambda\|_{L^q(M)} = \mathcal{O}(\lambda^{\delta(q)})$$

Spectral clusters and the wave kernel

- In practice, can replace Π_λ by $\chi_\lambda = \chi(\sqrt{\Delta_g} - \lambda)$, $\chi \in \mathcal{S}(\mathbb{R})$, $\text{supp}(\hat{\chi}) \subset (\varepsilon, 2\varepsilon)$, $\chi(0) = 1$

$$\begin{aligned}\chi_\lambda &= \chi(\sqrt{\Delta_g} - \lambda) = \int e^{it(\sqrt{\Delta_g} - \lambda)} \hat{\chi}(t) dt \\ &= \int_\varepsilon^{2\varepsilon} e^{-it\lambda} \cos(t\sqrt{\Delta_g}) \hat{\chi}(t) dt + \tilde{\chi}(\sqrt{\Delta_g} + \lambda)\end{aligned}$$

- Shows Schwartz kernel of χ_λ is *local*, main contribution is supported in $\varepsilon \leq d_g(x, y) \leq 2\varepsilon$
- The good: Opens up harmonic analysis toolbox, oscillatory integral operator estimates
- The bad: Ignores global behavior of the geodesic flow, wave kernel

Spectral clusters and the wave kernel

- In practice, can replace Π_λ by $\chi_\lambda = \chi(\sqrt{\Delta_g} - \lambda)$, $\chi \in \mathcal{S}(\mathbb{R})$, $\text{supp}(\hat{\chi}) \subset (\varepsilon, 2\varepsilon)$, $\chi(0) = 1$

$$\begin{aligned}\chi_\lambda &= \chi(\sqrt{\Delta_g} - \lambda) = \int e^{it(\sqrt{\Delta_g} - \lambda)} \hat{\chi}(t) dt \\ &= \int_\varepsilon^{2\varepsilon} e^{-it\lambda} \cos(t\sqrt{\Delta_g}) \hat{\chi}(t) dt + \tilde{\chi}(\sqrt{\Delta_g} + \lambda)\end{aligned}$$

- Shows Schwartz kernel of χ_λ is *local*, main contribution is supported in $\varepsilon \leq d_g(x, y) \leq 2\varepsilon$
- The good: Opens up harmonic analysis toolbox, oscillatory integral operator estimates
- The bad: Ignores global behavior of the geodesic flow, wave kernel

Spectral clusters and the wave kernel

- In practice, can replace Π_λ by $\chi_\lambda = \chi(\sqrt{\Delta_g} - \lambda)$, $\chi \in \mathcal{S}(\mathbb{R})$, $\text{supp}(\hat{\chi}) \subset (\varepsilon, 2\varepsilon)$, $\chi(0) = 1$

$$\begin{aligned}\chi_\lambda &= \chi(\sqrt{\Delta_g} - \lambda) = \int e^{it(\sqrt{\Delta_g} - \lambda)} \hat{\chi}(t) dt \\ &= \int_{\varepsilon}^{2\varepsilon} e^{-it\lambda} \cos(t\sqrt{\Delta_g}) \hat{\chi}(t) dt + \tilde{\chi}(\sqrt{\Delta_g} + \lambda)\end{aligned}$$

- Shows Schwartz kernel of χ_λ is *local*, main contribution is supported in $\varepsilon \leq d_g(x, y) \leq 2\varepsilon$
- The good: Opens up harmonic analysis toolbox, oscillatory integral operator estimates
- The bad: Ignores global behavior of the geodesic flow, wave kernel

Spectral clusters and the wave kernel

- In practice, can replace Π_λ by $\chi_\lambda = \chi(\sqrt{\Delta_g} - \lambda)$, $\chi \in \mathcal{S}(\mathbb{R})$, $\text{supp}(\hat{\chi}) \subset (\varepsilon, 2\varepsilon)$, $\chi(0) = 1$

$$\begin{aligned}\chi_\lambda &= \chi(\sqrt{\Delta_g} - \lambda) = \int e^{it(\sqrt{\Delta_g} - \lambda)} \hat{\chi}(t) dt \\ &= \int_{\varepsilon}^{2\varepsilon} e^{-it\lambda} \cos(t\sqrt{\Delta_g}) \hat{\chi}(t) dt + \tilde{\chi}(\sqrt{\Delta_g} + \lambda)\end{aligned}$$

- Shows Schwartz kernel of χ_λ is *local*, main contribution is supported in $\varepsilon \leq d_g(x, y) \leq 2\varepsilon$
- The good: Opens up harmonic analysis toolbox, oscillatory integral operator estimates
- The bad: Ignores global behavior of the geodesic flow, wave kernel

Model operator on $\mathbb{R}^n$

- Analog of χ_λ on $\mathbb{R}^n$: Fourier multiplier, symbol $\chi(|\xi| - \lambda)$

$$\chi_\lambda f = \int e^{ix \cdot \xi} \chi(|\xi| - \lambda) \widehat{f}(\xi) d\xi$$

- Schwartz kernel is a convolution kernel given by:

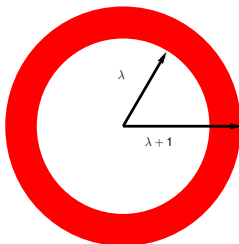
$$\int_0^\infty \left(\int_{\mathbb{S}^{n-1}} e^{ir(x-y) \cdot \omega} d\sigma(\omega) \right) \chi(r - \lambda) r^{n-1} dr$$

- Term in parentheses: $\widehat{d\sigma}(z) = |z|^{-\frac{n-1}{2}} \sum_{\pm} e^{\pm i|z|} a_{\pm}(|z|)$, setting $z = r(x - y) \approx \lambda(x - y)$, kernel is essentially

$$\lambda^{\frac{n-1}{2}} |x - y|^{-\frac{n-1}{2}} \sum_{\pm} e^{\pm i\lambda|x-y|} a_{\pm}(\lambda, |x - y|)$$

Examples saturating $\lambda^{\delta(q)}$ growth, $\frac{2(n+1)}{n-1} \leq q \leq \infty$

