MATH 240; FINAL EXAM, 150 points, December 21, 2004 (R.A.Brualdi)

TOTAL SCORE (14 problems; 150 points possible):

Name: BRIEF SOLUTIONS

TA: Darren Neubauer (circle time) Mon 12:05 Mon 1:20 Wed 12:05 Wed. 1:20

Do NOT compute binomial coefficients or factorials in any of the problems

1. [15 points] Let A be a set of 8 elements and let B be a set of 12 elements. Determine

   1. The number of functions $f : A \to B$:

      12^8

   2. The number of injective functions $f : A \to B$:

      $12!/4!$

   3. The number of surjective functions $f : A \to B$:

      0

2. [10 points] Use modular exponentiation (no other method acceptable) to compute 42^{55} mod 13. **(You must show your work; a calculator answer is unacceptable).**

   Using modular exponentiation (55 in base 2 is 110111) we get 3 mod 13.

3. [15 points] A sequence of numbers $a_1, a_2, \ldots, a_n, \ldots$ is defined recursively by:

$$a_1 = -4, a_2 = 14, a_n = 3a_{n-1} + 4a_{n-2}.$$

Prove by **mathematical induction** that $a_n \equiv 2 \pmod 6$. Be sure to specify both steps of the inductive proof.):

(a) Which form of induction are you using?

Strong or structural induction.

(b) Basis Step:

One easily checks that a_1 and a_2 are 2 mod 6.

(c) Inductive Step: **Set this up as a direct proof stating what the assumption is and what you are proving.** Very briefly, $a_n = 3a_{n-1} + 4a_{n-2}$ is mod 6, using the inductive hypothesis and the recurrence,

$$3(2) + 4(2) = 14 \text{ which is 2 mod 6.}$$

4. [8 points] The transitive closure of a relation on a set of six elements $\{1, 2, 3, 4, 5, 6\}$ is being computed by Warshall's Algorithm. The matrix W_3 is shown below. Determine W_4.

$$W_3 = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & & 1 & & 1 \\ & & & & 1 & \\ 1 & 1 & & 1 & & 1 \\ & & & & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

5. [8 points] The sequence of numbers $h_0, h_1, \ldots, h_n, \ldots$ satisfies a recurrence relation whose **characteristic roots** are $2, 2, 2$. **What is the recurrence relation?**

So the char. equation is $(x-2)^3 = x^3 - 6x^2 + 12x - 8$ and hence the recurrence relation is $h_n = 6h_{n-1} - 12h_{n-2} + 8h_{n-3}$

6. [12 points] Let $n \geq 1$. Determine, with initial conditions, a recurrence relation for the number a_n of n-bit strings of length n which do **not** contain 3 consecutive 1's. (Do **NOT** solve the recurrence relation.)

By considering how the sequence must end we see that $h_n = h_{n-1} + h_{n-2} + h_{n-3}, h_1 = 2, h_2 = 2, h_3 = 7$

7. [12 points] Let A be a set of size 15. Recall that order does not matter for the parts of a partition.

1. How many ways are there to partition A into three parts of size 3, 5, and 7, respectively?

   $\binom{15}{3}\binom{12}{5}$

2. How many ways are there to partition A into two parts with no part empty?

   Pick a subset not equal to A or the empty set, and divide by 2 since order doesn't matter: $\frac{2^{15}-2}{2}$

8. [8 points] Let R be the equivalence relation "$\equiv$ mod 12" on the set Z of integers, and let S be the relation "$\equiv$ mod 15."

(a) What integers are in the equivalence class of R that contains 3?

All integers of the form $3 \pm k$ where k is a nonnegative integer.

(b) Describe explicitly the binary relation on Z given by $R \cap S$?

$\equiv$ mod the LCM of 12 and 15 and so 60.

9. [12 points] Let R be a binary relation R on the set $\{a, b, c, d, e\}$.

(a) Express the transitive closure of R as the union of finitely many binary relations:

$R \cup R^2 \cup R^3 \cup R^4 \cup R^5$

(b) If R is given by the bit matrix $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$. Is R:

(circle if so) reflexive, symmetric, anti-symmetric, **none of these**.

(c) What is the bit matrix of the symmetric closure of the binary relation defined in (b)?

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

10. [6 points] Let R and S be a relations on $\{1, 2, 3, 4\}$ with bit matrices

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

Calculate the bit matrix $M_{S \circ R}$ of the relation $S \circ R$.

It's $M_R \times M_S$ using Boolean arithmetic and so

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

(11) [16 points] Consider the partial order | (is a divisor of) on the set $A = \{3, 4, \ldots, 14, 15\}$ of thirteen elements. Find, if they exist,

1. LUB $\{4, 6\} = 12$

2. LUB $\{3, 7\}$: doesn't exist

3. GLB $\{6, 9\} = 3$.

4. GLB $\{4, 6\}$: doesn't exist

12. [8 points] Let $A = \{a, b, c, \ldots, x, y, z\}$ and consider the poset $(S, \subseteq)$ where S is the power set $\mathcal{P}(A)$ of A. Let $X = \{c, f, k, p, q\}$ and let $Y = \{a, f, q, z\}$. In this poset, calculate

1. $LUB(X, Y) = X \cup Y = \{a, c, f, k, p, q, z\}$

2. $GLB(X, Y) = X \cap Y = \{f, q\}$

13. [10 points] Construct the binary rooted tree corresponding to the compound proposition:

$$((p \vee q) \wedge r) \rightarrow ((s \vee t) \wedge (u \wedge w)).$$

I won't draw this.

14. [10 points] The following is the prefix form (preorder) of an algebraic expression (note all number are single digit positive numbers). What is its value?

$$* \ * \ 5 \ + \ 4\,3 \ + \ - \ 6\,7\,2$$

Working from the right we get 35