Perfect Space-Time Block Codes and Ultra-Wideband

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1 Introduction

In this report, we present the techniques for constructing Space-Time Block Codes based on division algebra. Even though commutative division algebra can be applied for the construction, we focus on noncommutative division algebra, especially in the construction of so-called perfect space-time (ST) block codes. These codes provide full rate, full diversity and non-vanishing constant minimum determinant for increasing spectral efficiency, uniform average transmitted energy per antenna and good shaping. Even though the perfect codes exist only for 2, 3, 4 and 6 antenna systems, these codes may potentially be used in many wireless applications because of space limitation of the devices, e.g. cell phones and laptops. The design of these codes is based on the assumptions of flat fading channels where the channel gains are complex Gaussian random variables.

On the other hand, Ultra-Wideband (UWB) is a promising technology for indoor environment wireless applications and lately, research has focused on the possibility of using multiple antennas for these systems. Due its large bandwidth, UWB channels experience so-called frequency-selective phenomena and it violates the assumptions of flat fading for the design of the perfect codes. Because of different channel assumptions, it seems that the perfect codes cannot be used in UWB systems, however it was shown that ST codes designed for flat fading channels still provide at least the some advantage even in the presence of frequency-selective fading channels [4].

With the objective to give an answer or opinion whether the perfect codes can be used in UWB systems, we proceed as follow. We investigate the construction of ST block codes for frequency-selective channels and the validity of ST codes desinged for flat fading channels in frequency-selective channels. We also investigate the state of art of ST block codes in UWB systems. Based on this knowledge, we provide a short discussion and give some thought whether the perfect codes can be used in UWB systems.

2 Background

2.1 Division Algebra

In this section, we review some basic background of abstract algebra theory that are the main principles to construct space-time block codes. First, we review some basic definitions and then noncommuntative division algebras is discussed to give an inside of how cyclic division algebra can be the basic tool for constructing the perfect codes.

2.1.1 Definitions

A *division algebra* D, also called a "division ring" or "skew field," is a ring in which every nonzero element has a multiplicative inverse, but multiplication is not necessarily commutative. Every field is therefore also a division algebra.

We say that a field K is an *extension* of F, if F is a *subfield* of K, and we usually denote by K/F. If K is an extension of F, we can regard K as a vector space over F. The dimension of this space is denoted $[K : F]$, and called degree of the extension. A field extension K/F is *normal* if every irreducible polynomial $f \in F[x]$ which has at least one root in K splits (factors into a product of linear factors) in $K[x]$. An arbitrary field extension K/F is defined to be *separable* if every finitely generated intermediate field extension L/F has a transcendence basis $S \subset L$ such that L is a separable algebraic extension of $F(S)$. A field extension is *Galois* if it is normal and separable.

A Galois extension K/F is said to be a *cyclic extension* if the Galois group $Gal(K/F)$ is cyclic. The Galois group $Gal(K/F)$ of a field extension K/F is the group of all field automorphisms $\sigma\colon K \to K$ of K which fix F (i.e., $\sigma(x) = x$ for all $x \in F$). The group operation is given by composition: for two automorphisms $\sigma_1, \sigma_2 \in Gal(K/F)$, given by $\sigma_1\colon K \to K$ and $\sigma_2\colon K \to K$, the product $\sigma_1 \cdot \sigma_2 \in Gal(K/F)$ is the composite of the two maps $\sigma_1 \circ \sigma_2\colon K \to K$.

By a *subfield* of a division algebra, we will mean a field K, such that $F \subseteq K \subseteq D$. If K is a subfield of D, then K is a subspace of the F-vector space D, and $[K : F]$ divides $[D : F] = n^2$. It is known that the maximum possible value of $[K : F]$ is n, such a subfield is called a *maximal subfield* of D.

The ring of integers is the set of integers $..., -2, -1, 0, 1, 2, ...$, which form a ring. This ring is commonly denoted $\mathbb{Z}$. More generally, let K be a number field, then the *ring of integers* of K, denoted $\mathcal{O}_K$, is the set of algebraic integers in K, which is a ring of dimension d over $\mathbb{Z}$, where d is the extension degree of K over $\mathbb{Q}$. The Gaussian integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ is the ring of integers of $K = \mathbb{Q}(i)$, and the Eisenstein integers $\mathbb{Z}[w] = \{a + bw : a, b \in \mathbb{Z}\}$ is the ring of integers of $\mathbb{Q}(w)$, where $w = (-1 + \sqrt{-3})/2$ is a primitive cube root of unity.

An *ideal* is a subset $\mathcal{I}$ of elements in a ring R that forms an additive group and has the property that, whenever x belongs to R and y belongs to $\mathcal{I}$, then xy and yx belong to $\mathcal{I}$.

An n-dimensional *real lattice* $\Lambda(K)$ is a subset in $(R)^n$:

$$\Lambda(K) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = K \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} |z_i \in \mathbb{Z} \qquad \text{for} 1 \leq i \leq n \right\} \tag{1}$$

where (Z) is the ring of all integers and K is an $n \times n$ real matrix of full rank and called the generating matrix of the real lattice $\Lambda(K)$ and $\det(\Lambda(K))$=$|\det(K)|$.

An n-dimensional *complex lattice* $\Lambda^C(G)$ over K is a subset of $(C)^n$:

$$\Lambda^C(G) = \left\{ \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = G \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} |z_i \in K \qquad \text{for} 1 \leq i \leq n \right\} \tag{2}$$

where G is an $n \times n$ complex matrix of full rank and called the generating matrix of the complex lattice $\Lambda^C(G)$.

2.1.2 Cyclic Division Algebra

Given a cyclic division algebra D of index n, and with maximal cyclic subfield K/F, let $Gal(K/F)$ be generated by σ, then $\sigma^n = 1$. D is a right vector space over K, with the product of the $k \in K$(scalar) on the vector $d \in D$ defined to be dk. It is well known that we have the following decomposition of D as right K spaces:

$$D = K \oplus zK \oplus z^2K \oplus \cdots \oplus z^{n-1}K \tag{3}$$

where z some element of D that satisfies the relations

$$kz = z\sigma(k), \qquad \forall k \in K \tag{4}$$

$$z^n = \gamma, \qquad \text{for some } \gamma \in F^* \tag{5}$$

where F^* is the set of F excluding the zero element and z^iK stands for the set of all elements of the form z^k for $k \in K$. The division algebra D, with its decomposition, is often written as $(K/F, \sigma, \gamma)$.

2.1.3 Embedding Field Extension Into Matrices

Let K and F be fields, with $F \subset K$, and $[K : F] = n$, i.e., K is of dimension n over F. It has a map ς from K to $End_F(K)$, which is the set of $F-$linear transforms of the vector space K. This map is given by $k \mapsto \lambda_k$, where λ_k is the map on the $F-$vector space K that sends any $u \in K$ to the element k_u (i.e., λ_k is a left multiplication by k.) ς maps K isomorphically into $End_F(K)$, that is, K embeds in $M_n(F)$ which is the set of $n \times n$ matrices over the field F. This method of embedding K into $M_n(F)$ is known as the *regular representation* of K in $M_n(F)$.

