MATH 240; FINAL EXAM, 150 points, December 16, 2002 (R.A.Brualdi)

TOTAL SCORE:

Name: **TA (circle): Matthew Petro Dilip Raghavan**

Disc. (circle) TUES. THURS. TIME:

Part I (12 multiple choice, 84 points, 7 points each)
Part II (5 questions, 66 points)
Part III (1 bonus question, 10 points)

Part I. [84 points: 12 questions, 7 points each] Multiple Choice Questions.

1. Let $f(x_1, x_2, \ldots, x_n)$ be a Boolean function of n Boolean variables $x_1, x_2, \ldots, x_n$. The **number of elements in the domain of** f, the **number of elements in the co-domain of** f, and the **number of such Boolean functions** f are given, respectively, by:

(a) n, 2, and 2^n.

(b) 2, 2, and 2^{2n}.

(c) 2^n, 2, and 2^{2n}.

(d) 2^n, 2, and 2^{2^n}.

2. The **value of the sum** $\sum_{i=1}^{n} 5 \cdot 7^i$ equals

(a) $\frac{5}{6}(7^{n+1} - 1)$

(b) $\frac{35}{6}(7^n - 1)$.

(c) $5 + \frac{7^{n+1}-1}{6}$

(d) none of the above.

3. Which of the following is **NOT** correct?

(a) If p is a prime and a and b are integers such that $p|ab$, then $p|a$ or $p|b$.

(b) $100n^3 + 4n^2 = O(n^4)$.

(c) 1111001 (base 2) equals 171 (base 8).

(d) 21 has a multiplicative inverse mod 63.

4. Identify one of the examples below as **NOT** being an example of a recursive definition, calculation, construction, ... or circle (d) that all are recursive.

(a) The sequence $a_0, a_1, a_2, \cdots, a_n, \cdots$ is defined by $a_0 = 3, a_1 = 24$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

(b) The set A of all positive integers which are congruent to 6 mod 11 is given by: $6 \in A$, and if $n \in A$, then $n + 11 \in A$.

(c) Let a and b are nonnegative integers (not both zero) with $a \leq b$. Then GCD$\{a, b\}$ is given by: If $a = 0$, then GCD$\{a, b\} := b$. Else GCD$\{a, b\}$ =GCD$\{b \text{ mod } a, a)$.

(d) All are recursive.

5. There are 40 different cards each with one of the "ranks" $1, 2, \ldots, 10$ and one of the "colors" hearts, diamonds, clubs, and spades. (i.e the non-picture cards of an ordinary deck of cards). The **number of possible different hands of 5 cards** (i.e. a set of 5 cards from the 40 cards) which are "**full houses**" (i.e. three cards of one rank and two cards of a different rank) equals:

(a) C(10,3)C(10,2)

(b) C(10,2)C(4,3)C(4,2)

(c) P(10,2)C(4,3)C(4,2)

(d) none of the above

6. Identify the **incorrect statement**:

(a) If E and F are two events, the $\Pr(E \cup F) = \Pr(E)+ \Pr(F)- \Pr(E \cap F)$

(b) If A and B are sets, then $\overline{\overline{A} \cap B} = A \cup \overline{B}$

(c) The number of solutions in nonnegative integers of $x_1 + x_2 + \cdots + x_n = r$ equals $C(n + r - 1, n)$

(d) The solution of the recurrence relation $h_n = 2h_{n-1} + 1 (n \geq 2)$ with initial condition $h_1 = 1$ is $h_n = 2^n - 1$.

7. Consider a 2 by n board of unit squares and "pieces" which are either **dominoes** (1 by 2 and 2 by 1 pieces consisting of 2 squares in a row) or **squared-dominoes** (2 by 2 pieces) Let a_n denote the number of different ways to perfectly tile a 2 by n board with dominoes and squared-dominoes. Thus $a_1 = 1$ and $a_2 = 3$. The **recurrence relation** satisfied by a_n is:

(a) $a_n = a_{n-1} + a_{n-2} + 1$

(b) $a_n = a_{n-1} + 2a_{n-2}$

(c) $a_n = 2a_{n-1} + a_{n-2}$

(d) none of the above.

8. Let R and S be two relations on a finite set A of integers with n elements. Identify the statements below which are **NOT** correct. **In this problem you might circle more than one answer.**

(a) If R and S are transitive, then $R \cap S$ is also transitive.

(b) If R and S are transitive, then $R \cup S$ is also transitive.

