MATH 475; FINAL EXAM, 120 points, December 22, 2005 (R.A.Brualdi)

TOTAL SCORE (120 points possible):

Name: These R. Solutions

1. [20 points] Answer the following two questions.

- What is the **common value** of the Stirling number of the second kind $S(n, n-1)$ and the Stirling number of the first kind $s(n, n-1)$? **What do they count and why are the values the same**?

  Both have the value $\binom{n}{2}$. $S(n, 2)$ counts the number of ways to partition n distinct objects into $n-1$ indistinguishable, non-empty boxes. So all boxes contain one element, except for one which contains two; so the only choice to make is *which two elements are in a box together*, thus $\binom{n}{2}$. $s(n, 2)$ partitions into circles instead of boxes. So the only choice to make is *which two objects are in a circle together and what is the circle.* Since there is only one way to arrange two objects in a circle, we again get $\binom{n}{2}$.

- What is the ordinary generating function for the number h_n $(n \geq 0)$ of partitions of the integer n each of whose parts is one of $1, 2, 3, 4, 5$.

$$\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^4}\frac{1}{1-x^5}$$

2. [15 points] Determine the **number of permutations** of the set $\{1, 2, \ldots, 9\}$ consisting of the first 9 positive integers where at least one of the integers 1, 3, 5, 7, and 9 is in its natural position.

The answer is 9! minus the number of permutations in which none of the integers $1, 3, 5, 7, 9$ is in its natural position, and this can be found in the standard way from inclusion-exclusion using sets A_1, A_3, A_5, A_7, A_9.

3. [15 points] Let m and n be positive integers. Give a **combinatorial proof** of the following identity:

$$\binom{m+n}{p} = \sum_{k=0}^{p} \binom{m}{k}\binom{n}{p-k}.$$

Consider $m + n$ people of two types, e.g. m men and n women. The LHS counts the number of groups of size p that can be made. The RHS counts the same thing according to the number k $(k = 0, 1, \ldots, p)$ of men in the group.

4. [20 points] Let C_n denote the nth Catalan number.

- What is a **formula** for C_n?

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

- Use your formula to derive a **recurrence relation** for the Catalan numbers.

  By straightforward calculation (you were supposed to do it) we get

$$C_n = \frac{4n-2}{n+1}C_{n-1}, n \geq 2$$

- Consider an 12 by 12 grid where the lower left corner is labelled $(0,0)$ and the upper right corner is labelled $(12,12)$. **How many routes** using a total of 24 blocks are there that never dip below (touching is OK) the diagonal joining $(0,0)$ and $(12,12)$ but touch **neither** of the points $(5,5)$ and $(8,8)$?

  Use inclusion-exclusion to get

$$C_{12} - C_5C_7 - C_8C_4 + C_5C_3C_4$$

5. [30 points] First, state **in words** Burnsides formula for the number of inequivalent colorings of a set X under the action of a group of permutations:

If $\mathcal{C}$ is a collection of colorings of a set X and G is a group of permutations of X such that $f * c \in \mathcal{C}$ for all $f \in G$ and $c \in \mathcal{C}$, then the number of inequivalent colorings equals the average number of colorings fixed by the permutations in G.

Then, determine **the number of nonequivalent ways** to color the corners of a regular 8-gon with colors Red, White, and Blue under the action of the corner symmetry group of the 8-gon?

The group is the dihedral group D_8 with 8 rotations and 8 reflections. Using Burnsides theorem we get

$$\frac{1}{16}(3^8 + 4 \cdot 3^5 + 5 \cdot 3^4 + 2 \cdot 3^2 + 4 \cdot 3^1) = 498$$

6. [20 points] A two-sided colored tromino is a 1 by 3 board of three squares with each square (6 in all, because of the two sides) colored with one of the colors Red, White, Blue, Green, and Yellow. **How many different (inequivalent) two-sided colored trominoes are there**?

Here there are 4 permutations in G. Applying Burnsides theorem we get

$$\frac{1}{4}(5^6 + 5^4 + 5^3 + 5^3)$$