Consider f_λ with $\widehat{f}_\lambda$ the indicator of $A_\lambda = \{\lambda \leq |\xi| \leq \lambda + 1\}$



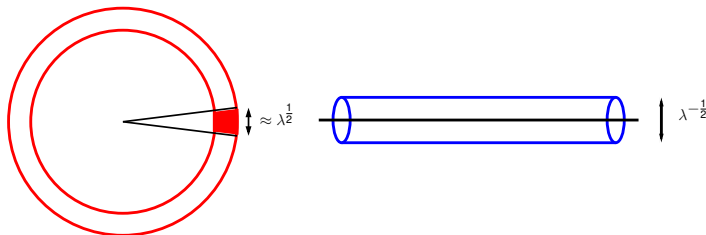
Previous slide: $|f_\lambda(x)| \approx \lambda^{n-1}(1 + \lambda|x|)^{-\frac{n-1}{2}}$, so for $q > \frac{2n}{n-1}$

$$\frac{\|f_\lambda\|_{L^q}}{\|f_\lambda\|_{L^2}} \approx \frac{\lambda^{n-1}}{(\text{Vol}(A_\lambda))^{\frac{1}{2}}} \left(\int (1 + \lambda|x|)^{-\frac{(n-1)q}{2}} dx \right)^{\frac{1}{q}} \approx \lambda^{\frac{n-1}{2} - \frac{n}{q}}$$

Examples saturating $\lambda^{\delta(q)}$ growth, $2 \leq q \leq \frac{2(n+1)}{n-1}$

Consider g_λ with $\widehat{g_\lambda}$ a bump function on

$$\Gamma_\lambda = A_\lambda \cap \left\{ \left| \frac{\xi}{|\xi|} - \mathbf{e}_n \right| \lesssim \lambda^{-\frac{1}{2}} \right\}$$



g_λ constant on the tube $\mathcal{T}_\lambda = \{|x'| \leq \lambda^{-\frac{1}{2}}, |x_n| \leq 1\}$, get

$$\frac{\|g_\lambda\|_{L^q}}{\|g_\lambda\|_{L^2}} \approx (\text{Vol}(\mathcal{T}_\lambda))^{\frac{1}{q} - \frac{1}{2}} \approx \lambda^{\frac{n-1}{2}(\frac{1}{2} - \frac{1}{q})}$$

Examples saturating $\lambda^{\delta(q)}$ growth, $q = \frac{2(n+1)}{n-1}$

Digression:

Consider $h_{\lambda,\theta}$ with $\widehat{h_{\lambda,\theta}}$ a bump function adapted to

$$\Gamma_{\lambda,\theta} = A_\lambda \cap \left\{ \left| \frac{\xi}{|\xi|} - e_n \right| \lesssim \theta \right\}$$

For any $\theta \in [\lambda^{-\frac{1}{2}}, 1]$,

$$\frac{\|h_{\lambda,\theta}\|_{L^{\frac{2(n+1)}{n-1}}}}{\|h_{\lambda,\theta}\|_{L^2}} \approx \lambda^{\frac{n-1}{2(n+1)}}$$

- More generally, on a compact manifold M , main contribution of the Schwartz kernel is

$$\lambda^{\frac{n-1}{2}} (d_g(x, y))^{-\frac{n-1}{2}} \sum_{\pm} e^{\pm i\lambda d_g(x, y)} a_{\pm, \lambda}(x, y)$$

with $\text{supp}(a_{\pm, \lambda}(\cdot, \cdot)) \subset \{\varepsilon \leq d_g(x, y) \leq 2\varepsilon\}$

- In coordinates, $d_g(x, y)$ satisfies the $n \times n$ Carleson-Sjölin condition, i.e. many of the salient features as $|x - y|$

- Sogge: can find families of clusters f_λ, g_λ on M with same behavior as above

$$|f_\lambda(x)| \approx \lambda^{n-1} (1 + \lambda d_g(x, x_0))^{-\frac{n-1}{2}} \quad \text{near } x_0 \in M$$

and hence $\|f_\lambda\|_{L^q(M)} / \|f_\lambda\|_{L^2(M)} \approx \lambda^{\frac{n-1}{2} - \frac{n}{q}}$

- On $\mathbb{S}^n$, these are zonal harmonics (exact eigenfunctions)

- $g_\lambda(x)$ concentrated in a $\lambda^{-\frac{1}{2}}$ -neighborhood of a geodesic segment γ of unit length

$$\mathcal{T}_{\lambda^{-1/2}}(\gamma) = \{x : d_g(x, \gamma) \lesssim \lambda^{-\frac{1}{2}}\}$$

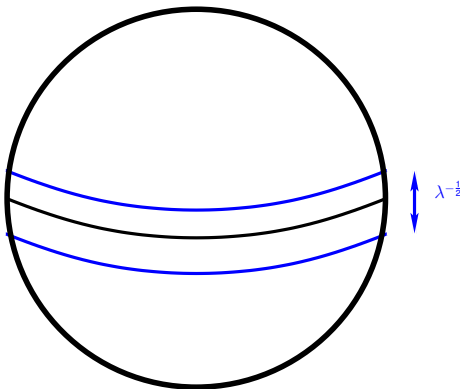
As before, this gives

$$\frac{\|g_\lambda\|_{L^q(M)}}{\|g_\lambda\|_{L^2(M)}} \approx (\text{Vol}(\mathcal{T}_{\lambda^{-1/2}}(\gamma)))^{\frac{1}{q}-\frac{1}{2}} \approx \lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})}$$

- On $\mathbb{S}^n$, these are the sectoral harmonics

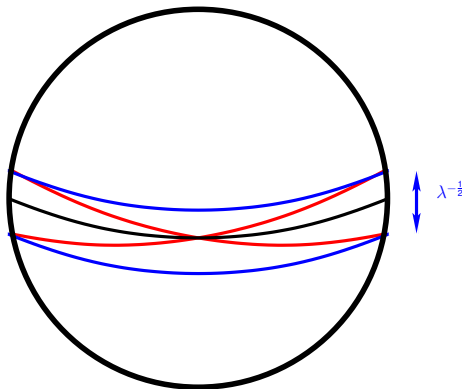
Sectoral harmonics on $\mathbb{S}^n$

- Mass of g_λ concentrated in a $\lambda^{-\frac{1}{2}}$ neighborhood of the equator