For a given F, the basis $\mathcal{B} = \{u_1, u_2, \ldots, u_n\}$ of K, one can write down the matrix corresponding to λ_k for any k as follow: for any given basis element $u_i (1 \leq i \leq n)$, and for any $u_j (1 \leq j \leq n)$, let

$$u_i u_j = \sum_{l=0}^{n} c_{ij,l} u_l.$$

Then, the jth column of λ_{u_i} is simply the coefficients $c_{ij,l}$. Once the matrix corresponding to each λ_{u_i}, call it M_i. In general, the matrix corresponding to a λ_k with $k = \sum_{l=1}^{n} f_i u_i$ is the linear combination $\sum_{l=1}^{n} f_i M_i$

When K is generated over F by a primitive α, the matrices in the particular basis $\mathcal{B}$ $= \{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ are easier to write down. Suppose that minimal polynomial of α over F is

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Then the matrix corresponding to λ_α is its companion form matrix M given by

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \tag{6}$$

and the matrices corresponding to the power α^i can be computed directly as the i-th power of M and the general element $k = f_0 + f_1\alpha + \cdots + f_{n-1}\alpha^{n-1}$ will be mapped to the matrix $f_0 I_n + f_1 M + \cdots + f_{n-1} M^{n-1}$.

2.1.4 Embedding Division Algebra Into Matrices

So far, we have developed embedding field extensions into matrices. In the similar way, we can develop embedding division algebras into matrices. Given an F division algebra D of degree n, D is an $F-$vector space of dimension n^2. Thus, we have a map $\varsigma : D \mapsto End_F(D)$. Similarly, this map is given by left multiplication, and it takes any $d \in D$ to λ_d, where λ_d is left multiplication by d, i.e., $\lambda_d(e) = de$ for all $e \in D$. The rest of the development is the same as the previous case, except that in the basis $\mathcal{B}$, we have n^2 elements instead of n.

2.1.5 Embedding Cyclic Division Algebra Into Matrices

Finally, embedding cyclic division algebras into matrices over the maximal cyclic subfield can be developed based on the previous cases. Let D be a cyclic division algebra over F of index n, with maximal cyclic subfield K. As we have have seen earlier, D is a right K space, of dimension n and let denote it as D_K. Now, D acts on D_K by multiplication of the left as follows: given $d \in D$, it sends an arbitrary $e \in D_K$ to de. Since this action is from the left, while the scalar action of K is from the right, these two actions commute, i.e., $d(ek) = (de)k$. Let us write λ_d for the map from D_K to D_K that sends $e \in D_K$ to de. The fact that the action λ_d and that of the scalars commute means that λ_d is a $K-$ linear transform of D_K. Thus, we have an embedding of D into $End_K(D_K)$, which, once one chooses a K basis for D_K, translates into the embedding of D into $M_n(K)$. We can choose the basis $\{1, z, z^2, \ldots, z^{n-1}\}$, where z is some element of D that satisfy the relations that we discussed earlier. A typical element $d = k_0 + zk_1 + \cdots + z^{n-1}k_{n-1}$ sends 1 to $k_0 + zk_1 + \cdots + z^{n-1}k_{n-1}$, so the first column of the matrix corresponding to λ_d in this basis is $[k_0, k_1, \ldots, k_{n-1}]^T$. For the second column, note that

$$\begin{aligned} dz &= (k_0 + zk_1 + \cdots + z^{n-1}k_{n-1})z \\ &= k_0 z + zk_1 z + \cdots + z^{n-1}k_{n-1}z \qquad (7) \\ &= z\sigma(k_0) + z^2\sigma(k_1) + \cdots + z^{n-1}\sigma(k_{n-2}) + \gamma\sigma(k_{n-1}). \qquad (8) \end{aligned}$$

Proceeding this manner, we can find the matrix corresponding to λ_d as

$$\begin{bmatrix} k_0 & \gamma\sigma(k_{n-1}) & \gamma\sigma^2(k_{n-2}) & \cdots & \gamma\sigma^{n-2}(k_2) & \gamma\sigma^{n-1}(k_1) \\ k_1 & \sigma(k_0) & \gamma\sigma^2(k_{n-1}) & \cdots & \gamma\sigma^{n-2}(k_3) & \gamma\sigma^{n-1}(k_2) \\ k_2 & \sigma(k_1) & \gamma\sigma^2(k_0) & \cdots & \gamma\sigma^{n-2}(k_4) & \gamma\sigma^{n-1}(k_3) \\ k_3 & \sigma(k_2) & \gamma\sigma^2(k_1) & \cdots & \gamma\sigma^{n-2}(k_5) & \gamma\sigma^{n-1}(k_4) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{n-1} & \sigma(k_{n-2}) & \sigma^2(k_{n-3}) & \cdots & \sigma^{n-2}(k_1) & \sigma^{n-1}(k_0) \end{bmatrix}. \qquad (9)$$

2.2 Wireless Channel Models

In this section, we quickly overview some main characteristics of wireless channel models. These characteristics are important because performance analysis and the design of space-time block codes depends on the channel models.

2.2.1 Multipath Channel Model

The complex baseband received signal $x(t)$ can be expressed as

$$\begin{aligned} x(t) &= h(t,\tau) * s(t) \\ &= \sum_{l=1}^{L_s} A_l s(t-\tau_l) e^{j[2\pi F_d \cos(\psi_l)t+\phi_l]}, \end{aligned} \tag{10}$$

where $s(t)$ is the complex baseband transmitted signal with bandwidth B, the symbol $*$ denotes the convolution operation, τ_l is the multipath delay, L_s is the number of scatters of the environments, ϕ_l is the uniformly distributed random phases over $[0, 2\pi]$, ψ_l is the random angles of arrival relative to the motion of the receiver of each scatter, and F_d is the maximum Doppler frequency due to mobility. Then the channel impulse response $h(t,\tau)$ can be shown to be

$$h(t,\tau) = \sum_{l=1}^{L_s} A_l e^{j[2\pi F_d \cos(\psi_l)t+\phi_l]} \delta(t-\tau_l). \tag{11}$$

2.2.2 Narrowband Channel Model

If the relative delay $\Delta_\tau = \tau_{L_s} - \tau_1$ (also known as delay spread) is much less than the inverse bandwidth of the signal, then $s(t-\tau_l) \approx s(t-\tau_1)$. Then the channel model (10) becomes,

$$\begin{aligned} x(t) &= s(t-\tau_1) \sum_{l=1}^{L_s} A_l e^{j[2\pi F_d \cos(\psi_l)t+\phi_l]} \\ &= s(t-\tau_1)\alpha(t)e^{j\Theta(t)} \qquad (12) \\ &= s(t-\tau_1)\beta(t), \qquad (13) \end{aligned}$$

where

$$\beta(t) = \alpha(t)e^{j\Theta(t)} = \sum_{l=1}^{L_s} A_l e^{j[2\pi F_d \cos(\psi_l)t+\phi_l]}. \tag{14}$$

If we assume that the $\{A_l\}$ are independent and identically distributed and independent of the ϕ_l, and L_s is large, then by central limit theorem $\beta(t)$ will be a complex Gaussian random variable. In this case, $\alpha(t)$ has a Rayleigh distribution. If there is light of sight (LOS) between the transmitter and the receiver, then $\alpha(t)$ will have a Rician distribution. If we assume τ_1 is the first arriving path (i.e., $\tau_1 = 0$), the channel response is reduced to

$$\begin{aligned} h(t) &= \sum_{l=1}^{L_s} \frac{1}{\sqrt{L_s}} e^{j[2\pi F_d \cos(\psi_l)t+\phi_l]} \\ &= \alpha(t)e^{j\Theta(t)} \\ &= \beta(t) \end{aligned} \tag{15}$$

This is channel model is known as *flat* fading channel.

