Fall 2003 HOMEWORK 5

17. Prove that a finite p-group has maximal class if and only if it has an element whose centralizer has order p^2.

HINT: Show that the property of having such an element is inherited by quotient groups of order $\geq p^2$.

18. If G is a finite p-group, define $\Omega_i(G) := < x \in G : x^{p^i} = 1 >$. Consider the so-called Ω-series of G, namely

$$1 = \Omega_0(G) \leq \Omega_1(G) \leq ... \leq \Omega_e(G) = G$$

where p^e is the exponent of G. Are these always proper inclusions? For "regular" p-groups, $\Omega_i(G) = \{x \in G : x^{p^i} = 1\}$. Give an example of a finite p-group that is not regular.

19. If subgroup H of G satisfies $H' = G'$, show that $L_i(H) = L_i(G)$ for all $i \geq 2$.

HINT: You can use without proof the Three-Subgroup Lemma, which says that if H, K, L, N are subgroups of G, with N normal, then if $[H, K, L]$ and $[K, L, H] \leq N$, then $[L, H, K] \leq N$.

20. If G is a p-group of order p^n and $\alpha \in$Aut(G) has order p^m, show that $m < n$.