Sectoral harmonics on $\mathbb{S}^n$

- Equator is a stable elliptic orbit



$\mathbb{S}^n$ is too special

No such elliptic orbits on the torus: e.g. image of lines with irrational slope under the quotient map are dense

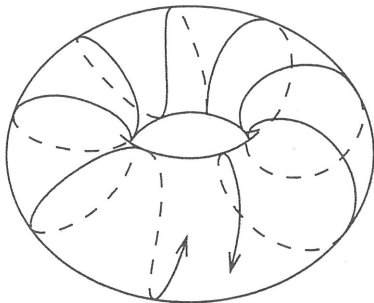


Image from J. Lee, *Introduction to Smooth Manifolds*

$\mathbb{S}^n$ is *too* special

Main question for general (M, g) : When can the $\mathcal{O}(\lambda^{\delta(q)})$ bounds be improved?

- Zygmund: If $\Delta e_\lambda = \lambda^2 e_\lambda$ on $\mathbb{T}^2$ is L^2 normalized, then $\|e_\lambda\|_{L^4(\mathbb{T}^2)} \lesssim 1$
- Bourgain, Bourgain-Demeter: improvements on exponent $\|e_\lambda\|_{L^q(\mathbb{T}^n)} \lesssim \lambda^{\sigma(q)}$, $\sigma(q) < \delta(q)$
- Iwaniec-Sarnak: power improvements in arithmetic cases
- Sogge-Zelditch: L^∞ norms of normalized efncs are $o(\lambda^{\frac{n-1}{2}})$ if set of closed loops forms a set of measure zero on cosphere bundle
- Bérard, Hassell-Tacy: Further improvements for nonpositively curved manifolds for $\frac{2(n+1)}{n-1} < q \leq \infty$

$\mathbb{S}^n$ is *too* special

Main question for general (M, g) : When can the $\mathcal{O}(\lambda^{\delta(q)})$ bounds be improved?

- Zygmund: If $\Delta e_\lambda = \lambda^2 e_\lambda$ on $\mathbb{T}^2$ is L^2 normalized, then $\|e_\lambda\|_{L^4(\mathbb{T}^2)} \lesssim 1$
- Bourgain, Bourgain-Demeter: improvements on exponent $\|e_\lambda\|_{L^q(\mathbb{T}^n)} \lesssim \lambda^{\sigma(q)}$, $\sigma(q) < \delta(q)$
- Iwaniec-Sarnak: power improvements in arithmetic cases
- Sogge-Zelditch: L^∞ norms of normalized efcs are $o(\lambda^{\frac{n-1}{2}})$ if set of closed loops forms a set of measure zero on cosphere bundle
- Bérard, Hassell-Tacy: Further improvements for nonpositively curved manifolds for $\frac{2(n+1)}{n-1} < q \leq \infty$

Main question for general (M, g) : When can the $\mathcal{O}(\lambda^{\delta(q)})$ bounds be improved?

- Zygmund: If $\Delta e_\lambda = \lambda^2 e_\lambda$ on $\mathbb{T}^2$ is L^2 normalized, then $\|e_\lambda\|_{L^4(\mathbb{T}^2)} \lesssim 1$
- Bourgain, Bourgain-Demeter: improvements on exponent $\|e_\lambda\|_{L^q(\mathbb{T}^n)} \lesssim \lambda^{\sigma(q)}$, $\sigma(q) < \delta(q)$
- Iwaniec-Sarnak: power improvements in arithmetic cases
- Sogge-Zelditch: L^∞ norms of normalized efncs are $o(\lambda^{\frac{n-1}{2}})$ if set of closed loops forms a set of measure zero on cosphere bundle
- Bérard, Hassell-Tacy: Further improvements for nonpositively curved manifolds for $\frac{2(n+1)}{n-1} < q \leq \infty$

Avenue for improvements when $2 < q < \frac{2(n+1)}{n-1}$

Theorem

Let Π denote the set of unit length geodesics on M . There exists $2 < q < \frac{2(n+1)}{n-1}$ such that if $\varepsilon > 0$ is any positive number and $\Delta_g e_\lambda = \lambda^2 e_\lambda$,

$$\|e_\lambda\|_{L^q(M)}^q \leq \varepsilon \lambda^{q\delta(q)} \|e_\lambda\|_{L^2(M)}^q + C_\varepsilon \lambda^{q\delta(q)} \|e_\lambda\|_{L^2(M)}^2 \sup_{\gamma \in \Pi} \left(\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda(z)|^2 dz \right)^{\frac{q-2}{2}},$$

- Sogge: $\dim(M) = 2, q = 4$
- MDB-Sogge: $\dim(M) \geq 3, \frac{2(n+2)}{n} < q < \frac{2(n+1)}{n-1}$

Avenue for improvements when $2 < q < \frac{2(n+1)}{n-1}$

Theorem

Let Π denote the set of unit length geodesics on M . There exists $2 < q < \frac{2(n+1)}{n-1}$ such that if $\varepsilon > 0$ is any positive number and $\Delta_g e_\lambda = \lambda^2 e_\lambda$,

$$\|e_\lambda\|_{L^q(M)}^q \leq \varepsilon \lambda^{q\delta(q)} \|e_\lambda\|_{L^2(M)}^q + C_\varepsilon \lambda^{q\delta(q)} \|e_\lambda\|_{L^2(M)}^2 \sup_{\gamma \in \Pi} \left(\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda(z)|^2 dz \right)^{\frac{q-2}{2}},$$

- Sogge: $\dim(M) = 2, q = 4$
- MDB-Sogge: $\dim(M) \geq 3, \frac{2(n+2)}{n} < q < \frac{2(n+1)}{n-1}$

