Math 846 Exercises

1. Let $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ be a family of subsets of a finite set E and let $m \geq 1$. Prove that $\mathcal{A}$ has m pairwise disjoint transversals iff

$$|A(K)| \geq m|K| \quad (K \subseteq \{1, 2, \ldots, n\}.$$

2. Let $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ be a family of subsets of a finite set E and let $m \geq 1$. Prove that $\mathcal{A}$ has a SR in which no element occurs more than m times if and only if

$$m|A(K)| \geq |K| \quad (K \subseteq \{1, 2, \ldots, n\}.$$

3. Let $a_1, a_2, \ldots, a_{n^2+1}$ be a permutation of $\{1, 2, \ldots, n^2 + 1\}$. **Use Dilworth's Theorem** to show that $a_1, a_2, \ldots, a_{n^2+1}$ has a monotone sequence of length $n + 1$. (Note: there ia a nice proof of this using the pigeon-hole principle; if you have never seen it you should try that too.)

4. Use Rado's theorem on the existence of an independent transversal and the rank function of a transversal matroid to show the following: If $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ and $\mathcal{B} = (B_1, B_2, \ldots, B_n)$ are two families of subsets of E, then $\mathcal{A}$ and $\mathcal{B}$ have a *common transversal* (i.e. a set that is a transversal of both families) if and only if

$$|A(K) \cap B(L)| \geq |K| + |L| - n \quad (K.L \subseteq \{1, 2, \ldots, n\}).$$

5. (Another way to compute the Möbius function of a poset $P : X, \leq$.) Let $\hat{0}$ and $\hat{1}$ be two new elements and extend P to $\hat{P} : \hat{X}, \leq$ by defining $\hat{0} < x < \hat{1}$ for all $x \in X$. (Of course, if P is already a lattice, one doesn't have to do the extension.)

   Prove that

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \cdots$$

   where c_i equals the number of chains (not necessarily saturated) of the form $\hat{0} = x_0 < x_1 < \cdots < x_i = \hat{1}$.

   HINT: Consider $\zeta = 1 + (1 - \zeta)$.

6. Let n be a positive integer, let q be a prime power and let F be a field of q elements. The subspaces of $V = F^n$ ordered by inclusion determine a finite lattice P_n.

   (a) Prove that that the Móbius function for this lattice satisfies

$$\mu(0.1) = (-1)^n q^{\binom{n}{2}},$$

   from which one obtains the complete Möbius function since every interval is isomorphic to P_m for some $m \leq n$.

   HINT: Use our Lemma: If L is a lattice and $a \in L$ with $a > 0$, then

$$\mu(0, 1) = - \sum_{1 \neq x : x \vee a = 1} \mu(0, x)$$

   by choosing a to be a 1-dimensional space and considering what x satisfy $x \vee a = 1$.

(b) Use Möbius inversion to show that the number of *spanning* sets of vectors of V is

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} (2^{q^k} - 1).$$

where

$$\binom{n}{k}_q = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}.$$

7. Let $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ be a family of subsets of a set E on which there is defined a matroid with rank function $r(\cdot)$. Prove that $\mathcal{A}$ has an independent partial transversal of size t if and only if

$$r(A(K)) \geq |K| - (n - t) \quad (K \subseteq \{1, 2, \ldots, n\}).$$

HINT: See the outline I gave in class on October 3, 2006

8. Let Π be a system of points and lines (and an incidence relation) such that Π has a total of $n^2 + n + 1$ points with exactly $n + 1$ points on a line, and exactly $n + 1$ lines through a point. Assume that two distinct lines intersect in exactly one point, and that there exist 4 points no 3 on a common line. Prove that Π is a projective plane.

9. For each integer $n \geq 4$ find a partial latin square of order n which cannot be embeddded in a latin square of any order $m < 2n$.

10. Complete the following partial latin square of order 6 to a latin square of order 6:

$$\begin{bmatrix} 1 & 2 & 3 & * & * & * \\ * & * & * & 4 & * & * \\ * & * & * & * & 5 & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}.$$

11. Let $G = \{a_1, a_2, \ldots, a_n\}$ be an abelian group of odd order n, written additively. Let A be the addition table for G and let B be the subtraction table for G. Prove that A and B are orthogonal latin squares of order n.

12. Let λ and μ be two partitions of the same integer t. Prove that the Kostka number $K_{\lambda,\mu} \neq 0$ if and only if $\mu \preceq \lambda$. Conclude that the number of nonnegative integral matrices with row sum vector R and column sum vector S is given by:

$$\kappa'(R, S) = \sum_{R,S \preceq \lambda} K_{\lambda,R} K_{\lambda,S}$$