MATH 240; EXAM # 2, 100 points, November 21, 2006 (R.A.Brualdi)

TOTAL SCORE (7 problems; 100 points possible):

Name: These R. Solutions

TA: Luanlei Zhao (circle time) Mon 9:55 — Mon 11:00 — Wed 9:55 — Wed. 11:00

1. [14 points] Give a **recursive definition with initial conditions** of the following function and set:

(a) If $a_1, a_2, \ldots, a_n, \ldots$ is an infinite sequence, then for $n \geq 1$, $f(n) = a_1 + a_2 + \cdots + a_n$ (the nth partial sum).

$f(n) = f(n-1) + a_n, (n \geq 2), f(1) = a_1.$

(b) The set A of bit sequences of any length at least 1 such that the sequence does not contain two consecutive 0's.

$1, 0, 01, 10, 11$ are in A; if $a_1a_2 \ldots a_{n-1}1$ is in A then so is $a_1a_2 \ldots a_{n-1}0$ and $a_1a_2 \ldots a_{n-1}1$; if $a_1a_2 \ldots a_{n-1}0$ is in A then so is $a_1a_2 \ldots a_{n-1}01$.

2. [5 points] Compute the following **Boolean product**:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

3. [10 points] Complete the following:

(a) $\sum_{k=0}^{n} \binom{n}{k}$ =: 2^n

(b) $\binom{n-1}{k} + \binom{n-1}{k-1}$ =: $\binom{n}{k}$

(c) The coefficient of x^8 in $(1+x)^{20}$ equals: $\binom{20}{8}$.

(d) $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$ =: 0

4.[15 points] Recall that the Fibonacci numbers are defined recursively by

$$f_0 = 0, f_1 = 1, f_n = f_{n-2} + f_{n-1} \ (n \geq 2).$$

We can thus compute these numbers using the following **recursive algorithm**:

If $n = 0$, then $f_0 = 0$.
Else, if $n = 1$, then $f_1 = 1$.
Else, $f_n = f_{n-2} + f_{n-1}$.

Let $g(n)$ be the total number of additions used in computing f_n by this algorithm. Thus, for instance, $g(0) = 0, g(1) = 0, g(2) = 1, g(3) = 2, g(4) = 4, g(5) = 7$. **Prove by mathematical induction** (whatever kind that works) that

$$g(n) = f_{n+1} - 1.$$

We use strong induction:

Basis Step: We have $g(0) = 0 = f_1 - 1 = 1 - 1 = 0$ and $g(1) = 0 = f_2 - 1 = 1 - 1 = 0$.

Inductive step: Suppose that $g(k) = f_{k+1} - 1$ for $k = 0, 1, 2, \ldots, n$. We need to show that $g(n+1) = f_{n+2} - 1$. But by the algorithm (since $f_{n+1} = f_{n-1} + f_n$),

$$g(n+1) = g(n-1) + g(n) + 1 = f_n - 1 + f_{n+1} - 1 + 1 = f_n + f_{n+1} - 1 = f_{n+2} - 1.$$

Thus the result follows from strong induction.

5.[20 points] (a) A Student Organization has 20 members. A committee of 5 is to be formed.

(a) **How many different committees are possible?** $\binom{20}{5}$

(b) Let Mary be one of the member of the student organization, and suppose all choices are equally likely. **What is the probability that Mary is on the committee?** $\frac{\binom{19}{4}}{\binom{20}{5}}$.

(c) Now suppose that the committee is to have two officers, one person designated as Chair and a different person designated as Fund Raiser? **How many different committees are there now?** $\binom{20}{5} \cdot 5 \cdot 4$.

(d) **What is the probability that Mary is an officer if you have the added information that Mary is on the committee?** $\frac{2\cdot 4}{5\cdot 4} = 2/5$.

6.[16 points] At the end of the day, a Bagel Factory has left only 50 plain bagels. The factory has 4 employees Alice, Bob, Cathy, and Dave who at the end of the day take home the leftover bagels.

(a) **How many ways are there to distribute the leftover bagels to the employees?**

The number of solutions of $x_1 + x_2 + x_3 + x_4 = 50$ in nonnegative integers, so

$$\binom{53}{3} = \binom{53}{50}.$$

(b) **Assuming all possibilities are equally likely, what is the probablility that Dave gets at least 3 bagels?**

The number of ways to distribute the bagels so that Dave gets at least 3 is the number of solutions of $y_1 + y_2 + y_3 + y_4 = 47$ in nonnegative integers (give Dave 3 and then distribute the remaining 47 to the 4 people), so

$$\binom{50}{3}.$$

Thus the probability equals

$$\frac{\binom{50}{3}}{\binom{53}{3}}.$$

7. [20 points] A biased coin has a probability of coming up Heads with probability 3/4 and Tails with probability 1/4. The coin is flipped 12 times in succession. Let E be the event that there are 8 Heads and let F be the event that at least one flip results in a Head.

(a) **What is expected number of Heads?**

$$(3/4) \cdot 12 = 9.$$

(b) **What is $P(E)$?**

$$\binom{12}{8}(3/4)^8(1/4)^4.$$

(c) **What is $P(F)$?** It's 1 minus the probability of $\overline{F}$ and thus $1 - (1/4)^{12}$.

(d) If G is the event that the first 4 flips results in 3 Heads, **what is $P(E|G)$?**

$$\binom{8}{5}(3/4)^5(1/4)^3.$$

(e) **Are E and G independent? Justify your answer.** Since $P(E) \neq P(E|G)$, E and G are not independent.