2.2.3 Wideband Channel Model

As the signal bandwidth B increases such that $\Delta_\tau \approx B^{-1}$, the received signal becomes a sum of multiple copies of the original signal, with each copy delayed by τ_l, attenuated by α_l and phase shifted by $\Theta_l(t)$. In the wideband channel model, if the incoming paths from the same scatter arrive within $1/B$ seconds, they are not *resolvable* at the receiver. Thus, they are combined into a single path. If a finite number of resolvable paths is present due to multiple dominant reflectors, then the received signal can be written as

$$\begin{aligned} x(t) &= \sum_{l=1}^{L} \alpha_l(t-\tau_l) e^{j\Theta_l(t-\tau_l)} s(t-\tau_l) \\ &= \sum_{l=1}^{L} \beta_l(t-\tau_l) s(t-\tau_l) \end{aligned} \tag{16}$$

where L is the number of resolvable paths, τ_l, $\alpha_l(t)$ and $\Theta_l(t)$ are the delay, amplitude and phase of each resolvable path respectively. In the absence of LOS, $\beta(t)$ can still assume to be a complex Gaussian random variable because each unresolvable path can contain a relative large number of subpaths, therefore central limit theorem still apply. However, this is no longer valid for ultra-wideband channels, since the number of subpaths is small, therefore, the statistics is hard to be determined. Then the normalized channel response for a wideband channel becomes

$$\begin{aligned} h(t,\tau) &= \sum_{l=1}^{L} \alpha(t) e^{j\Theta(t)} \delta(\tau-\tau_l) \\ &= \sum_{l=1}^{L} \beta(t) \delta(\tau-\tau_l) \end{aligned} \tag{17}$$

This channel model is known as *frequency-selective* fading channel model.

2.3 Space-Time Block Codes

2.3.1 Space-Time Coding

First, we discuss different diversity techniques that can be applied in wireless systems. The idea behind space-time coding is to apply one or more of these diversity techniques to combat fading. Then we discuss briefly the space-time coding model and derive the pairwise error probability which leads the design criteria for ST block codes.

2.3.2 Diversity

Diversity techniques are effective ways of combating channel fading and improve reliable system performance in wireless communications. Diversity techniques include time, frequency, antenna and multipath diversity. Due to space limitation of the report, we limit on the diversities that applicable to STBC. Time diversity is provided by channel coding in combination with limited interleaving. Multiple replicas of the transmitted signal are spaced in time and the time spacing between transmission exceeds the coherence time of the channel. It essentially provides a form of redundancy in time domain to the receiver. In frequency diversity, the replicas of the signals are transmitted over different frequency band. The frequency spacing between channels has to exceed the coherence bandwidth of the channel. It provide a form of redundancy in the frequency domain. Note that time and frequency diversity normal induce loss in bandwidth efficiency. Antenna diversity which is also known an space diversity, the transmitter and/or the receiver employs multiple antennas that are spatially separated or differentially polarized to create independent fading channels. It provides a form of redundancy in space domain and the performance gain can be provided without any penalty to system's bandwidth efficiency. Depending on whether multiple antennas are used for transmission or reception, it is either transmit-antenna diversity or receive-antenna diversity. In receive-antenna diversity schemes, multiple antennas are deployed at the receiver to acquire separate copies of the transmitted signal which are then properly combined to mitigate channel fading. On the other hand, transmit-antenna diversity relies on the multiple antennas at the transmitter. Two major obstacles to implement transmit-antenna diversity are the following: i) unlike the receiver, the transmitter does not have instantaneous information about the channels; and ii) the transmitted signals are mixed spatially before they arrive at the receiver. In order to exploit the embedded diversity from multiple transmissions, transmit-antenna diversity must rely on some additional processing. Two categories of transmit-antenna diversity schemes have been proposed, open loop and closed loop. Open loop schemes does not require channel knowledge at the transmitter whereas closed loop relies on some channel information at the transmitter that is acquired through feedback channels. In this case, the transmitter may not able of acquiring and tracking the channel variations. Lastly, in the frequency selective fading channels, usually multipath diversity is available. The received signal consists of multiple copies of the transmitted signal and each copy experiences a different path delay and amplitude produced by the channel.

2.3.3 Space-Time Coding Model

Suppose we have a MIMO system with N_t transmit antennas and N_r receive antennas. At the transmitter, information symbols $s(n)$ belonging to the constellation set $\mathcal{A}_s$ are parsed into block $\mathbf{s}(n) = [s(nK), \ldots, s(nK + K - 1)]^T$ of size $K \times 1$. The block $\mathbf{s}(n)$ is encoded by the ST encoder which uniquely maps $\mathbf{s}(n)$ to the following $N_t \times P$ ST code matrix

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1P} \\ c_{21} & c_{22} & \cdots & c_{2P} \\ \vdots & \vdots & \vdots & \vdots \\ c_{N_t1} & c_{N_t2} & \cdots & c_{N_tP} \end{bmatrix} \tag{18}$$

where the coded symbol c_{ij} belongs to the constellation set $\mathcal{A}_c$ and P is the frame(block) length. At each time slot t, signals c_{it}, $i = 1, 2, \ldots, N_t$ are transmitted simultaneously from the N_t transmit antennas. Because the P coded symbols sent from each transmit-antenna in each time slot correspond to K information symbols, the transmission rate R is

$$R = \frac{K}{P} \log_2 |\mathcal{A}_s| \tag{19}$$

where $|\mathcal{A}_s|$ denotes the cardinality of $\mathcal{A}_s$. For binary, the coding rate is defined as $R = K/P$. If $K = P$, the code is termed *full rate* or rate 1 code.

2.3.4 Pairwise Error Probability (PEP)

The pairwise error probability (PEP) is the probability that the maximum likelihood decoder selects as its estimate a signal $\mathbf{e}$ when in fact the signal c was transmitted. We consider the received signal is composed of two terms, the first term is the convolution of the channel and the transmitted matrix and the second term is additive Gaussian noise with variance $\frac{N_o}{2}$ per dimension. With the ideal channel state information at the receiver and for a given realization of the channel, the PEP conditioned on the channel is given by [1],

$$P(\mathbf{c} \to \mathbf{e} | h_{i,j}, i = 1, \ldots, i = 1, \ldots) = Q\left(\frac{d(\mathbf{c}, \mathbf{e})}{2\sigma}\right) \quad (20)$$

$$\leq \exp(-d^2(\mathbf{c}, \mathbf{e}) E_s / 4N_o) \quad (21)$$

where $Q(x)$ is the complementary error function given by $Q(x) = \int_x^{\infty} e^{-t^2/2} dt$. Note that $\{h_{ij}\}$ depends on the assumptions of the channel, e.g., flat fading. The distance d depends on the channel assumptions and the ST block codes and it must be defined in the appropriate metric. To compute the unconditional PEP, we take the expectation with respect to the channel coefficients.

