**TOTAL SCORE (90 points possible):**

**MATH 340; EXAM # 1, February 28, 2005 (R.A.Brualdi)**

Discussion Section (**circle one**): Mon 8:50 Mon 12:05 Wed 8:50 Wed 12:05

**Name: These R. Solutions**

1. (12 points) Let $A$ be the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 5 & 1 & 2 \end{bmatrix}.$$

Let $x$ be a $4 \times 1$ matrix (a column vector) $(x_1, x_2, x_3, x_4)^T$, let $y$ be a $1 \times 3$ matrix $(y_1, y_2, y_3)$.

(i) **Express $Ax$ as a linear combination of the columns of $A$:**

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

(ii) **Express $yA$ as a linear combination of the rows of $A$:**

$$y_1 \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} + y_2 \begin{bmatrix} 2 & 4 & 1 & 3 \end{bmatrix} + y_3 \begin{bmatrix} 3 & 5 & 1 & 2 \end{bmatrix}$$

(iii) **Is $yAx$ in the row space, column space, both the row and column space or neither? Circle one.**

**NEITHER**: $yAx$ is a scalar (a $1 \times 1$ matrix).

2. (14 points) Let $A$ be a $4 \times 3$ matrix.

(i) Let $E$ be the elementary matrix such that $EA$ interchanges rows 2 and 4 of $A$. **What is $E$ and, if $E$ is invertible, what is the inverse of $E$?**

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We have $E^{-1} = E$.

(ii) Now let $E$ be the elementary matrix such that $EA$ is obtained from $A$ by adding 10 times row 2 of $A$ to row 4. **What is $E$ and, if $E$ is invertible, what is the inverse of $E$?**

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 10 & 0 & 1 \end{bmatrix}$$

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -10 & 0 & 1 \end{bmatrix}$$

3. (15 points) Let $A$ be an $n \times n$ invertible matrix.

(i) **Define what it means for $A$ to be invertible**.

That there exists a matrix $B$ such that $AB = BA = I_n$.

(ii) **Prove that $A^2$ is invertible if $A$ is invertible.**

Since $A$ is invertible we know $A^{-1}$ exists. Consider the matrix $B = (A^{-1})^2$. Then using properties of matrix multiplication, one easily checks (do it!) that $AB = BA = I_n$

One could also use determinants as in part (iii).

(iii) **If $A^2$ is invertible, is $A$ invertible? Why or why not?**

We know that $A$ is invertible if and only if $\det A \neq 0$. We also know that $\det A^2 = (\det A)^2$ since the deteminant is multiplicative. So since $\det A^2 \neq 0$, $\det A \neq 0$ and $A$ is invertible.

4. (18 points) Let $A$ be a $5 \times 6$ matrix whose reduced row echelon form is

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 5 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) **What is the dimension of the row space of $A$. What is a basis for it.**

This dimension is 3 and the first three rows of the matrix above form a basis.

(ii) **What is the dimension of the column space of $A$. What is a basis for it?**

This dimension is also 3 and columns 1, 3 and 5 of $A$ form a basis.

(iii) **What is the dimension of the null space of $A$. What is a basis for it?**

This dimension is $6 = 3 = 3$ and a basis can be found as follows. the above rref tells us that the solutions of $Ax = 0$ (the null space of $A$) is:

$$\begin{aligned} x_1 &= -2c - 3d - 2f \\ x_2 &= c \\ x_3 &= -5d - 4f \\ x_4 &= d \\ x_5 &= -6f \\ x_6 &= f \end{aligned}$$

where $d, e, f$ are arbitrary. Separating $c$, $d$ and $e$, we get as a basis:

$$(-2, 1, 0, 0, 0, 0)^T, (-3, 0, -5, 1, 0, 0)^T, (-2, 0, -4, 0, -6, 1)^T.$$

5. (14 points) Consider the vector space $M_{4,4}(R)$ of all $4 \times 4$ real matrices. Let $U$ be the subset of $M_{4,4}(R)$ consisting of all **symmetric matrices with zeros on the main diagonal**.

(i) **Prove that $U$ is a subspace of $M_{4,4}(R)$.**

The zero matrix is in $U$ so $U$ is nonempty. One easily checks (do it) that the sum of two matrices in $U$ is also in $U$ and a scalar multiple of a matrix in $U$ is in $U$. So $U$ is a subspace.

(ii) **Exhibit a basis of $U$ and then give dim $U$.**

A basis consist of the six $4 \times 4$ symmetric matrices with 0's on the main diagonal, having exactly one 1 above the main diagonal (and so by symmetry exactly one 1 below the diagonal). Write the matrices out. The dimension is 6.

6. (8 points) **Use Cramer's rule to find $x_2$ of the solution of the following system. Calculator answer is not acceptable.**

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 2 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 2 \\ 3 & 1 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

By Cramer's rule,

$$x_2 = \frac{\det \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 2 \\ 3 & 0 & 0 & 0 & 4 \end{bmatrix}}{\det \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 2 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 2 \\ 3 & 1 & 0 & 0 & 4 \end{bmatrix}} = \frac{-30}{-18} = \frac{5}{3}$$

(You must show your work in computing the two determinants.)

7. (9 points) **What is the $(1,3)$-entry of the inverse of the matrix**

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 4 \\ 2 & 4 & 1 \end{bmatrix}.$$

One easily caculates that $\det B = -15$. The cofactor of the $(3,1)$-entry of $B$ is used to get the $(1,3)$-entry of $\text{adj}(B)$. This cofactor equals

$$(-1)^{3+1} \det \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} = -7.$$

Hence the $(1,3)$-entry of $B^{-1}$ equals $\frac{-7}{-15} = \frac{7}{15}$.

**Do the columns of $B$ form a basis of $R^3$? Why or why not?**

Yes, because $\det B \neq 0$ and hence the columns of $B$ are linearly independent, Since $R^3$ has dimension 3, these columns must form a basis of $R^3$.