Avenue for improvements when $2 < q < \frac{2(n+1)}{n-1}$

Theorem

Let Π denote the set of unit length geodesics on M . There exists $2 < q < \frac{2(n+1)}{n-1}$ such that if $\varepsilon > 0$ is any positive number and $\Delta_g e_\lambda = \lambda^2 e_\lambda$,

$$\|e_\lambda\|_{L^q(M)}^q \leq \varepsilon \lambda^{q\delta(q)} \|e_\lambda\|_{L^2(M)}^q + C_\varepsilon \lambda^{q\delta(q)} \|e_\lambda\|_{L^2(M)}^2 \sup_{\gamma \in \Pi} \left(\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda(z)|^2 dz \right)^{\frac{q-2}{2}},$$

- Sogge: $\dim(M) = 2$, $q = 4$
- MDB-Sogge: $\dim(M) \geq 3$, $\frac{2(n+2)}{n} < q < \frac{2(n+1)}{n-1}$

Avenue for improvements when $2 < q < \frac{2(n+1)}{n-1}$

Hence if $\|e_\lambda\|_{L^2(M)} = 1$,

$$\lambda^{-q\delta(q)} \|e_\lambda\|_{L^q(M)}^q \leq \varepsilon + C_\varepsilon \sup_{\gamma \in \Pi} \left(\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda(z)|^2 dz \right)^{\frac{q-2}{2}}$$

This gives sufficient conditions to see that $\|e_\lambda\|_{L^q(M)} = o(\lambda^{\delta(q)})$:
Given $\varepsilon > 0$ show that there exists $\Lambda(\varepsilon)$ such that

$$\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda(z)|^2 dz \leq \varepsilon \quad \text{for every } \lambda \geq \Lambda(\varepsilon)$$

Corollary

The following are equivalent for any subsequence of L^2 -normalized eigenfunctions $\{e_{\lambda_{j_k}}\}_{k=1}^\infty$:

$$\limsup_{k \rightarrow \infty} \left(\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda_{j_k}^{-1/2}(\gamma)}} |e_{\lambda_{j_k}}(z)|^2 dz \right) = 0$$
$$\limsup_{k \rightarrow \infty} \lambda_{j_k}^{-\delta(q)} \|e_{\lambda_{j_k}}\|_{L^q(M)} = 0 \quad \text{for } 2 < q < \frac{2(n+1)}{n-1}.$$

Consequence of Snirelman, Zelditch, Colin de Verdière

Suppose the sectional curvatures of M are negative or more generally, the geodesic flow is ergodic. Then there exists a density one subsequence of eigenvalues λ_{j_k} such that

$$\lim_{k \rightarrow \infty} \int_E |e_{\lambda_{j_k}}(z)|^2 dz \rightarrow \frac{\text{Vol}_g(E)}{\text{Vol}_g(M)}$$

for any Jordan measurable E .

- cp. sectoral harmonics, if the distance from E to the equator is positive then LHS tends to 0

Snirelman, Zelditch, Colin de Verdière

Suppose the sectional curvatures of M are negative or more generally, the geodesic flow is ergodic. Then there exists a density one subsequence of eigenvalues λ_{j_k} such that $|e_{\lambda_{j_k}}(z)|^2 dz \rightarrow dz$. Moreover, for $a(x, \xi) \in C_c^\infty(S^*M)$

$$\int \left((\text{Op } a) e_{\lambda_{j_k}} \right) (z) \overline{e_{\lambda_{j_k}}(z)} dz \rightarrow \int_{S^*M} a(x, \xi) dx d\xi$$

- cp. sectoral harmonics, $|e_{\lambda_{j_k}}(z)|^2 dz$ tends to arclength measure on the equator

Nonpositive Curvature

Theorem (MDB-Sogge)

Suppose $\dim(M) \geq 3$, (M, g) has everywhere nonpositive sectional curvatures, and $\{e_{\lambda_j}\}$ is any orthonormal sequence

$$\limsup_{j \rightarrow \infty} \left(\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda_j^{-1/2}}(\gamma)} |e_{\lambda_j}(z)|^2 dz \right) = 0$$

and hence

$$\limsup_{j \rightarrow \infty} \lambda_j^{-\delta(q)} \|e_{\lambda_j}\|_{L^q(M)} = 0 \quad \text{for any } 2 < q < \frac{2(n+1)}{n-1}.$$

- Builds on results of Sogge-Zelditch ($\dim(M) = 2$) and Chen-Sogge (geodesic restrictions)

Avenue for improvements when $2 < q < \frac{2(n+1)}{n-1}$

Theorem

Let Π denote the set of unit length geodesics on M . There exists $2 < q < \frac{2(n+1)}{n-1}$ such that if $\varepsilon > 0$ is any positive number and $\Delta_g e_\lambda = \lambda^2 e_\lambda$,

$$\|e_\lambda\|_{L^q(M)}^q \leq \varepsilon \lambda^{q\delta(q)} \|e_\lambda\|_{L^2(M)}^q + C_\varepsilon \lambda^{q\delta(q)} \|e_\lambda\|_{L^2(M)}^2 \sup_{\gamma \in \Pi} \left(\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda(z)|^2 dz \right)^{\frac{q-2}{2}},$$

- Sogge: $\dim(M) = 2$, $q = 4$
- MDB-Sogge: $\dim(M) \geq 3$, $\frac{2(n+2)}{n} < q < \frac{2(n+1)}{n-1}$

Corollary

The following are equivalent for any subsequence of L^2 -normalized eigenfunctions $\{e_{\lambda_{j_k}}\}_{k=1}^{\infty}$:

$$\limsup_{k \rightarrow \infty} \left(\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda_{j_k}^{-1/2}(\gamma)}} |e_{\lambda_{j_k}}(z)|^2 dz \right) = 0$$
$$\limsup_{k \rightarrow \infty} \lambda_{j_k}^{-\delta(q)} \|e_{\lambda_{j_k}}\|_{L^q(M)} = 0 \quad \text{for } 2 < q < \frac{2(n+1)}{n-1}.$$

Hörmander's theorem

Theorem (Hörmander)