2.3.5 Design Criteria for Flat Fading Channels

Detail analysis of optimal ST codes can be found in [2]. By assuming the channel is a slow fading channel, as we discussed earlier, the channel gains $\{h_{i,j}\}$ are i.i.d. complex Gaussian random variables with zero mean and variance 1/2 per dimension. We also assume that ML decoding is performed at the receiver and the channel knowledge is perfectly known. Let us defined the error matrix $B = (\mathbf{c} - \mathbf{e})(\mathbf{c} - \mathbf{e})^H$. At high SNR, it was shown in [2] that PEP is upper bounded by

$$P(\mathbf{c} \to \mathbf{e}) \leq \left(\prod_{i=1}^{rank(B)} \lambda_i\right)^{-N_r} \cdot \left(\frac{E_s}{4N_o}\right)^{-rank(B) N_r}, \quad (22)$$

where $\lambda_i, i = 1, \ldots$,rank(B) are nonzero eigenvalues of B. Conventionally, the performance of ST coded systems is measured by two parameters, diversity gain and coding gain. From this expression, the diversity gain is $G_d = \text{rank}(B).N_r$ and the coding gain is $G_c = \left(\prod_{i=1}^{rank(B)} \lambda_i\right)^{1/rank(B)}$. Then at high SNR, the PEP is upper bounded by

$$P(\mathbf{c} \to \mathbf{e}) \leq \left(G_c \frac{E_s}{4N_o}\right)^{-G_d}, \quad (23)$$

In order to minimize $P(\mathbf{c} \to \mathbf{e})$, it is clear that STBC should be designed to maximize both G_d and G_c, which leads to the following two design criteria:

Rank Criterion

In order to achieve the maximum diversity gain, the matrix B has to be full rank over all possible distinct $\mathbf{c}$ and $\mathbf{e}$. Note that the rank of B is equivalent to the rank of $(\mathbf{c} - \mathbf{e})$.

Derterminant Criterion

Suppose B is full rank. In order to achieve the maximum coding gain, the minimum determinant of B over all possible $\mathbf{c}$ and $\mathbf{e}$ should be maximized.

Additional design criteria can be found in [2] for correlated flat fading channels and fast flat fading channels.

2.4 Space-Time Codes over Frequency-Selective Channels

Up to now, we have reviewed the performance and the design criteria of space-time codes considering a flat fading channel. This model is valid for the case when the delay spread is small enough compared to the symbol duration. As we discuss earlier, if the delay spread is comparable or larger than the symbol duration, the channel will distort the signal resulting in what is known as frequency-selective channels. The channel response is no longer constant and not all frequency components fade simultaneously. In this case, the delayed paths overlap will cause intersymbol interference (ISI).

The effect of frequency-selectivity on ST coding designs have been investigated in [3, 4], where the pairwise error block error probability is re-derived based on a simple two-ray channel model and assuming ML decoding. It was shown that ST codes designed to provide a certain diversity gain in flat fading channels still provide at least the same diversity gain even in the presence of frequency-selective fading channels [4]. With regards to coding gain, the theoretical analysis reveals that the coding gain might decrease considerably in the presence of ISI induced by the frequency-selective channels [3], unless additional processing is employed. For instance, Alamouti's linear ML decoding for STBC is not directly applicable in the presence of frequency-selectivity. In order to maintain the decoding simplicity and take advantage of existing ST coding designs for flat fading channels, most work on ST coding design pursue (suboptimal) two-step approaches. The basic idea behind these approaches is to first mitigate the ISI and convert frequency-selective fading channel into flat fading and the second step is to design ST coder and decoder for the resulting flat fading channels. One approach to mitigate the ISI is to employ equalization at the receiver. The equalizer equalizes the frequency-selective channels into temporal ISI-free channels, where STBC designed for flat fading channels can be applied directly. One of the drawback of this approach is high receiver complexity. In order to maintain receiver simple, most work on ST coding for frequency-selective fading channels employ Orthogonal Frequency Division Multiplexing (OFDM) modulation with a long enough symbol interval to eliminate the need for equalization at the receiver. Since part of the ISI is eliminated at the transmitter, the receiver complexity for this approach is relatively low.

Results in [4] suggest that optimal ST codes in frequency-selective fading channel may achieve higher diversity gain than those in flat fading channels. However, these results are obtained based on ML decoding that is computationally heavy. Practically, feasible ST decoding designs for frequency-selective channels are challenging because the transmitted signals are mixed both temporally and spatially. Lately, researchers have shown that STBC over frequency selective fading channels can achieve a maximum diversity order N_tN_rL in rich scattering environments, where L is number of multipath [5, 6]. In [6], they claim that their schemes enable simple linear processing that is comparable to single-antenna transmissions. They also suggest that joint exploitation of space-multipath diversity leads to significantly improved performance in the presence of frequency-selective multipath channels.

2.5 Ultra Wideband

Even though UWB channels experience the same characteristics of frequency-selective channels as we described earlier, UWB channels have several differences compared to the traditional wideband frequency-selective channels. First of all, it is a baseband transmission system and the channel gains are reals numbers instead of complex numbers. Secondly, because of its large bandwidth, central limit theorem does not apply, therefore the gains are no longer Gaussian random variables and they are difficult to be determined. In addition, due to its large bandwidth and sharing same frequency bands with existing systems, it has to limit its instantaneous transmission power to avoid causing interference to the existing systems. In this scenario, one way to achieve certain level of Signal-to-Noise Ratio (SNR) with the limited transmission power is to reduce the data rate, i.e., send each symbol several times. The transmitted waveform is usually a nonsinusoidal pulse which is one of the main differences compared to the narrowband and wideband systems. Moreover, the number of multipaths, L, is usually a random number whereas in wideband systems, it is deterministic. Even though UWB channels experience other characteristics, e.g., clustering phenomena, frequency-selective channel models with a fixed number of multipaths can be a first approximation. To make analysis possible, some researchers have treated the UWB channels as Rayleigh fading channels.

There are two methods to transmit the repeated pulses, one is to repeat them consecutively and another one to is repeat them with certain time separations. The first method is the same as CDMA systems, where the pulse amplitudes are based upon a certain spreading code, a PN sequence. In the second method, the symbol duration is divided into N_f frames, and each frame is divided into N_c slots. In each slot, a pulse is transmitted according to the modulation technique. This method is known as Time Hopping (TH) since each pulse can take any slots within a frame and different slots in different frames and it seems that it "hops" inside the frames. Two main modulation techniques are widely used for UWB systems are Pulse Position Modulations (PPM) and Pulse Amplitude Modulation (PAM).

It is clear that in the first method, each pulse will cause interference to others and it is known as InterChip Interference (ICI). This is the same problem as ISI that we have discussed earlier. In the second method, if the separation between pulses is long enough, ICI problem can be avoided at the expense of data rate.

3 Perfect Space Time Block Codes

In this section, we present the construction of the perfect space-time block codes based on paper [15] where the main tool for constructing the Perfect codes is the cyclic division algebra.

3.1 Construction of Perfect Space-Time Block Codes

Consider a field $\mathbb{F}$ and let $\mathbb{K}/\mathbb{F}$ be a cyclic extension of degree n, with Galois group $Gal(\mathbb{K}/\mathbb{F}) =< \sigma >\simeq \mathbb{Z}/n$, where σ is the generator of the cyclic group. Let $\mathbf{D} = \{\mathbb{K}/\mathbb{F}, \sigma, \gamma\}$ be its corresponding cyclic algebra of degree n, i.e.,

$$\mathbf{D} \triangleq \mathbb{K} \cdot 1 \oplus \mathbb{K} \cdot e \oplus \ldots \mathbb{K} \cdot e^{n-1} \tag{24}$$

with $e \in \mathbf{D}$ such that for some $\gamma \in \mathbb{F}^*$, $e^n = \gamma$ and $\forall x \in \mathbb{K}$, $e^{-1}xe = \sigma(x)$.

