TOTAL SCORE (90 points possible):

MATH 340; EXAM # 2, April 11, 2006 (R.A.Brualdi)

Discussion Section (**circle one**): Mon 8:50 Mon 12:05 Wed 8:50 Wed 12:05

NAME:

1. (9 points) Let $T : R^3 \to R^3$ be a linear transformation such that

$$T(1,0,0) = (1,2,3),\ T(0,1,0) = (2,1,4),\ T(0,0,1) = (3,5,6).$$

Calculate $T(4,-1,2)$.

$(4,-1,2) = 4(1,0,0) - 1(0,1,0) + 2(0,0,1)$ so that

$$T(4,-1,2) = 4T(1,0,0) - 1T(0,1,0) + 2T(0,0,1) = 4(1,2,3) - 1(2,1,4) + 2(3,5,6) = (8,17,20).$$

2. [8 points] For each of the following pairs A, B of matrices, **determine whether or not** A **and** B **are similar**. Justify your answer in each case.

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$. **Yes No. Why?**

NO: they have different determinants

(b) $A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -4 \\ 1 & 1 \end{bmatrix}$. **Yes No. Why?**

No: they have different traces

3. (10 points) Let V and W be vector spaces of the **same dimension** n. Let $v_1, v_2, \ldots, v_n$ be a basis of V. Let $T : V \to W$ be a bijective linear transformation (isomorphism).

PROVE that $T(v_1), T(v_2), \ldots, T(v_n)$ **is a basis of** W**.**

Suppose that $c_1T(v_1) + c_2T(v_2) + \ldots + c_nT(v_n) = 0$. Then using properties of linear transformations we get that $T(c_1v_1 + c_2v_2 + \ldots + c_nv_n) = T(0) = 0$. Since T is bijective, this implies that $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$. Since $v_1, v_2, \ldots, v_n$ is a basis of V, we conclude that $c_1, c_2, \ldots, c_n$ all equal 0, and hence $T(v_1), T(v_2), \ldots, T(v_n)$ are linearly independent. Since W has dimension n, $T(v_1), T(v_2), \ldots, T(v_n)$ is a basis of W.

4. [10 points] Let A and B be square matrices of order n with A similar to B. **Prove that** A^3 **is similar to** B^3.

We have $B = PAP^{-1}$ for some nonsingular matrix P. Calculating we get that $B^3 = PAP^{-1}PAP^{-1}PAP^{-1} = PA^3P^{-1}$. Thus A^3 is similar to B^3

5. (10 points) Consider the standard ordered basis

$$\alpha : e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1) \text{ of } R^3$$

and the ordered basis

$$\beta : e_2, e_3, e_1 \text{ of } R^3.$$

Let $T : R^3 \to R^3$ be the linear transformation given by

$$T(a, b, c) = (2a - b + c, a + 2b + 4c, 3a + 5c).$$

Determine $[T]_\alpha^\beta$.

We need to write $T(e_1), T(e_2), T(e_3)$ as linear combinations of e_2, e_3, e_1 in that order. Using the given formula for T we have

$$T(e_1) = (2, 1, 3) = 1e_2 + 3e_3 + 2e_1$$

$$T(e_2) = (-1, 2, 0) = 2e_2 + 0e_3 - 1e_1$$

$$T(e_3) = (1, 4, 5) = 4e_2 + 5e_3 + 1e_1.$$

Hence

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 5 \\ 2 & -1 & 1 \end{bmatrix}.$$

6. (15 points) Let ℓ be the line in the plane through the origin making an angle of $\theta = \pi/6$ (30 degrees) with the positive x-axis. Let T be the linear transformation on R^2 given by the reflection in line ℓ. **Determine the matrix $[T]_\alpha^\alpha$ of T with respect to the standard basis $\alpha : e_1 = (1, 0), e_2 = (0, 1)$ of R^2.**

We can do T by rotating by $-\pi/6$ to bring ℓ to the horizontal axis, reflect about the horizontal axis, and then rotate back by $\pi/6$. Thus the matrix is

$$\begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos -\pi/6 & -\sin -\pi/6 \\ \sin -\pi/6 & \cos -\pi/6 \end{bmatrix}.$$

It is then an easy matter to substitute for the sines and cosines and carry out the multiplication.

7. [10 points] Let $T : R^3 \to R^3$ be the linear transformation given by

$$T(x_1, x_2, x_3) = (x_1 + x_3, x_2 + x_3, 2x_1 + x_3)$$

Determine, with justification, whether or not

1. **T is injective (one to one),**
2. **T is surjective (onto),**
3. **T is an isomorphism?**

Circle those that are correct and then explain why the statements are correct or not correct.

The matrix of T relative to the standard basis is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix},$$

whose determinant is -1. Hence A is nonsingular (invertible) and so T is injective, surjective, and hence an isomorphism.

8. [18 points] Consider R^4 and the (standard) Euclidean inner product (the dot product). **Answer the following questions (no reason necessary):**

(a) Are the vectors $x = (1, 2, -1, 4)$ and $y = (3, 2, 3, -1)$ **orthogonal**? **YES**

(b) The **length** $||x||$ of X equals:

$\sqrt{22}$

(c) If a vector z in R^4 is orthogonal to x and y, then does z have to be orthogonal to $3x + 2y$? **YES**

(d) What is the statement of Cauchy-Schwarz inequality for this dot product on R^4.

$$|x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4| \leq \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}\sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}.$$

(e) Four nonzero vectors in R^4 which are **mutually orthogonal must** be a basis of R^4. **YES**

(f) Four nonzero vectors of R^4 which are a basis **must** be **mutually orthogonal**. **No**