Consider R_λ defined by

$$R_\lambda f(z) = \int_{\mathbb{R}^n} e^{i\lambda\phi(z,w)} b(z,w) f(w) dw$$

where $b \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and $\phi \in C^\infty$ in a nbd. of $\text{supp}(b)$ satisfies

$$\det \left(\nabla_{zw}^2 \phi(z,w) \right) = \det \left(\partial_{z_i w_j}^2 \phi(z,w) \right) \neq 0.$$

Then

$$\|R_\lambda f\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}$$

Hörmander's theorem

Corollary (Hörmander)

Consider R_λ defined by

$$R_\lambda f(z) = \int_{\mathbb{R}^n} e^{i\lambda\phi(z,w)} b(z,w) f(w) dw$$

where $b \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and $\phi \in C^\infty$ in a nbd. of $\text{supp}(b)$ satisfies $\det(\nabla_{zw}^2 \phi) \neq 0$. Then for $2 \leq p \leq \infty$

$$\|R_\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim \lambda^{-\frac{n}{p}} \|f\|_{L^{p'}(\mathbb{R}^n)}$$

- Result of interpolation with $\|R_\lambda f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}$.
- Taking $\phi = z \cdot w$, generalizes Plancherel/Hausdorff-Young

Carleson-Sjölin integral operators

Now consider the mapping from functions in $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$,
 $b \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$:

$$S_\lambda h(x, s) = \int_{\mathbb{R}^{n-1}} e^{i\lambda\phi(x, s, y)} b(x, s, y) h(y) dy, \quad x, y \in \mathbb{R}^{n-1}, s \in \mathbb{R}$$

$\phi(x, s, y)$ satisfies a Carleson-Sjölin type condition:

- Mixed Hessian $\nabla_{xy}^2 \phi(x, s, y)$ is invertible
- If $\theta \in \mathbb{S}^{n-1}$ is such that $\nabla_y \langle \nabla_{(x,s)} \phi, \theta \rangle = 0$, then

$\nabla_{yy}^2 \langle \nabla_{(x,s)} \phi, \theta \rangle$ has eigenvalues of the same sign.

Model phases

Equivalent: for fixed (x_0, s_0) , the following $n - 1$ dim'l submanifold has positive curvatures

$$\{d_{(x,s)}\phi(x_0, s_0, y) : y \in \text{supp}(b(x_0, s_0, \cdot))\} \subset T_{(x_0, s_0)}^* \mathbb{R}^n$$

Examples:

- Fourier extension phases for the paraboloid

$$\phi(x, s, y) = x \cdot y + \frac{1}{2}s|y|^2$$

- Fourier extension phases for the sphere

$$\phi(x, s, y) = x \cdot y + s\sqrt{1 - |y|^2}$$

(provided $\text{supp}(b(x, s, \cdot)) \subset B_1(0)$)

More examples:

Fix t and assume $|t - s| \approx \varepsilon > 0$, $|x - y| \ll \varepsilon$ in $\text{supp}(b)$

- Restricted Bochner-Riesz phases,

$$\phi(x, s, y) = \sqrt{(x - y)^2 + (s - t)^2} = |(x - y, s - t)|$$

- Restricted Riemannian distance functions,

$$\phi(x, s, y) = d_g((x, s), (y, t))$$

Carleson-Sjölin integral operators

After coordinate change, can assume

$$\phi(x, s, y) = x \cdot y + \frac{1}{2}s|y|^2 + \mathcal{O}\left((|x| + |s|)^2|y|^2 + (|x| + |s|)|y|^3\right)$$

Theorem (Hörmander, Stein)

Suppose ϕ is a Carleson-Sjölin phase and for $(x, s) \in \mathbb{R}^n, y \in \mathbb{R}^{n-1}$, define

$$S_\lambda h(x, s) = \int_{\mathbb{R}^{n-1}} e^{i\lambda\phi(x, s, y)} b(x, s, y) h(y) dy.$$

Then for $\frac{2(n+1)}{n-1} \leq q \leq \infty$,

$$\|S_\lambda h\|_{L^q(\mathbb{R}^n)} \lesssim \lambda^{-\frac{n}{q}} \|h\|_{L^2(\mathbb{R}^{n-1})}$$

Proof of Sogge's χ_λ bounds

$$\|\chi_\lambda f\|_{L^q(M)} \lesssim \lambda^{\frac{n-1}{2} - \frac{n}{q}} \|f\|_{L^2(M)}$$

- Recall: Schwartz kernel of χ_λ is supported in $\{\varepsilon \leq d_g(x, y) \leq 2\varepsilon\}$

$$\lambda^{\frac{n-1}{2}} (d_g(x, y))^{-\frac{n-1}{2}} \sum_{\pm} e^{\pm i\lambda d_g(x, y)} a_{\pm, \lambda}(x, y)$$

- After a partition of unity, localize χ_λ so that $|t - s| \approx \varepsilon$ and $|x - y| \ll \varepsilon$, can replace $\chi_\lambda e_\lambda$ with $\lambda^{\frac{n-1}{2}} S_\lambda$

$$S_\lambda h(x, s) = \int_{\mathbb{R}^{n-1}} e^{i\lambda \phi(x, s, y)} b(x, s, y) h(y) dy, \quad x, y \in \mathbb{R}^{n-1}, s \in \mathbb{R}$$

$$h(y) = f(y, t), \text{ some fixed } t, \text{ and } \phi(x, s, y) = d_g((x, s), (y, t))$$

Proof of Keakeya-Nikodym theorem

$$\|e_\lambda\|_{L^q(M)}^q \leq \varepsilon \lambda^{q\delta(q)} \|e_\lambda\|_{L^2(M)}^q + C_\varepsilon \lambda^{q\delta(q)} \|e_\lambda\|_{L^2(M)}^2 \sup_{\gamma \in \Pi} \left(\int_{\mathcal{T}_{\lambda^{-1/2}(\gamma)}} |e_\lambda(z)|^2 dz \right)^{\frac{q-2}{2}},$$

- Using $\chi_\lambda e_\lambda = e_\lambda$, want to estimate

$$\|e_\lambda\|_{L^q}^q = \int |e_\lambda|^q = \int |\chi_\lambda e_\lambda|^2 |e_\lambda|^{q-2}$$