Through the embedding division algebra into matrices section, we can obtain the left regular representation of $\mathbf{D}$. For any element $x \in \mathbf{D}$, where $x = x_0 + x_1 e + \ldots + x_n e^{n-1}$ with $x_i \in \mathbb{K}$, let λ_x be the linear map that sends $\mathbf{D}$ to itself via multiplication by x, i.e.,

$$\lambda_x : \mathbf{D} \to \mathbf{D} \qquad y \mapsto \lambda_x(y) = xy \qquad \text{for any } y \in \mathbf{D}.$$

Then the matrix that corresponding λ_x is given by

$$\mathbf{x} = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-1} \\ \gamma\sigma(x_{n-1}) & \sigma(x_0) & \sigma(x_1) & \dots & \sigma(x_{n-2}) \\ \vdots & & & \ddots & \vdots \\ \gamma\sigma(x_2) & \gamma\sigma(x_3) & \gamma\sigma(x_4) & \dots & \sigma^{n-2}(x_1) \\ \gamma\sigma^{n-1}(x_1) & \gamma\sigma^{n-1}(x_2) & \gamma\sigma^{n-1}(x_3) & \dots & \sigma^{n-1}(x_0) \end{bmatrix} \tag{25}$$

The space-time block codes is then obtained by

$$\mathcal{C}_\infty = \left\{ \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-1} \\ \gamma\sigma(x_{n-1}) & \sigma(x_0) & \sigma(x_1) & \dots & \sigma(x_{n-2}) \\ \vdots & & & \ddots & \vdots \\ \gamma\sigma(x_2) & \gamma\sigma(x_3) & \gamma\sigma(x_4) & \dots & \sigma^{n-2}(x_1) \\ \gamma\sigma^{n-1}(x_1) & \gamma\sigma^{n-1}(x_2) & \gamma\sigma^{n-1}(x_3) & \dots & \sigma^{n-1}(x_0) \end{bmatrix} | x_i \in \mathbb{K}, i = 0, \dots, n-1 \right\} \tag{26}$$

All the coefficients of such matrices are in $\mathbb{K}$. Based on the definition of the perfect codes and the code design criteria, the construction steps are the following:

- The symbol information constellation of QAM or HEX is finite subset of the ring of integers $\mathcal{O}_\mathbb{F} = \mathbb{Z}[i](\mathcal{O}_\mathbb{F} = \mathbb{Z}[j]$ for HEX) therefore, we take the field $\mathbb{F} = \mathbb{Q}(i)$ ($\mathbb{F} = \mathbb{Q}(j)$ for HEX).

- We take the cyclic extension $\mathbb{K}/\mathbb{F}$ of degree $n = N_t$ with Galois group $Gal(\mathbb{K}/\mathbb{F}) = \langle\sigma\rangle$ and build the corresponding cyclic algebra: $\mathbf{D} = \{\mathbb{K}/\mathbb{F}, \sigma, \gamma\}$. We choose γ such that $|\gamma| = 1$ in order to satisfy the constraint on the uniform transmitted energy per antenna.

- To obtain non-vanishing determinants, we choose $\gamma \in \mathbb{Z}[i]$ and adding the previous constraint, γ can only be $\pm 1, \pm i$ (or $\pm 1, \pm j, \pm j^2$ for $\gamma \in \mathbb{Z}[j]$).

- We choose an ideal $\mathcal{I} \subseteq \mathcal{O}_\mathbb{K}$ so that the signal constellation on each layer is a finite subset of the rotated version of the lattice $\mathbb{Z}^{2n}$ or A_2^n.

- Among all elements of $\mathbf{D}$, we consider the discrete set of codewords, where $x_i \in \mathcal{I}$, an *ideal* of $\mathbb{K}$, the ring of integers of $\mathbb{K}$. This restriction guarantee a discrete minimum determinant. Thus, the ST block codes get the form

$$\mathcal{C}_\mathcal{I} = \left\{ \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-1} \\ \gamma\sigma(x_{n-1}) & \sigma(x_0) & \sigma(x_1) & \dots & \sigma(x_{n-2}) \\ \vdots & & & \ddots & \vdots \\ \gamma\sigma(x_2) & \gamma\sigma(x_3) & \gamma\sigma(x_4) & \dots & \sigma^{n-2}(x_1) \\ \gamma\sigma^{n-1}(x_1) & \gamma\sigma^{n-1}(x_2) & \gamma\sigma^{n-1}(x_3) & \dots & \sigma^{n-1}(x_0) \end{bmatrix} | x_i \in \mathcal{I} \subseteq \mathcal{O}_\mathbb{K}, i = 0, \dots, n-1 \right\} \tag{27}$$

- In order to ensure that $\mathbf{D}$ is a division algebra, which means every nonzero element has a multiplicative inverse, the necessary and sufficient condition is $\gamma^k \neq N_{\mathbb{K}/\mathbb{F}}(y)$ for any $y \in \mathbb{K}$ and $1 \leq k \leq n-1$, where $N_{\mathbb{K}/\mathbb{F}}(y)$ is the *norm* of y over $\mathbb{F}$ which is given by

$$N_{\mathbb{K}/\mathbb{F}}(y) = \prod_{\sigma \in Gal(\mathbb{K}/\mathbb{F})} \sigma(y).$$

The idea behind the construction of the perfect codes based on the steps is to determine $\mathbb{F}$, $\mathbb{K}$, γ and $\mathcal{I}$ to optimize this code. Now, Let us start working in detail with these steps satisfying the design criteria.

In order to fulfill the rank criterion to achieve full diversity, the codes must satisfy the following condition,

$$\zeta(\mathcal{C}_{\mathcal{I}}) = \min_{\mathbf{x},\mathbf{y}\in C_{\mathcal{I}},\mathbf{x}\neq\mathbf{y}} |\det(\mathbf{x}-\mathbf{y})|^2 > 0. \tag{28}$$

On the other hand, it can be shown that in the construction of the division algebra, the difference of any two elements is still in it and will be full rank. In this case, the rank criterion becomes

$$\zeta(\mathcal{C}_{\mathcal{I}}) = \min_{\mathbf{x}\in C_{\mathcal{I}},\mathbf{x}\neq 0} |\det(\mathbf{x})|^2 > 0. \tag{29}$$

Since the determinant of any invertible matrix is nonzero, we can satisfy the rank criterion for any element of the infinite code. In the construction steps, we restrict the codes to be a subset of infinite code, i.e., $\mathcal{C}_{\mathcal{I}}$ by restricting the choice of γ, and therefore the perfect codes satisfy the rank criterion. It can be shown that by choosing $\gamma \in \mathcal{O}_{\mathbb{F}}$, then the determinant of any $\mathbf{x} \in \mathcal{C}_\infty$ belongs to $\mathcal{O}_{\mathbb{F}}$. As a consequence, the minimum determinant of the infinite code with $\mathcal{I} = \mathcal{O}_{\mathbb{K}}$ is $\zeta(\mathcal{C}_\infty) = 1$. Now, we know that $\zeta(\mathcal{C}_{\mathcal{I}}) \geq \zeta(\mathcal{C}_\infty)$ and if $\mathcal{I}$ is principal, the minimum determinant of the infinite code $\mathcal{C}_{\mathcal{I}}$ can be computed easily. Otherwise, we can find the lower bound. The minimum determinant is essentially the coding advantages or the coding gain of the ST codes.

For having uniform average transmitted energy per antenna in all time slots, all coded symbols in the code matrix should have the same energy. As as consequence, $|\gamma| = 1$ and therefore γ must be a unit in $\mathcal{O}_{\mathbb{F}}$. The only units in $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$ are ± 1 unless $\mathbb{F} = \mathbb{Q}(i)$ or $\mathbb{Q}(j)$ where j is a nonreal third root of unity.

