MATH 340; EXAM # 1, 100 points, November 13 , 2007 (R.A.Brualdi)

TOTAL SCORE :

Name: These R. Solutions

I. (42 points; 3 points each) Answer the following questions as **True (T)** or **False (F)** by circling **T** or **F** below (no justification wanted):

1. **T** If u_1, u_2, u_3, u_4, u_5 are linearly independent vectors in a 5-dimensional subspace U of R^8, then they are a basis of U.

2. **T** If v_1, v_2, v_3, v_4, v_5 spans a 5-dimensional subspace V of R^8, then they are a basis of V.

3. **T** If A is a 6 by 8 matrix then you can be sure that the homogeneous system $Ax = 0$ has a non-trivial solution.

4. **T** A linearly independent set of 3 vectors in 5-dimensional vector space V can always be enlarged to a basis of V.

5. **F** A set of 8 vectors in a 6-dimensional vector space U always contains 6 vectors that form a basis of U.

6. **F** The set of all singular 4 by 4 matrices is a subspace of the vector space $M_{4,4}$ of all 4 by 4 matrices.

7. **F** A 3 by 5 matrix A could have rank 4.

8. **F** The set of all vectors $[a\ b\ c]$ in R_3 with $a + b + c \geq 0$ forms a subspace of R_3.

9. **F** If the nullity (dimension of null space) of the 5 by 5 matrix A is 0, then A is singular.

10. **T** If A and B are 4 by 6 matrices with the same row space, then A and B have the same column space.

11. **T** If A is a 4 by 5 matrix and B is a 4 by 6 matrix and their column spaces have the same dimension, then their row spaces have the same dimension.

12. **T** If v_1, v_2, v_3, v_4 span a subspace U of a vector space V, and v_1 is a linear combination of v_2, v_3, v_4, then v_2, v_3, v_4 span U.

13. **F** The set of all real polynomials of degree ≤ 4, using standard polynomial addition and scalar multiplication of a polynomial, is a vector space of dimension 4.

14. **T** The set of all 3 by 3 symmetric matrices forms a subspace of dimension 6 of the vector space $M_{3,3}$ of 3 by 3 matrices.

II. (18 points) A matrix A has row vectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and column vectors $\beta_1, \beta_2, \ldots, \beta_6$. If A has row-reduced echelon form

$$\begin{bmatrix} 1 & 0 & 3 & -2 & 0 & 3 \\ 0 & 1 & -4 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Answer the following questions (your answer **may** use the αs and βs:

1. A **basis for the null space** of A is:

$$\begin{bmatrix} -3 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

2. A **basis for the row space** of A is:

   The first three rows of the rref.

3. A **basis for the column space** of A is:

   Columns 1, 2, and 5 of A (not of the rref).

III. (16 points; 8 points each) **Prove** the following two assertions:

1. If A is a n by n nonsingular matrix, and v_1, v_2, v_3, v_4 are linearly independent vectors in R^n, then Av_1, Av_2, Av_3, Av_4 are linearly independent vectors in R^n.

   Suppose that
$$c_1(Av_1) + c_2(Av_2) + c_3(Av_3) + c_4(Av_4) = 0.$$
   Then using properties of matrix algebra, we get
$$A(c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4) = 0.$$
   Since A is nonsingular,
$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0.$$
   Since v_1, v_2, v_3, v_4 are linearly independent, all the c's are 0. Hence Av_1, Av_2, Av_3, Av_4 are linearly independent.

2. The intersection $U \cap V$ of two subspaces U and V of R^n is also subspace of R^n. (Keep in mind what has to be checked for a set of vectors in a vector space to be a subspace.)

   First $U \cap V \neq \emptyset$ since both U and V contain the zero vector of R^n. So we need only show the two closure rules to conclude $U \cap V$ is a subspace.

   Let u and v be vectors in the intersection. Then u and v are in both U and V. Since both are subspaces $u + v$ is also in U and in V. Hence $u + v$ is in $U \cap V$.

   In the same way (you have to do it), one shows that $U \cap V$ is closed under scalar multiplicaion.

IV. (24 points) Consider the set S of three vectors

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

and the set T of three vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

1. Verify, in any legitimate way you can, that S and T are both bases of R^3.

   Maybe the easiest way is to take the determinants of the 3 by 3 matrices whose columns are the u's and v's, respectively. Both are nonzero, One could also show that the rrefs of these matrices are both equal to I_3.

2. Determine the transition matrix $P_{S \leftarrow T}$ from the T basis to the S basis.

   We have to express the v's as linear combinations of the u's, that is, solve three systems of three equations in three unknowns, which we can do simultaneously by EROS:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{array}\right].$$

   The 3 by 3 matrix on the right of the rref is $P_{S \leftarrow T}$

3. If a vector v in R_3 satisfies $[v]_T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, determine $[v]_S$.

$$[v]_S = P_{S \leftarrow T}[v]_T = \begin{bmatrix} 12 \\ -4 \\ -5 \end{bmatrix}.$$