**21.** A finite $p$-group $G$ is called generalized extra-special if $Z(G)$ is cyclic and $G'$ has order $p$. Show that if $G$ is a finite nonabelian $p$-group such that every proper quotient of $G$ is abelian (i.e. "minimally nonabelian"), then $G$ is generalized extra-special.

**22.** Show that the automorphism group of $Q_8$ is isomorphic to $Sym(4)$. Deduce that there is no finite group whose Frattini subgroup is isomorphic to $Q_8$.

HINT: Let $C = C_G(\Phi(G))$ and look at the image of $\Phi(G)$ in $G/C$.

**23.** A critical subgroup of a finite group $G$ is a characteristic subgroup $H$ of $G$ such that $\Phi(H) \leq Z(H) \geq [G, H]$ and $C_G(H) = Z(H)$. Show that every finite $p$-group has a critical subgroup.

HINT: Consider the set of all characteristic subgroups $H$ satisfying $\Phi(H) \leq Z(H) \geq [G, H]$.

**24.** If $R$ is a commutative ring with 1, let $U(n, R)$ denote the group of $n \times n$ upper unitriangular matrices over $R$ (i.e. 1 on the diagonal, 0 below it). Show that $U(n, \mathbf{F_p})$ is a Sylow $p$-subgroup of $GL(n, p)$ and deduce that every finite $p$-group is isomorphic to a subgroup of some $U(n, \mathbf{F_p})$.