Based on our discussion, we are able to determine the number of antennas for which the perfect codes exists. By the restriction on γ, which can only be -1, $\pm i$, $\pm j$, $\pm j^2$. Then $\gamma^k = 1$ for $k \leq 6$, thus $\mathbf{D}$ can only be a division algebra for n $\leq$ 6. From this information we get:

γ	smallest k	possible n	$\mathbb{F}$
-1	2	2	$\mathbb{Q}(\sqrt{-d})$
$\pm i$	4	2,3,4	$\mathbb{Q}(i)$
j, j^2	3	2,3	$\mathbb{Q}(j)$
$-j$, $-j^2$	6	2,3,4,5,6	$\mathbb{Q}(j)$

(30)

If $n = 5$ and by the choice of γ would have to be $-j$ or $-j^2$ which is the relative norm, then it prevents $\mathbf{D}$ from being a division algebra as we have seen in the last step of the construction. Therefore, any code constructed from this $\mathbf{D}$ would not be full rank. As conclusion, the number of antennas can be 2,3,4 or 6, see discussion in [15] for details.

So far, we are able to determine $\mathbb{K}, \mathbb{F}$ and γ based on the design criteria. The last parameter to determine is $\mathcal{I}$ which involves with the shaping of the codes. The shaping property is one of the new design criteria for the construction of the perfect codes. The idea behind the shaping is to optimize the energy efficiency of the codes. This can be achieved by introducing a shaping constraint on the signal constellation.

Basically, the idea is to think $\mathcal{O}_{\mathbb{K}}$ as a complex lattice in $\mathbb{C}^n$ and then we rotate it to obtain the desired shape. In each case above,

$$\mathcal{O}_{\mathbb{K}} = \{a_1 + a_2\theta + \ldots + a_n\theta^{n-1} | a_i \in \mathcal{O}_{\mathbb{F}}\} \tag{31}$$

where $\mathbb{F} = \mathbb{Q}(i)$ for n=2,4 and $\mathbb{Q}(j)$ for n=3,6. It is clear that $\{1, \theta, \ldots, \theta^{n-1}\}$ is the $\mathcal{O}_{\mathbb{K}}$ over $\mathcal{O}_{\mathbb{F}}$. Recall that $Gal(\mathbb{K}/\mathbb{F}) = <\sigma> \simeq \mathbb{Z}/n$, and we can define an embedding:

$$\begin{aligned} \varphi : & \mathcal{O}_{\mathbb{K}} \longmapsto \mathbb{C}^n \\ & x \longmapsto \varphi(x) = (x, \sigma(x), \ldots, \sigma^{n-1}(x)) \end{aligned} \tag{32}$$

Note that the image of this embedding is a lattice in $\mathbb{C}^n$ with basis of $\mathcal{O}_{\mathbb{K}}$. For every ideal $\mathcal{I}$ of $\mathcal{O}_{\mathbb{K}}$ we can restrict the embedding to provide a lattice $\Lambda(\mathcal{I})$ with basis of $\mathcal{I}$. We know that every such ideal has an integral basis over $\mathcal{O}_{\mathbb{F}}$. More specifically, suppose $\mathcal{I}$ has basis $\{\beta_k\}_{k=1}^n$. Then $\{\varphi(\beta_k)\}$ is a collection of n vectors in $\mathbb{C}^n$. We form a matrix M with the $\varphi(\beta_k)$ as columns so that the l, kth entry is $\sigma^{l-1}(\beta_k)$

$$M = c \begin{pmatrix} \beta_1 & \beta_2 & \ldots & \beta_n \\ \sigma(\beta_1) & \sigma(\beta_2) & \ldots & \sigma(\beta_n) \\ \vdots & & \ddots & \vdots \\ \sigma^{n-1}(\beta_1) & \sigma^{n-1}(\beta_2) & \ldots & \sigma^{n-1}(\beta_n) \end{pmatrix}. \tag{33}$$

Consequently, the l, kth entry of the Gram matrix G is given by

$$|c|^2 \sum_{m=1}^{n-1} \sigma^m(\beta_l \overline{\beta}_k) = |c|^2 Tr_{\mathbb{K}/\mathbb{F}}(\beta_l \overline{\beta}_k) \tag{34}$$

where $\bar{\beta}$ denotes the complex conjugation of β and $Tr_{\mathbb{K}/\mathbb{F}}(y)$ is the *trace* of y over $\mathbb{F}$ which is given by

$$Tr_{\mathbb{K}/\mathbb{F}}(y) = \sum_{\sigma \in Gal(\mathbb{K}/\mathbb{F})} \sigma(y).$$

The computation of G is fundamental to determine if we have chosen $\mathcal{I}$ so that ΛI is a rotated, scaled version of $\Lambda(\mathcal{O}_{\mathbb{F}})$, $\mathbb{Z}[i]^n$. In order for the codes we defined above to be perfect, for each we need an ideal $\mathcal{I}$ with basis $\{\beta_k\}_{k=1}^n$ and a c so that

$$|c|^2 Tr_{\mathbb{K}/\mathbb{F}}(\beta_l \overline{\beta}_k) = \delta_{lk}. \tag{35}$$

The idea to obtain the $\Lambda(\mathcal{I})$ that satisfies the shaping constraint is to guess it based on the following two conditions

$$vol(\Lambda(\mathcal{I})) = vol(\Lambda(\mathcal{O}_{\mathbb{F}})) = \begin{cases} k^{2n} & if \quad \mathcal{O}_{\mathbb{F}} = \mathbb{Q}(i) \\ (k^2 \frac{\sqrt{3}}{2})^n & if \quad \mathcal{O}_{\mathbb{F}} = \mathbb{Q}(j) \end{cases} \tag{36}$$

and

$$vol(\Lambda(\mathcal{I})) = \mathcal{N}(\mathcal{I}) vol(\Lambda(\mathcal{O}_{\mathbb{K}})) \tag{37}$$

$$vol(\Lambda(\mathcal{O}_{\mathbb{K}})) = 2^{-n} \sqrt{|d_{\mathbb{K}}|} \tag{38}$$

where $\mathcal{N}(\mathcal{I})$ is the norm of $\mathcal{I}$ and $d_{\mathbb{K}}$ is the absolute discriminant of $\mathbb{K}$.

3.2 Golden Codes

After describing the general method of constructing the perfect codes, in this section, we develop the construction of two-antenna perfect codes. Even though there are a lot of perfect codes for 2 antennas, we concentrate on the golden codes which is a special case of the perfect codes. It is so called golden code because of the appearance of $\theta = \frac{1+\sqrt{5}}{2}$ which is known as the *Golden Number*, in the construction of the code. Let $\mathbb{F} = \mathbb{Q}(i)$, $\mathbb{K} = \mathbb{F}(\theta) = \mathbb{Q}(i, \sqrt{5})$. Since θ has minimal polynomial $x^2 - x - 1$ over $\mathbb{Q}$, $[\mathbb{Q}(\theta) : \mathbb{Q}] = 2$. Also, $\mathbb{Q}(\theta) \subset \mathbb{R}$, so the minimal polynomial of i, $x^2 + 1$, remains irreducible over $\mathbb{Q}(\theta)$. Thus, $[\mathbb{K} : \mathbb{Q}] = 4$, which implies that $[\mathbb{K} : \mathbb{Q}(\theta)] = 2$. We then have $\mathcal{O}_{\mathbb{F}} = \mathbb{Z}[i]$ and $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[i][\theta] = \{a + b\theta | a, b \in \mathbb{Z}[i]\}$.