- Left to control $\int |(S_\lambda h)^2| |e_\lambda|^{q-2}$, this time

$$h(y) = e_\lambda(y, t)$$

Squaring $S_\lambda h$

- Recall: want to estimate $\int |(S_\lambda h)^2| |e_\lambda|^{q-2}$
- Consider $(S_\lambda h(x, s))^2$ given by

$$\int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} e^{i\lambda(\phi(x,s,y)+\phi(x,s,y'))} b_\lambda(x, s, y) b_\lambda(x, s, y') h(y) h(y') dy dy'$$

- For N large, split integral into “diagonal”, “off-diagonal” terms $|y - y'| \lesssim N\lambda^{-\frac{1}{2}}$, $|y - y'| \gtrsim N\lambda^{-\frac{1}{2}}$
- Diagonal term: Partition the y -support into cubes Q_ν with sidelength $\lambda^{-\frac{1}{2}}$, can dominate by

$$N^{n-1} \sum_\nu |S_\lambda(h \mathbf{1}_{Q_\nu})(x, s)|^2$$

- Later, we will see that off-diagonal term is better behaved

Squaring $S_\lambda h$

- Recall: want to estimate $\int |(S_\lambda h)^2| |e_\lambda|^{q-2}$
- Consider $(S_\lambda h(x, s))^2$ given by

$$\int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} e^{i\lambda(\phi(x, s, y) + \phi(x, s, y'))} b_\lambda(x, s, y) b_\lambda(x, s, y') h(y) h(y') dy dy'$$

- For N large, split integral into “diagonal”, “off-diagonal” terms $|y - y'| \lesssim N\lambda^{-\frac{1}{2}}$, $|y - y'| \gtrsim N\lambda^{-\frac{1}{2}}$
- Diagonal term: Partition the y -support into cubes Q_ν with sidelength $\lambda^{-\frac{1}{2}}$, can dominate by

$$N^{n-1} \sum_{\nu} |S_\lambda(h \mathbf{1}_{Q_\nu})(x, s)|^2$$

- Later, we will see that off-diagonal term is better behaved

On diagonal term

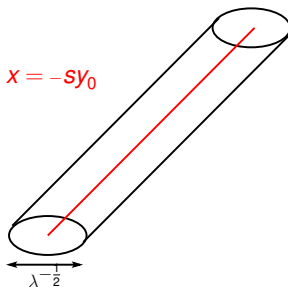
- If we had $\phi = x \cdot y + \frac{1}{2}s|y|^2$

$$\int_{\mathbb{R}^{n-1}} e^{i\lambda(x \cdot y + \frac{1}{2}s|y|^2)} b_\lambda(x, s, y) \mathbf{1}_{Q_\nu}(y) h(y) dy \quad x, y \in \mathbb{R}^{n-1}, s \in \mathbb{R}$$

- Let $y_0 = \text{center of } Q_\nu$,

$$\lambda \left(x \cdot y + \frac{1}{2}s|y|^2 \right) = \lambda x \cdot y_0 + \lambda(y - y_0) \cdot (x + sy_0) + \mathcal{O}(\lambda|y - y_0|^2)$$

- Oscillations negligible when $|x + sy_0| \lesssim \lambda^{-\frac{1}{2}}$, rapid outside



- Rapid oscillations outside the tube allows us to dominate

$$N^{n-1} \sum_{\nu} \int |S_{\lambda}(h\mathbf{1}_{Q_{\nu}})|^2 |e_{\lambda}|^{q-2} \lesssim$$

$$N^{n-1} \lambda^{q\delta(q)-(n-1)} \|h\|_{L^2(M)}^2 \sup_{\gamma \in \Pi} \left(\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_{\lambda}(z)|^2 dz \right)^{\frac{q-2}{2}}$$

- Morally,

$$\int |S_{\lambda}(h\mathbf{1}_{Q_{\nu}})|^2 |e_{\lambda}|^{q-2} \approx \|S_{\lambda}(h\mathbf{1}_{Q_{\nu}})\|_{L^{\infty}}^2 \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma(Q_{\nu}))} |e_{\lambda}|^{q-2},$$

so exploit gains from $L^1 \rightarrow L^{\infty}$ bounds and Hölder

$$\int_{|y-y'| \gtrsim N\lambda^{-\frac{1}{2}}} e^{i\lambda(\phi(x,s,y)+\phi(x,s,y'))} b_\lambda(x,s,y) b_\lambda(x,s,y') h(y) h(y') dy dy'$$

- Call the contribution here $B_\lambda(h, h)$ and use Hölder to reduce to $L^{\frac{q}{2}}$ estimates

$$\int |B_\lambda(h, h)|^2 |e_\lambda|^{q-2} \lesssim \|B_\lambda(h, h)\|_{L^{\frac{q}{2}}} \|e_\lambda\|_{L^q}^{q-2}$$

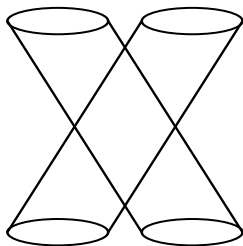
- Main estimate: for some $k > 0$ and $\frac{2(n+2)}{n} < q < \frac{2(n+1)}{n-1}$

$$\|B_\lambda(h, h)\|_{L^{\frac{q}{2}}} \lesssim N^{-k} \lambda^{2\delta(q)-(n-1)} \|h\|_{L^2}^2$$

so take $\varepsilon \approx N^{-k}$

Off diagonal term

Expect bilinear interactions to satisfy better estimates



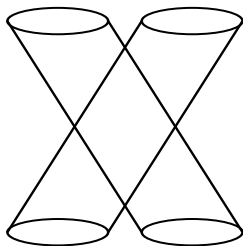
Tao-Vargas-Vega: Take a Whitney decomposition of

$$\{(y, y') : |y - y'| \geq N\lambda^{-\frac{1}{2}}\} = \bigcup_{\epsilon_0 \geq 2^{-j} \gtrsim N\lambda^{-\frac{1}{2}}} \bigcup_{d(Q_{\nu}^j, Q_{\nu'}^j) \approx 2^{-j}} Q_{\nu}^j \times Q_{\nu'}^j$$

