Math 240, Fall Semester 2001-02 NAME:
(Prof.) R.A. Brualdi
Exam 2: November 16, 2001, Total Points:

1. [24 points] A **ternary string** is a string (sequence) whose terms are 0's, 1's, and 2's. **Give**

(a) the number of ternary strings of length n: 3^n
(b) the number with no 0's: 2^n

(c) the number with exactly one 0: $n2^{n-1}$

(d) the number with exactly three 0's: $\binom{n}{3}2^{n-3}$

(e) the number with at least two 0's: $3^n - 2^n - n2^{n-1}$

(f) the number with exactly one 0 and exactly one 1: $n(n-1)$

2. [5 points] The coefficient of x^4y^9 in the expansion of $(2x-3y)^{13}$ is: $\binom{13}{4}2^4(-3)^9$

3. [10 points] A case of sodapop contains 24 cans (order unimportant) taken from Pepsi, Coke, 7-Up, and Dr. Pepper. The number of different cases of sodapop possible is:

The number of nonnegative integer solutions of $x_1+x_2+x_3+x_4 = 24$, and so $\binom{24+4-1}{24} = \binom{24+4-1}{3}$.

4. [15 points] Consider an ordinary deck of cards consisting of 13 ranks and 4 suits. A **pair** consists of two cards of the same rank. A **hand** is an (unordered) set of 6 cards. The number of hands with

(a) exactly one pair is: $13\binom{4}{2}\binom{12}{4}4^4$
(choose a rank, choose two cards of that rank, choose 4 of the remaining 12 ranks, choose one of the 4 cards of the 4 chosen ranks.)

(b) exactly two pairs is: $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{2}4^2$

(c) exactly three pairs is: $\binom{13}{3}\binom{4}{2}^3$

5. [12 points] Find but **do not solve** a recurrence relation for the number a_n of bit strings of length n that do not contain the pattern 001 (i.e. two 0's followed by a 1):

The beginning can be 1 (and so completed in a_{n-1} ways, 01 (and so completed in a_{n-2} ways, or 00 (and so completed in only one way, i.e. $00\cdots 0$. So $a_n = a_{n-1} + a_{n-2} + 1, n \geq 3$.

6. [8 points] Let R be the relation among points in the plane defined by: XRY (i.e. $(X, Y) \in R$) provided the distance between X and Y is at most 3.

Is R an equivalence relation? Why or why not? It is clearly not transitive (but it is reflexive and symmetric), and so not an equivalence relation.

Describe the relation R^2, i.e.

XR^2Y if and only if: XR^2Y if and only if the distance between X and Y is at most 6.

7. [12 points] The number of permutations of the 10 letters B,R,A,D,S,U,E,T,I,M that do not contain any of the names BRAD, SUE, or TIM equals:

Let A_1 be the set of those permutations containing BRAD, A_2 containing SUE, and A_3 containing TIM. Want $\bar{A}_1 \cap \bar{A}_2 \cap \ \bar{A}_3$. Now by DeMorgan's Law, this is $\overline{A_1 \cup A_2 \cup A_3}$. Using the inclusion-exclusion principle we get

$$10! - (7! + 8! + 8!) + (5! + 5! + 6!) - 3!.$$

8. [12 points] Use **Warshall's algorithm** to determine the transitive closure of the relation R on 6 elements whose Boolean matrix is

$$M_R = \left[\begin{array}{c|c|c|c|c|c} 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \cdots\cdots W_6 = \left[\begin{array}{c|c|c|c|c|c} 1 & 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 1 & 1 & 1 \end{array}\right].$$