It can be proved that $\gamma = i$ is not a relative norm form $\mathbb{K}$ to $\mathbb{F}$, therefore, $\mathbb{D}$ is a division algebra. Note that in the design of perfect codes for 2 antenna systems, we are not restricted to the Golden code. If we choose θ to be $\frac{1+\sqrt{p}}{2}$ where p is a prime $\equiv 5 \pmod 8$ [17]. To guess $\mathcal{I}$, first we compute $vol(\Lambda(\mathcal{O}_{\mathbb{K}}))$ which is

$$vol(\Lambda(\mathcal{O}_{\mathbb{K}})) = 5$$

since $d_{\mathbb{K}} = 2^4 5^2$.

While (36) gives us that $vol(\Lambda(\mathcal{I})) = k^2$. Putting these together with (37) we get

$$k^2 = 5\mathcal{N}(\mathcal{I})$$

Now, we look in $\mathcal{O}_{\mathbb{K}}$ for an ideal with norm 5. Since $5 \equiv 1 \pmod 4$, 5 splits in $\mathbb{Z}[i]$, and since $d_{\mathbb{Q}(\theta)} = 5$, 5 ramifies in $\mathcal{O}_{\mathbb{Q}(\theta)}$. This means that as an ideal in $\mathcal{O}_{\mathbb{K}}$, $(5) = (\mathcal{I}_1)^2(\mathcal{I}_2)^2$. By inspection, we find

$$\mathcal{I}_1 = (\alpha) = (1 + i - i\theta) \tag{39}$$

$$\mathcal{I}_2 = (\sigma(\alpha)) = (1 + i - i\sigma(\theta)) \tag{40}$$

$$\mathcal{N}(\mathcal{I}_1) = N_{\mathbb{K}/\mathbb{Q})}(\alpha) = \mathcal{N}(\mathcal{I}_2) = N_{\mathbb{K}/\mathbb{Q}}(\sigma(\alpha)) = 5 \tag{41}$$

It can be shown that $\mathbf{\Lambda}(\alpha) = (\sqrt{\mathbf{5}}\mathbb{Z}[\mathbf{i}])^{\mathbf{2}}$. Then $Tr_{\mathbb{K}/\mathbb{F}}(\alpha\overline{\alpha}) = Tr_{\mathbb{K}/\mathbb{F}}(\alpha\theta\overline{\alpha\theta}) = 5$ and $Tr_{\mathbb{K}/\mathbb{F}}(\alpha\overline{\alpha\theta}) = Tr_{\mathbb{K}/\mathbb{F}}(\alpha\theta\overline{\alpha}) = 0$

Hence $G = 5Id$ and $\Lambda(\alpha)$ is a rotated version of $(\mathbb{Z}[i])^2$, scaled by $\sqrt{5}$. The normalizing constant we choose to be $\frac{1}{\sqrt{5}}$ and we have verified

$$\mathcal{C}_{(\alpha)} = \left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} x_0 & x_1 \\ i\sigma(x_0) & \sigma(x_1) \end{pmatrix} \middle| x_i \in (\alpha) \right\} \tag{42}$$

is Perfect Code. The *Golden Code* is

$$\mathcal{C}_{(\alpha)} = \left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha(a + b\theta) & \alpha(c + d\theta \\ i\sigma(\alpha)(c + d\sigma(\theta)) & \sigma(\alpha)(a + b\sigma(\theta)) \end{pmatrix} \middle| a, b \in \mathbb{Z}[i] \right\} \tag{43}$$

4 Space-Time Block in UWB

In the literature, only two main papers had been published in ST codes for UWB systems [7, 8]. One is the analog space-time coding for multiantenna for UWB transmissions [8] inspired by Alamouti's code and the second one [7] is based on Real Orthogonal Design for Time Hopping (TH) and Direct Sequence (DS) UWB-MIMO systems inspired by Tarokh's work [9].

4.1 Analog Space-Time Coding

In [8], ST coding is developed for analog modulated UWB-MIMO systems. Traditional ST coding schemes operate on digital symbols, e.g., Alamouti's code, whereas this approach encodes pulses within symbol waveforms. This aspect of the codes enhances space-multipath diversity gains. Existing ST codes are designed for either flat fading channel or for ISI inducing frequency-selective channels. These codes are for non-ISI inducing channels that are rich in multipath diversity. They proposed two ST coding schemes and for simplicity, they started with $N_t = 2$ transmit antennas, and $N_r = 1$ receive antenna and then they generalized the results. In order to avoid ISI, the frame duration is chosen appropriately and to make the analysis simpler, the number of frame is even. Note that in order to exploit both spatial and multipath diversity, the received signal is combined coherently with the Rake receiver and Maximum-Ratio combining. Under this receiver scheme, the channel information is assumed perfectly known. In addition, the distribution of the channel gains is assumed to be Rayleigh distributed.

STC Scheme I

During each symbol duration $N_f T_f$, it transmit from the zeroth antenna

$$s_o(t) = s\sqrt{\frac{E_s}{2N_f}} \sum_{n=0}^{N_f-1} (-1)^n w(t - nT_f) \tag{44}$$

where s is the symbol and $w(t)$ is the pulse waveform. From the second antenna, it transmits

$$s_1(t) = s\sqrt{\frac{E_s}{2N_f}} \sum_{n=0}^{N_f-1} w(t - nT_f) \tag{45}$$

where the factor $\sqrt{2}$ is the normalized constant. To illustrate the idea, suppose that the waveform is a positive rectangular pulse and let us denote "1" and "-1" for the positive and negative rectangular pulse respectively. First, suppose that the symbol s is 1, then the STC scheme in matrix form becomes

$$\mathbf{c} = \begin{bmatrix} +1 & -1 \\ +1 & +1 \end{bmatrix}. \tag{46}$$

Therefore, this matrix is repeated $N_f/2$ times to send the symbol $s = 1$. To send the symbol $s = -1$, the matrix becomes

$$\mathbf{c} = \begin{bmatrix} -1 & +1 \\ -1 & -1 \end{bmatrix}. \tag{47}$$

In this scheme, the Binary Error Rate (BER) is upper bounded at high SNR by

$$P_e \leq \left(\frac{\beta_L}{2}\rho\right)^{-L} \tag{48}$$

where L is the number of multipaths (note that they consider it as a deterministic), $\rho = E_s/\sigma^2$ (SNR in AWGN channel) and $\beta_L = (\prod_{l=0}^{L-1} E[h_l^2])^{1/L}$. It implies a diversity order L and a coding gain $\beta/2$. Compare it with single antenna system, this scheme doubles the diversity order at the expense of a 3-dB loss in coding gain.

STC Scheme II

Instead of transmitting the same symbol simultaneously from the two transmit antennas, it can transmits two consecutive symbols s_a and s_b alternately from each of the two transmit antennas. Therefore, over two symbol duration $2N_fT_f$, it transmits

$$s_o(t) = \sqrt{\frac{E_s}{2N_f}} \sum_{n=0}^{N_f-1} [s_a w(t - 2nT_f) - s_b w(t - 2nT_f - T_f)] \tag{49}$$

from the first antenna and

$$s_1(t) = \sqrt{\frac{E_s}{2N_f}} \sum_{n=0}^{N_f-1} [s_b w(t - 2nT_f) + s_a w(t - 2nT_f - T_f)] \tag{50}$$

from the second antenna. In matrix form, it is

$$\mathbf{c} = \begin{bmatrix} s_a & -s_b \\ s_b & s_a \end{bmatrix}. \tag{51}$$

In this scheme, BER is upper bounded at high SNR by

$$P_e \leq \left(\frac{\beta_L}{4}\rho\right)^{-2L} \tag{52}$$

which implies a diversity order 2L and a coding gain $\beta_L/4$. This ST coding scheme provide twice the diversity order compared to single antenna system without increasing either the number of Rake receiver finger, or channel estimation burden. It has longer delay than the first scheme though. The symbol is the same for all cases, one symbol per N_f frames.