Q_{ν}^j is a cube of sidelength 2^{-j}

Off diagonal term

Expect bilinear interactions to satisfy better estimates



Tao-Vargas-Vega: Take a Whitney decomposition of

$$\{(y, y') : |y - y'| \geq N\lambda^{-\frac{1}{2}}\} = \bigcup_{\varepsilon_0 \geq 2^{-j} \gtrsim N\lambda^{-\frac{1}{2}}} \bigcup_{d(Q_\nu^j, Q_{\nu'}^j) \approx 2^{-j}} Q_\nu^j \times Q_{\nu'}^j$$

Q_ν^j is a cube of sidelength 2^{-j}

Off diagonal term

Left to consider

$$B_\lambda(\mathbf{1}_{Q_\nu^j} h, \mathbf{1}_{Q_{\nu'}^j} h)(x, s) = \int_{Q_\nu^j \times Q_{\nu'}^j} e^{i\lambda(\phi(x, s, y) + \phi(x, s, y'))} h(y) h(y') \tilde{b}_\lambda dy dy'$$

Refining the almost orthogonality arguments of
Tao-Vargas-Vega, S. Lee reduce matters to

$$\begin{aligned} & \|B_\lambda(\mathbf{1}_{Q_\nu^j} h, \mathbf{1}_{Q_{\nu'}^j} h)\|_{L^{\frac{q}{2}}(\mathbb{R}^n)} \\ & \lesssim \lambda^{-\frac{2n}{q}} 2^{j(\frac{2(n+1)}{q} - (n-1))} \|\mathbf{1}_{Q_\nu^j} h\|_{L^2(\mathbb{R}^{n-1})} \|\mathbf{1}_{Q_{\nu'}^j} h\|_{L^2(\mathbb{R}^{n-1})} \end{aligned}$$

Can check: Geometric series in j yields

$$\lambda^{-\frac{2n}{q}} \sum_{1 \leq j \leq \log_2(N^{-1} \lambda^{\frac{1}{2}})} 2^{j(\frac{2(n+1)}{q} - (n-1))} \lesssim N^{-k} \lambda^{2\delta(q) - (n-1)}$$

- Bilinear bounds for restriction problem: Hörmander, Tao-Vargas-Vega, Wolff, Tao
- S. Lee: bilinear bounds for “variable-coefficient” Carleson-Sjölin integrals above ($n \geq 3$)
- MDB-Sogge: Adapt “ ε -removal” lemma of Tao-Vargas to variable-coefficient setting, avoids losses in Lee’s bounds

Nonpositive Curvature

Theorem (MDB-Sogge)

Suppose $\dim(M) \geq 3$, (M, g) has everywhere nonpositive sectional curvatures, and $\{e_{\lambda_j}\}$ is any orthonormal sequence

$$\limsup_{j \rightarrow \infty} \left(\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda_j^{-1/2}}(\gamma)} |e_{\lambda_j}(z)|^2 dz \right) = 0$$

and hence

$$\limsup_{j \rightarrow \infty} \lambda_j^{-\delta(q)} \|e_{\lambda_j}\|_{L^q(M)} = 0 \quad \text{for any } 2 < q < \frac{2(n+1)}{n-1}.$$

- Builds on results of Sogge-Zelditch ($\dim(M) = 2$) and Chen-Sogge (geodesic restrictions)

Projecting onto smaller frequency windows

- Consider a geodesic segment γ of unit length, and claim that given $\varepsilon > 0$, there exists $\Lambda(\varepsilon)$

$$\int_{\mathcal{T}_{\lambda_j^{-1/2}}(\gamma)} |\mathbf{e}_{\lambda_j}|^2 \leq \varepsilon \quad \text{for } \lambda_j \geq \Lambda(\varepsilon)$$

- Let $\chi \in \mathcal{S}$ be as above, this time consider for some $T \gg 1$

$$\chi_{\lambda,T} = \chi(T(\sqrt{\Delta_g} - \lambda)) \implies \chi_{\lambda,T} \mathbf{e}_\lambda = \mathbf{e}_\lambda$$

(regularized projection onto $\lambda_j \in [\lambda - \frac{1}{T}, \lambda + \frac{1}{T}]$)

- Want to show there exists constants A and A_T such that

$$\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |\chi_{\lambda,T} f|^2 \leq \left(A T^{-\frac{1}{4}} + A_T \lambda^{-\frac{1}{8}} \right) \|f\|_{L^2(M)}$$

Projecting onto smaller frequency windows

- Consider a geodesic segment γ of unit length, and claim that given $\varepsilon > 0$, there exists $\Lambda(\varepsilon)$

$$\int_{\mathcal{T}_{\lambda_j^{-1/2}}(\gamma)} |\mathbf{e}_{\lambda_j}|^2 \leq \varepsilon \quad \text{for } \lambda_j \geq \Lambda(\varepsilon)$$

- Let $\chi \in \mathcal{S}$ be as above, this time consider for some $T \gg 1$

$$\chi_{\lambda,T} = \chi(T(\sqrt{\Delta_g} - \lambda)) \implies \chi_{\lambda,T} \mathbf{e}_\lambda = \mathbf{e}_\lambda$$

(regularized projection onto $\lambda_j \in [\lambda - \frac{1}{T}, \lambda + \frac{1}{T}]$)

- Want to show there exists constants A and A_T such that

$$\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |\chi_{\lambda,T} f|^2 \leq \left(A T^{-\frac{1}{4}} + A_T \lambda^{-\frac{1}{8}} \right) \|f\|_{L^2(M)}$$

Projecting onto smaller frequency windows

- Consider a geodesic segment γ of unit length, and claim that given $\varepsilon > 0$, there exists $\Lambda(\varepsilon)$

$$\int_{\mathcal{T}_{\lambda_j^{-1/2}}(\gamma)} |\mathbf{e}_{\lambda_j}|^2 \leq \varepsilon \quad \text{for } \lambda_j \geq \Lambda(\varepsilon)$$