(c) The transitive closure of R is $R \cup R^2 \cup R^3 \cup \cdots \cup R^n$

(d) If R is the relation *congruent modulo* 20 and S is the relation *congruent modulo* 8, then $R \cap S$ is the relation *congruent modulo* 160.

9. Let $(S, \preceq)$ be a partially ordered finite set. Which of the following is **always correct**.

(a) $(S, \preceq)$ is a totally ordered set.

(b) Each pair of distinct elements of S has an upper bound.

(c) S contains at least one minimal element.

(d) The (Hasse) diagram of $(S, \preceq)$ when viewed as a graph is a tree.

10. Identify a property below which is **NOT** a property of every tree T with $n > 1$ vertices, or else say (e) that all are properties of trees.

(a) T has $n - 1$ edges.

(b) $\chi(T) = 2$

(c) Inserting a new edge into a tree (but no new vertices) creates exactly one simple cycle.

(d) If we adjoin a new vertex x to T and insert a new edge joining x to a vertex y of T, then the result is also a tree.

(e) Properties (a), (b), (c), and (d) are properties of every tree with $n > 1$ vertices

11. Identify the proposition below which is **NOT** a tautology or circle (e) all are tautologies.

(a) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

(b) $(p \to q) \leftrightarrow (\neg p \vee q)$

(c) $\neg(\forall x\ P(x)) \leftrightarrow \exists\ x \neg P(x)$

(d) $(\forall x \exists y\ P(x, y)) \leftrightarrow (\exists\ y \forall x P(x, y))$

(e) All are tautologies.

12. Identify which of the following is **NOT** an identity or say that (e) all are identities.

(a) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, $(1 \le k \le n)$

(b) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ $(n \ge 1)$

(c) $(x + 3y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} 3^k y^k$.

(d) $1 + 2 + 3 + \cdots + (n - 1) = \binom{n}{2}$, $(n \ge 2)$

(e) All are identities.

Part II. (**5 questions; 66 points**)

13. [12 points] Use **Warshall's Algorithm** to construct the matrices W_1, W_2, W_3 to **PARTLY** determine the transitive closure of the relation R on a set of 5 elements, whose matrix reprsentation is given by:

$$W_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \end{bmatrix}$$

$$W_2 = \begin{bmatrix} \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \end{bmatrix}$$

$$W_3 = \begin{bmatrix} \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \end{bmatrix}$$

14. [16 points]

(a) What are the three (defining) properties of an equivalence relation R on a set A?

(b) Consider the partition of $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ into the three sets $A_1 = \{1, 4, 6, 8\}$, $\{2, 5, 9\}$, and $A_2 = \{3, 7\}$ and the equivalence relation R on A determined by it. Determine the equivalence class $[5]_R$

(c) By using partitions, give all possible equivalence relations on the set $\{1, 2, 3\}$ of 3 elements?

15. [12 points]

(a) Consider the graph shown below with vertices labeled by $1, 2, 3, 4, 5, 6, 7$. Use the **Greedy Coloring Algorithm** to color the vertices of the graph using **only** colors $1, 2, 3, 4$.

vertex	color
1	
2	
3	
4	
5	
6	
7	

(b) Is the following graph a bipartite graph? If so, label each vertex as L (for Left) or R (for Right).

16. [12 points] **Dijktra's Algorithm** is being used with the following graph with edge weights as shown, in order to find the shortest path from vertex a to every other vertex in the graph. The algorithm has been started in the table shown below, Complete the **next two lines (and only the next two lines)** in the following table.

k	S_k	a	b	c	d	e	f	g	h
0	$\emptyset$	0	∞	∞	∞	∞	∞	∞	∞
1	a	0	1	2	4	∞	∞	∞	∞
2	b	0	1	2	4	4	∞	2	∞
3	c	0	1	2	3	4	∞	2	∞
4									
5									
6									
7									
8									

17. [14 points]

(a) The following is the postfix form of an algebraic expression. What is its **value**? (**In the expression all numbers are single digit numbers.**)

$$8\,2\ \div\ 6\ \times\ 6\,4\,5\ \times\ +\ -\ 5\ \times$$

(b) In what order are the vertices visited if **preorder** is used for traversing the following tree?

Part III (10 points, 1 bonus question). **Prove that for every positive integer** n**, 21 divides** $4^{n+1} + 5^{2n-1}$**.** Little or no partial credit will be given for this bonus problem.