Generalization of the Schemes

STC scheme II can add an interleaver at the transmitter, by deploying an $N_f \times N_i$ block interleaver, the average BER is upper bounded at high SNR by

$$P_e \leq \left(\frac{\beta_L}{2N_i}\rho\right)^{-N_iL}. \tag{53}$$

It achieve a diversity order N_i time that provided by STC scheme I with identical L and the same estimation complexity at the expense of decoding delay by N_i symbols and loss in coding gain by a factor N_i.

By deploying more receive antennas, the average BER for STC scheme I is given by

$$P_e \leq \left(\frac{\beta_L}{2}\rho\right)^{-N_rL}, \tag{54}$$

and for STC scheme II with an interleaver is given by

$$P_e \leq \left(\frac{\beta_L}{2N_i}\rho\right)^{-N_rN_iL}. \tag{55}$$

4.2 Real Orthogonal Design ST Codes

In [7], ST coding is analyzed for multiuser environment. Even though it is for multiuser schemes, it can be reduced to a single user system since the multiple accesss interference is considered as Gaussian noise. They assumed that the channel is a frequency-selective channel and the gains are Nagakami-m distributed. It describes 3 multiple access techniques, TH-BPPM, TH-BPSK and DS-BPSK. Suppose a UWB-MIMO system is equipped with N_t transmit antennas and with N_r receive antennas. The input is a binary symbol sequence which is divided into blocks of N_b symbols. The block is encoded into a ST codeword to be transmitted over N_t transmit antennas during K time slots. The codeword matrix can be expressed as an $K \times N_t$ matrix

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1N_t} \\ d_{21} & d_{22} & \cdots & d_{2N_t} \\ \vdots & \vdots & \vdots & \vdots \\ d_{K1} & c_{K2} & \cdots & c_{KN_t} \end{bmatrix} \tag{56}$$

In this case, the code rate is $R = N_b/K$.

TH-BPPM

The PEP at high SNR is given by

$$P(\mathbf{D} \to \mathbf{e}) \leq \left(G_0(\tilde{m}) \frac{\rho}{4N_t} \right)^{-mr\tilde{N}_r L} \tag{57}$$

where $G_0(\tilde{m} = (\tilde{m})^{-1} \left(\prod_{l=0}^{L-1} \Omega_0(l) \right)^{1/L} \left(\prod_{i=0}^{r-1} \lambda_i \right)^{1/r}$, r is the rank and $\{\lambda_i\}$ represent nonzero eigenvalues of matrix $\mathbf{Z}$. The matrix $\mathbf{Z} = (\mathbf{D}\text{-}\mathbf{e})^T (\mathbf{D}\text{-}\mathbf{e})$,$\rho = [1 - \gamma(T_d)]^2 E_0/(2\sigma^2)$ and $\tilde{m} = mN_t/(mN_t - m + 1)$. Ω_0 and m is the parameters of Nakagami distribution. Therefore, the minimum values of $\tilde{m} r N_r L$ and $G_0(\tilde{m})$ over all pairs of distinct codewords define diversity gain and the coding gain. Since $r \leq N_t$, then the maximum achievable diversity gain is $\tilde{m} N_t N_r L$. Based on real orthogonal design, TH-BPPM signals for two symbols and two frames can be specified as

$$\mathbf{X}(t) = \frac{E_s}{N_t} \begin{bmatrix} \tilde{w}(t - c(0)T_c - \frac{1-d_{11}}{2} T_d) & \tilde{w}(t - c(0)T_c - \frac{1-d_{12}}{2} T_d) \\ \tilde{w}(t - T_f - c(1)T_c - \frac{1+d_{12}}{2} T_d) & \tilde{w}(t - T_f - c(1)T_c - \frac{1-d_{11}}{2} T_d) \end{bmatrix}. \tag{58}$$

In this case, the code is full rate, $R = 1$. It can be generalized to K frames, then the diversity gain is increased and the coding gain is decreased, $R = 1/K$.

TH-BPSK

The PEP is the same as the TH-BPPM case and the TH-BPSK Space-Time signal is given by

$$\mathbf{X}(t) = \frac{E_s}{N_t} \begin{bmatrix} d_{11}\tilde{w}(t - c(0)T_c) & d_{12}\tilde{w}(t - c(0)T_c -) \\ d_{12}\tilde{w}(t - T_f - c(1)T_c) & d_{11}\tilde{w}(t - T_f - c(1)T_c) \end{bmatrix}. \tag{59}$$

.

DS-BPSK

In this case, the PEP is a hard to find a close form expression and it is,

$$P(\mathbf{D} \to \mathbf{e}|\mathbf{H}) \leq \frac{1}{2} \exp\left(-\frac{\rho}{4N_t} \mathbf{q}^T \mathbf{U} \Delta \mathbf{U}^T \mathbf{q} \right) \tag{60}$$

See details in [7] for the derivations. The main point is that maximum diversity gain can be achieved by maximizing the rank of Δ which is

$$rank(\Delta) = N_r L rank(\Lambda) = N_r L rank(\mathbf{Z})$$

Therefore, the rank criterion for DS-UWB ST system is identical to that of TH-UWB ST system, i.e., the diversity gain can be maximized when $\mathbf{Z}$ is full rank. In this case, it is very difficult to quantify the coding gain. The TH-BPSK Space-Time signal is given by

$$\mathbf{X}(t) = \frac{E_s}{N_t N_c} \sum_{n=0}^{N_c-1} c(n) \begin{bmatrix} d_{11}\tilde{w}(t - nT_c) & d_{12}\tilde{w}(t - nT_c -) \\ d_{12}\tilde{w}(t - T_f - nT_c) & d_{11}\tilde{w}(t - T_f - nT_c) \end{bmatrix}. \tag{61}$$

.

Performance Analysis

They concluded that TH-BPSK and DS-BPSK tend to outperform TH-BPPM system for every code rate in a single user scenario. In a multiuser environment, DS-BPSK perform better than the other two systems. In addition, reducing the rate of UWB ST code would not improve the performance of single user systems for all modulation schemes. However, in multiuser scenario, reducing the code rate improves the performance of TH systems, while the improvement in DS system is not significantly.

5 Discussion

Base on the literature survey, the perfect codes could be used directly in UWB systems depending on the multiple access techniques. Since UWB symbol constellations are reals, it might put more restrictions to the developed perfect codes. If time hopping scheme is employed, then it is possible to use the perfect codes directly in the analog space-time coding schemes. The time hopping pattern can avoid interference (ICI) at the expense of data rate.

In the direct sequence multiple access scheme, it is *not* possible to use the perfect codes directly because of interchip interference. One possible solution is to use an equalizer that can reduce the frequency-selective effect and convert UWB channels into approximately flat fading channels. In this case, it would require high complexity receivers.

When OFDM modulation is employed in UWB systems, each frequency band is a flat fading channel, and then we can apply directly the perfect codes. However, we are not taking advantages of frequency diversity. This approach would need to change the perfect codes in order to achieve full diversity.

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