- Let $\chi \in \mathcal{S}$ be as above, this time consider for some $T \gg 1$

$$\chi_{\lambda,T} = \chi(T(\sqrt{\Delta_g} - \lambda)) \implies \chi_{\lambda,T} \mathbf{e}_\lambda = \mathbf{e}_\lambda$$

(regularized projection onto $\lambda_j \in [\lambda - \frac{1}{T}, \lambda + \frac{1}{T}]$)

- Want to show there exists constants A and A_T such that

$$\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |\chi_{\lambda,T} f|^2 \leq \left(A T^{-\frac{1}{4}} + A_T \lambda^{-\frac{1}{8}} \right) \|f\|_{L^2(M)}$$

- Duality: suffices to show that if $m_{\lambda,T} = \chi^2(T(\sqrt{\Delta_g} - \lambda))$

$$\|m_{\lambda,T}f\|_{L^2(\mathcal{T}_{\lambda^{-1/2}}(\gamma))} \leq \left(CT^{-\frac{1}{2}} + C_T\lambda^{-\frac{1}{4}}\right) \|f\|_{L^2(\mathcal{T}_{\lambda^{-1/2}}(\gamma))}$$

- Up to negligible error

$$m_{\lambda,T} = \frac{1}{T} \int_{-T}^T e^{i\tau\lambda} \cos(\tau\sqrt{\Delta_g}) \widehat{\chi^2}(\tau/T) d\tau$$

- Main differences:
 - Can take $\text{supp}(\widehat{\chi^2}) \subset (-1, 1)$, but doesn't vanish near $\tau = 0$,
 - Normalizing factor of $1/T$

Lifting to the universal cover

- Cartan-Hadamard: Given $x \in M$, $\exp_x : T_x M \cong \mathbb{R}^n \rightarrow M$ is a covering map, Γ is the group of deck transformations,

$$\Gamma = \{ \alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n \mid \exp_x \circ \alpha = \exp_x \}$$

- Lifting g and $x, y \in M$ to the cover as $\tilde{g}$ and $\tilde{x}, \tilde{y} \in \mathbb{R}^n$,

$$\cos \left(\tau \sqrt{\Delta_g} \right) (x, y) = \sum_{\alpha \in \Gamma} \cos \left(\tau \sqrt{\Delta_{\tilde{g}}} \right) (\tilde{x}, \alpha(\tilde{y}))$$

Cosine transform on the cover

Main lemma (Sogge-Zelditch, Chen-Sogge, Sogge)

Up to acceptable error, for $d_g(\tilde{x}, \tilde{z}) \leq T$

$$\begin{aligned} \frac{1}{T} \int_{-T}^T e^{-i\tau\lambda} \cos\left(\tau\sqrt{\Delta_{\tilde{g}}}\right)(\tilde{x}, \tilde{z}) \widehat{\chi^2}(\tau/T) d\tau \\ \approx (d_{\tilde{g}}(\tilde{x}, \tilde{z}))^{-\frac{n-1}{2}} \frac{\lambda^{\frac{n-1}{2}}}{T} \sum_{\pm} a_{\pm}(\lambda, T, \tilde{x}, \tilde{z}) e^{\pm i\lambda d_{\tilde{g}}(\tilde{x}, \tilde{z})} \quad (1) \end{aligned}$$

- Uses stationary phase, Hadamard parametrix for $\cos\left(\tau\sqrt{\Delta_{\tilde{g}}}\right)$

Oscillatory and non-oscillatory terms

- Up to acceptable error, kernel of $m_{\lambda,T}$ is thus

$$\sum_{\alpha \in \Gamma, \pm} [d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))]^{-\frac{n-1}{2}} \frac{\lambda^{\frac{n-1}{2}}}{T} a_{\pm}(\lambda, T, \tilde{x}, \alpha(\tilde{y})) e^{\pm i\lambda d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))}$$

- Can restrict attention to $x, y \in \mathcal{T}_{\lambda^{-1/2}}(\gamma)$
- γ lifts to geodesic segment $\tilde{\gamma}$, have 2 cases:
 - $\alpha(\tilde{\gamma}), \tilde{\gamma}$ lie on the same geodesic: no oscillations, but only $\mathcal{O}(T)$ terms and good control over $(d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})))^{-\frac{n-1}{2}}$
 - Otherwise, there are $\mathcal{O}(e^{cT})$ such terms, but oscillations are nontrivial. Use Hörmander's theorem to yield a gain of $\lambda^{-\frac{1}{4}}$
- End result

$$\|m_{\lambda,T} f\|_{L^2(\mathcal{T}_{\lambda^{-1/2}}(\gamma))} \leq \left(C T^{-\frac{1}{2}} + C_T \lambda^{-\frac{1}{4}} \right) \|f\|_{L^2(\mathcal{T}_{\lambda^{-1/2}}(\gamma))}$$

Oscillatory and non-oscillatory terms

- Up to acceptable error, kernel of $m_{\lambda,T}$ is thus

$$\sum_{\alpha \in \Gamma, \pm} [d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))]^{-\frac{n-1}{2}} \frac{\lambda^{\frac{n-1}{2}}}{T} a_{\pm}(\lambda, T, \tilde{x}, \alpha(\tilde{y})) e^{\pm i\lambda d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))}$$

- Can restrict attention to $x, y \in \mathcal{T}_{\lambda^{-1/2}}(\gamma)$
- γ lifts to geodesic segment $\tilde{\gamma}$, have 2 cases:
 - $\alpha(\tilde{\gamma}), \tilde{\gamma}$ lie on the same geodesic: no oscillations, but only $\mathcal{O}(T)$ terms and good control over $(d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})))^{-\frac{n-1}{2}}$
 - Otherwise, there are $\mathcal{O}(e^{cT})$ such terms, but oscillations are nontrivial. Use Hörmander's theorem to yield a gain of $\lambda^{-\frac{1}{4}}$
- End result

$$\|m_{\lambda,T}f\|_{L^2(\mathcal{T}_{\lambda^{-1/2}}(\gamma))} \leq \left(CT^{-\frac{1}{2}} + C_T\lambda^{-\frac{1}{4}}\right) \|f\|_{L^2(\mathcal{T}_{\lambda^{-1/2}}(\gamma))}$$